2016 VCAA Specialist Mathematics Exam 2 Solutions

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SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
E	B	C	E	A	C	A	D	C	B

11	12	13	14	15	16	17	18	19	20
D	C	A	B	D	D	E	A	B	E

Q1 $\operatorname{cosec}^2 t - \cot^2 t = 1$, $\frac{x}{3} - \frac{(y+1)^2}{16} = 1$, $(y+1)^2 = \frac{16(x-3)}{3}$ E

Q2 $-1 \le \frac{x-a}{b} \le 1$, $a-b \le x \le a+b$ B

Q3 $f(x) = x - \frac{a}{x}$ where $x \ne 0$, asymptotes are $y = x$ and $x = 0$ C

Q4 $z^3 + bz^2 + cz = z(z^2 + bz + c)$
Consider the quadratic: $c =$ product of roots , $b = -($sum of roots$)$
.: $c = (3-2i)(3+2i) = 13$ and $b = -6$ E

Q5 z is in the third quadrant. A

Q6 C

Q7 $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos 2t}{\cos t + \sin t} = \frac{(\cos t - \sin t)(\cos t + \sin t)}{\cos t + \sin t}$
$= \cos t - \sin t$ A

Q8 $u = x^4$, $\frac{du}{dx} = 4x^3$, $\frac{1}{4}\frac{du}{dx} = x^3$
$\int_a^b x^3 e^{2x^4}\, dx = \int_a^b \frac{1}{4} e^{2u} \frac{du}{dx}\, dx = \frac{1}{4}\int_{a^4}^{b^4} e^{2u}\, du$ D

Q9
$x = 2$, $y_0 = 0$, $\frac{dy}{dx} = f(2) = 6$
$x = 2.1$, $y_1 \approx 0 + 0.1 \times 6 = 0.6$, $\frac{dy}{dx} = f(2.1)$
$x = 2.2$, $y_2 \approx 0.6 + 0.1 \times f(2.1) = 1.272$, $\frac{dy}{dx} = f(2.2)$
$x = 2.3$, $y_3 \approx 1.272 + 0.1 \times f(2.2)$ C

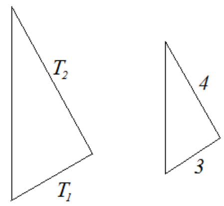
Q10 B

Q11 $\tilde{a}.\hat{b} = \frac{10+\alpha^3}{\sqrt{17+\alpha^4}} = \frac{74}{\sqrt{273}}$, $\alpha = 4$ D

Q12 $\tilde{a} - \tilde{b} = (-2+m)\tilde{i} - 2\tilde{j} + \tilde{k}$
$(\tilde{a} - \tilde{b}).\tilde{b} = ((-2+m)\tilde{i} - 2\tilde{j} + \tilde{k}).(-m\tilde{i} + \tilde{j} + 2\tilde{k}) = 0$
$-m(-2+m) - 2 + 2 = 0$, .: $m = 0$ or 2 C

Q13 Net force $= \sqrt{9^2 + 12^2} = 15 = 5a$, $a = 3$ ms^{-2} A

Q14 $\frac{T_1}{T_2} = \frac{3}{4}$ B



Q15 $v = 3 - x^2$, $\frac{1}{2}v^2 = \frac{1}{2}(3-x^2)^2$,
$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2x(3-x^2) = 2x^3 - 6x$, $F = ma = 6x^3 - 18x$ D

Q16 Vertically: $u = 20\sin 30° = 10$, $v = -10$, $a = -g$,
$t = \frac{v-u}{a} = \frac{20}{g}$
Horizontally, $u = 20\cos 30° = 10\sqrt{3}$, $s = ut = 10\sqrt{3} \times \frac{20}{g} = \frac{200\sqrt{3}}{g}$ D

