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SPECIALIST MATHS 3 & 4 TRIAL EXAMINATION 1 SOLUTIONS 2016

Question 1 (4 marks)

a.

$$\frac{\bar{z}_1}{1+i} = \frac{-2i}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{-2i-2}{1+1}$$
$$= -1-i$$

(1 mark)

b. Let $P(z) = z^3 - 3z^2 + 4z - 12$.

Substitute $z_1 = 2i$ into $P(z)$.

$$P(2i) = (2i)^3 - 3(2i)^2 + 4(2i) - 12$$
$$= -8i + 12 + 8i - 12$$
$$= 0 \text{ as required}$$

(1 mark)

c. Since the coefficients of the terms in the equation are real we know that $-2i$ is also a solution (conjugate root theorem). **(1 mark)**

So $z-2i$ and $z+2i$ are factors.
So $(z-2i)(z+2i) = z^2 + 4$ is also a factor.

Method 1 – using the constant term

$P(z) = z^3 - 3z^2 + 4z - 12$
$P(3) = 27 - 27 + 12 - 12 = 0$
So $z = 3$ is the third solution.
The solutions are $z = \pm 2i, 3$.

(1 mark)

Method 2 – using long division

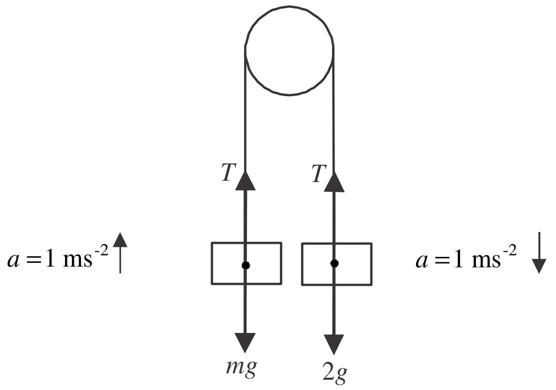
$$\begin{array}{r} z-3 \\ z^2+4 \overline{\smash{\big)}\, z^3 - 3z^2 + 4z - 12} \\ \underline{z^3 \qquad\quad + 4z \quad\;\;} \\ -3z^2 \qquad - 12 \\ \underline{-3z^2 \qquad - 12} \\ 0 \end{array}$$

So $z^3 - 3z^2 + 4z - 12 = 0$
becomes $(z^2 + 4)(z - 3) = 0$
The solutions are $z = \pm 2i, 3$.

(1 mark)

Question 2 (4 marks)

a. Mark in the forces. The tension force in the string is T.



<u>Around the m kg particle</u>

$T - mg = m \times 1$

$T = m + mg$

<u>Around the 2 kg particle</u>

$2g - T = 2 \times 1$

$T = 2g - 2$

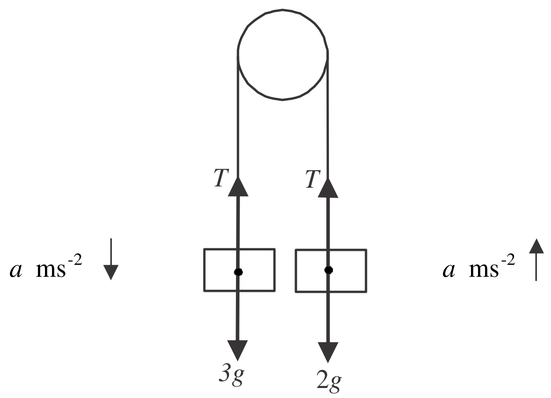
(1 mark)

So $m(1+g) = 2g - 2$

$$m = \frac{2g-2}{1+g}$$

(1 mark)

b. Mark in the forces.



<u>Around the 3 kg particle</u>

$3g - T = 3a$

$T = 3g - 3a$

<u>Around the 2 kg particle</u>

$T - 2g = 2a$

$T = 2g + 2a$

(1 mark)

So $3g - 3a = 2g + 2a$

$5a = g$

$$a = \frac{g}{5}$$

(1 mark)

Question 3 (3 marks)

$f(x) = \arcsin(3x)$

Method 1

$$f'(x) = \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}$$
$$= \frac{1}{\sqrt{\frac{1-9x^2}{9}}}$$
$$= \frac{3}{\sqrt{1-9x^2}}$$
$$= 3(1-9x^2)^{-\frac{1}{2}}$$

Method 2

Let $y = \arcsin(u)$ where $u = 3x$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \text{(chain rule)}$$
$$= \frac{1}{\sqrt{1-u^2}} \times 3$$
$$= \frac{3}{\sqrt{1-9x^2}}$$
$$= 3(1-9x^2)^{-\frac{1}{2}}$$

(1 mark)

$$f''(x) = 3 \times -\frac{1}{2}(1-9x^2)^{-\frac{3}{2}} \times -18x \qquad \text{(chain rule)}$$
$$= \frac{27x}{\sqrt{(1-9x^2)^3}}$$

(1 mark)

$$f''\left(\frac{1}{6}\right) = \frac{27}{6} \div \sqrt{\left(\frac{3}{4}\right)^3}$$
$$= \frac{9}{2} \div \sqrt{\frac{27}{64}}$$
$$= \frac{9}{2} \times \frac{8}{3\sqrt{3}}$$
$$= \frac{12}{\sqrt{3}}$$
$$= 4\sqrt{3}$$

(1 mark)

Question 4 (3 marks)

$$xy^2 + y\log_e(x) - 2y - 3 = 0$$

$$y^2 + 2xy\frac{dy}{dx} + \frac{dy}{dx}\log_e(x) + \frac{y}{x} - 2\frac{dy}{dx} = 0$$

(1 mark)

$$(2xy + \log_e(x) - 2)\frac{dy}{dx} = -y^2 - \frac{y}{x}$$
$$= \frac{-xy^2 - y}{x}$$
$$\frac{dy}{dx} = \frac{-xy^2 - y}{x(2xy + \log_e(x) - 2)}$$

At (1, 3),

$$\frac{dy}{dx} = \frac{-9-3}{6+\log_e(1)-2}$$

(1 mark)

$$= \frac{-12}{4}$$
$$= -3$$

(1 mark)

Question 5 (6 marks)

a. The approximate 95% confidence interval for μ is $\left(\bar{x}-1.96\frac{s}{\sqrt{100}},\ \bar{x}+1.96\frac{s}{\sqrt{100}}\right)$.

So $\bar{x}-1.96\frac{s}{10}=15.02 \quad -(1)$

and $\bar{x}+1.96\frac{s}{10}=16.98 \quad -(2)$ **(1 mark)**

Solve these two equations simultaneously.

$(1)+(2)$ gives $2\bar{x}=32$

$\bar{x}=16$

In (1), $16-0.196s=15.02$

$-0.196s=-0.98$

$s=\frac{0.98}{0.196}$

$s=\frac{980}{196}$

$s=5$

(1 mark)

b. **i.** $H_0: \quad \mu=5.2$

$H_1: \quad \mu>5.2$

(1 mark)

ii. $\text{E}(\bar{X})=\mu=5.2 \qquad \text{sd}(\bar{X})=\frac{1.5}{\sqrt{25}}$

$=\frac{1.5}{5}$

$=0.3$

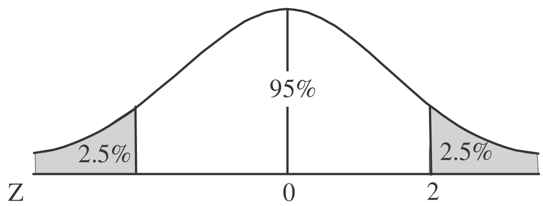
p value $=\Pr(\bar{X}\geq 5.8 \mid \mu=5.2)$ **(1 mark)**

$=\Pr\left(Z\geq\frac{5.8-5.2}{0.3}\right)$

$=\Pr\left(Z\geq\frac{0.6}{0.3}\right)$

$=\Pr(Z\geq 2)$

$=0.025$



(1 mark)

iii. From part **ii.**, p value = 0.025.
Since p value < 0.05 there is good evidence to reject the null hypothesis at the 5% level.