Q17 $\Delta p = m(v-u) = 3(2-(-2)) = 12$ kg ms^{-1} E

Q18 Assume that the fruits are randomly selected, .: the masses of the fruits are independent.
Let random variable O be the mass of an orange and L be the mass of a lemon.
$\mathrm{E}(O_1 + O_2 + O_3 + L_1 + L_2) = \mathrm{E}(O_1) + \mathrm{E}(O_2) + \mathrm{E}(O_3) + \mathrm{E}(L_1) + \mathrm{E}(L_2)$
$= 3\times\mathrm{E}(O) + 2\times\mathrm{E}(L) = 3\times 204 + 2\times 76 = 764$
$\mathrm{Var}(O_1 + O_2 + O_3 + L_1 + L_2)$
$= \mathrm{Var}(O_1) + \mathrm{Var}(O_2) + \mathrm{Var}(O_3) + \mathrm{Var}(L_1) + \mathrm{Var}(L_2)$
$= 3\times\mathrm{Var}(O) + 2\times\mathrm{Var}(L) = 3\times 9^2 + 2\times 3^2 = 261$
$\mathrm{sd}(O_1 + O_2 + O_3 + L_1 + L_2) = \sqrt{261} = 3\sqrt{29}$ A

Q19 $\left(210 - 1.96\times\frac{16}{\sqrt{100}}, 210 + 1.96\times\frac{16}{\sqrt{100}}\right) \approx (206.9, 213.1)$ B

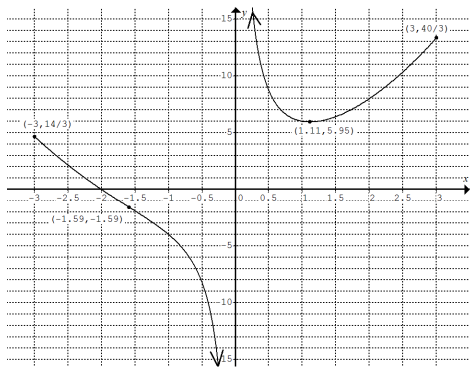
Q20 Normal distribution of the sample mean $\bar{X}$:
$\mathrm{E}(\bar{X}) = \mu = 20$, $\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{2}{5}$, $\Pr(\bar{X} > 19.3) \approx 0.9599$ E

SECTION B

Q1a $y = \dfrac{4 + x^2 + x^3}{x}$, stationary point $(1.11, 5.95)$ approx.

Q1b Find (x, y) such that $\dfrac{d^2y}{dx^2} = 0$, point of inflection $(-1.59, -1.59)$

Q1c



Q1di $\displaystyle\int_{-3}^{-0.5} \sqrt{1 + (f'(x))^2}\, dx = \int_{-3}^{-0.5} \sqrt{1 + \left(\frac{2x^3 + x^2 - 4}{x^2}\right)^2}\, dx$

Q1dii 13.18 units

Q1e $a = \pi$, $b = f(-3) = \dfrac{14}{3}$, $c = f(-0.5) = -\dfrac{33}{4}$

Q2a Let $z = x + yi$, $|(x-1) + yi| = |(x+2) + (y-3)i|$

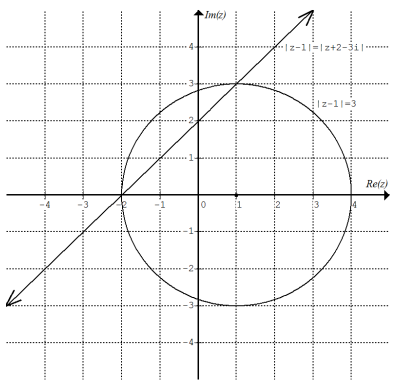
$(x-1)^2 + y^2 = (x+2)^2 + (y-3)^2$, $y = x + 2$

Q2b Circle $(x-1)^2 + y^2 = 9$, line $y = x + 2$

$\therefore$ $x = -2$, 1 and $y = 0$, 3 respectively

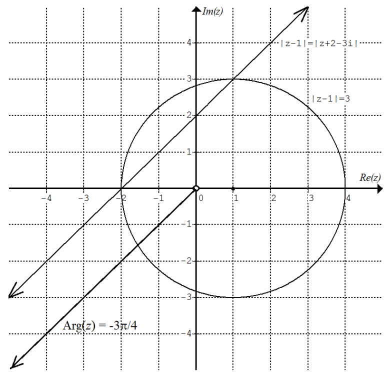
Points of intersection are $(-2, 0)$ and $(1, 3)$