(1 mark)

Question 6 (3 marks)

$$\int_0^{\frac{\pi}{4}} \cos^2(x)\sin(2x)dx$$

$$= \int_0^{\frac{\pi}{4}} 2\sin(x)\cos^3(x)dx$$ since $\sin(2x) = 2\sin(x)\cos(x)$ **(1 mark)**

$$= -2\int_1^{\frac{1}{\sqrt{2}}} u^3 \frac{du}{dx} dx$$

Let $u = \cos(x)$ also $x = \frac{\pi}{4}$ so $u = \frac{1}{\sqrt{2}}$

$\frac{du}{dx} = -\sin(x)$ and $x = 0$ so $u = 1$

$$= -2\int_1^{\frac{1}{\sqrt{2}}} u^3 du$$ **(1 mark)**

$$= -2\left[\frac{u^4}{4}\right]_1^{\frac{1}{\sqrt{2}}}$$

$$= -2\left(\frac{1}{16} - \frac{1}{4}\right)$$

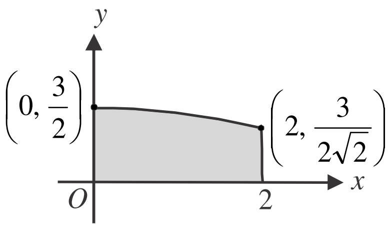
$$= -\frac{1}{8} + \frac{1}{2}$$

$$= \frac{3}{8}$$

(1 mark)

Question 7 (3 marks)

a. Do a quick sketch.



$$\text{volume} = \pi\int_0^2 y^2 dx$$

$$= \pi\int_0^2 \frac{9}{x^2+4} dx$$

(1 mark)

b.

$$\text{volume} = \frac{9\pi}{2}\int_0^2 \frac{2}{x^2+4} dx$$

$$= \frac{9\pi}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$ **(1 mark)**

$$= \frac{9\pi}{2}(\tan^{-1}(1) - \tan^{-1}(0))$$

$$= \frac{9\pi}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{9\pi^2}{8} \text{ cubic units}$$

(1 mark)

Question 8 (5 marks)

a. We require $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$

$$2 + m + 8 = 0$$
$$m = -10$$

(1 mark)

b.

$$\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$$
$$|\underset{\sim}{a}| = \sqrt{2^2 + 1^2 + (-2)^2}$$
$$= 3$$
$$\hat{\underset{\sim}{a}} = \frac{1}{3}(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$$

(1 mark)

So $\underset{\sim}{d} = 6\hat{\underset{\sim}{a}}$

$$= 6 \times \frac{1}{3}(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$$
$$= 4\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$$

(1 mark)

c. If $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent then $\alpha\underset{\sim}{a} + \beta\underset{\sim}{b} = \underset{\sim}{c}$ where α and $\beta \in R$.

So we require $\alpha(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) + \beta(\underset{\sim}{i} + m\underset{\sim}{j} - 4\underset{\sim}{k}) = -\underset{\sim}{i} + 3\underset{\sim}{j}$.

For the $\underset{\sim}{i}$ components,

$$2\alpha + \beta = -1 \qquad (1)$$

For the $\underset{\sim}{j}$ components,

$$\alpha + m\beta = 3 \qquad (2)$$

For the $\underset{\sim}{k}$ components,

$$-2\alpha - 4\beta = 0$$

So $\alpha = -2\beta$ **(1 mark)**

In (1) $-3\beta = -1$

$$\beta = \frac{1}{3}$$

So $\alpha = -\frac{2}{3}$

In (2) $-\frac{2}{3} + \frac{m}{3} = 3$

$m = 11$ **(1 mark)**

If you have time, check your answer.

$$-\frac{2}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b} = \underset{\sim}{c}$$
$$LS = -\frac{4}{3}\underset{\sim}{i} - \frac{2}{3}\underset{\sim}{j} + \frac{4}{3}\underset{\sim}{k} + \frac{1}{3}\underset{\sim}{i} + \frac{11}{3}\underset{\sim}{j} - \frac{4}{3}\underset{\sim}{k}$$
$$= -\underset{\sim}{i} + 3\underset{\sim}{j}$$
$$= RS$$

Question 9 (5 marks)

$$\frac{dy}{dx}=\frac{x\sqrt{x^2-1}}{e^{2y}}$$

$$\int e^{2y}dy=\int x\sqrt{x^2-1}\,dx \quad \text{(separation of variables)}$$ **(1 mark)**

$$\frac{e^{2y}}{2}+c_1=\int \frac{1}{2}\frac{du}{dx}u^{\frac{1}{2}}dx \qquad u=x^2-1$$

$$=\frac{1}{2}\int u^{\frac{1}{2}}du \qquad \frac{du}{dx}=2x$$

$$=\frac{1}{2}u^{\frac{3}{2}}\times\frac{2}{3}+c_2$$

So $\frac{e^{2y}}{2}=\frac{1}{3}(x^2-1)^{\frac{3}{2}}+c$ where $c=c_2-c_1$

(1 mark) – left side
(1 mark) – right side

When $x=1,\ y=0$.

So $\frac{e^0}{2}=\frac{1}{3}\times 0+c$

$$c=\frac{1}{2}$$ **(1 mark)**

So $\frac{e^{2y}}{2}=\frac{1}{3}(x^2-1)^{\frac{3}{2}}+\frac{1}{2}$

$$e^{2y}=\frac{2}{3}(x^2-1)^{\frac{3}{2}}+1$$

$$2y=\log_e\left(\frac{2}{3}(x^2-1)^{\frac{3}{2}}+1\right)$$

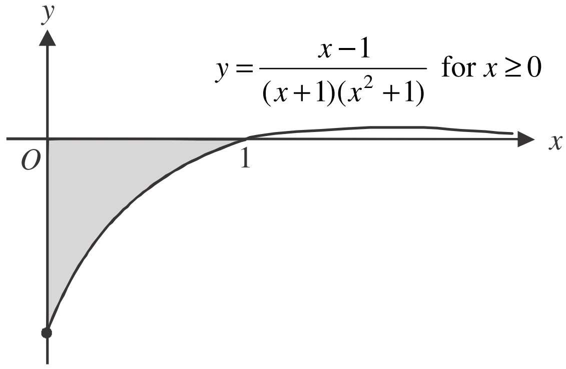
$$y=\log_e\sqrt{\frac{2}{3}(x^2-1)^{\frac{3}{2}}+1}$$

So $a=2, b=3$ and $c=1$.

(1 mark)

Question 10 (4 marks)

A sketch of $y=\dfrac{x-1}{(x+1)(x^2+1)}$ for $x \ge 0$ is shown below.



The shaded region shown above is equal in area to the shaded region shown in the question, but is below the *x*–axis.
The *x*-intercept for both graphs occurs when $x-1=0$ i.e. at $(1,0)$.

Let $\dfrac{(x-1)}{(x+1)(x^2+1)} \equiv \dfrac{A}{(x+1)}+\dfrac{Bx+C}{x^2+1}$ **(1 mark)**

$$\equiv \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

True iff $x-1=A(x^2+1)+(Bx+C)(x+1)$

Put $x=-1$, $-2=2A$, $A=-1$
Put $x=1$, $0=-2+2B+2C$ ________(1)
Put $x=0$, $-1=-1+C$, $C=0$
In (1) so $B=1$ **(1 mark)**

So $\dfrac{x-1}{(x+1)(x^2+1)}=\dfrac{-1}{x+1}+\dfrac{x}{x^2+1}$

area required $=-\displaystyle\int_0^1\left(\frac{-1}{x+1}+\frac{x}{x^2+1}\right)dx$ **(1 mark)**

$$=\left[\log_e|x+1|\right]_0^1-\left[\frac{1}{2}\log_e(x^2+1)\right]_0^1$$
$$=\log_e(2)-\log_e(1)-\frac{1}{2}\log_e(2)+\frac{1}{2}\log_e(1)$$
$$=\log_e(2)-\log_e\sqrt{2}$$
$$=\log_e\left(\frac{2}{\sqrt{2}}\right)$$
$$=\log_e(\sqrt{2})\text{ square units}$$

(1 mark)