Q2c



Q2d Area of major segment $= \dfrac{1}{2} \times 3 \times 3 + \dfrac{3}{4} \times \pi \times 3^2 = \dfrac{9(2 + 3\pi)}{4}$ unit^2

Q2e



Q2f $\left(-1, -\dfrac{3}{4}\right) \cup \left(\dfrac{1}{4}, 1\right]$

Q3a $\dfrac{dx}{dt} = -\dfrac{x}{20 + t}$, $x > 0$, $t \ge 0$

$\displaystyle\int \frac{1}{x}\, dx = \int -\frac{1}{20 + t}\, dt$, $\log_e x = -\log_e(20 + t) + c$

When $t = 0$, $x = 20$, $c = 2\log_e 20 = \log_e 400$,

$\log_e x = \log_e 400 - \log_e(20 + t)$ $\therefore$ $x = \dfrac{400}{20 + t}$

Q3b y kg of salt in $(100 + 10t)$ L of solution after t min

Concentration $= \dfrac{y}{100 + 10t}$ kg per L

Q3c Rate of inflow of salt $= \dfrac{1}{60} \times 20 = \dfrac{1}{3}$ kg per min

Rate of outflow of salt $= \dfrac{y}{100 + 10t} \times 10 = \dfrac{y}{10 + t}$ kg per min

$\therefore$ $\dfrac{dy}{dt} = \dfrac{1}{3} - \dfrac{y}{10 + t}$, $\therefore$ $\dfrac{dy}{dt} + \dfrac{y}{10 + t} = \dfrac{1}{3}$

Q3d $y = \dfrac{t^2 + 20t + 900}{6(10 + t)}$, $\dfrac{y}{10 + t} = \dfrac{t^2 + 20t + 900}{6(10 + t)^2}$

$\dfrac{dy}{dt} = \dfrac{1}{6} \times \dfrac{(10 + t)(2t + 20) - (t^2 + 20t + 900)}{(10 + t)^2} = \dfrac{t^2 + 20t - 700}{6(10 + t)^2}$

$\therefore$ $\dfrac{dy}{dt} + \dfrac{y}{10 + t} = \dfrac{1}{3}$

$y = \dfrac{t^2 + 20t + 900}{6(10 + t)}$, when $t = 0$, $y = \dfrac{900}{60} = 15$

Q3e Concentration $= \dfrac{y}{100 + 10t} = \dfrac{y}{10(10 + t)} = \dfrac{t^2 + 20t + 900}{60(10 + t)^2} = 0.095$

$\therefore$ $t \approx 3.05$ min

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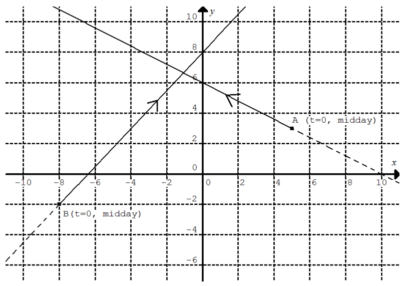
Q4a Consider the time when the ships have the same *x*-coordinate. $5(1-t)=4(t-2)$, $t=\frac{13}{9}$. At $t=\frac{13}{9}$, *y*-coordinates of *A* and *B* are different, $y_A=\frac{22}{3}$, $y_B=\frac{47}{9}$, .: the two ships will not collide.

Q4b *A*: $x=5(1-t)$, $y=3(1+t)$, eliminate t, $y=-\frac{3}{5}x+6$

B: $x=4(t-2)$, $y=5t-2$, .: $y=\frac{5}{4}x+8$



Q4c Let α be the obtuse angle between the two paths, and the paths of *A* and *B* make angles θ and ϕ with the positive *x*-axis respectively. $\alpha=\theta-\phi$,

$$\tan\alpha=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{m_A-m_B}{1+m_Am_B}=\frac{-\frac{3}{5}-\frac{5}{4}}{1-\frac{3}{5}\times\frac{5}{4}}=-\frac{37}{5}, \text{ .: } \alpha\approx 97.7^\circ$$

Q4di $\tilde{r}_B-\tilde{r}_A=(9t-13)\tilde{i}+(2t-5)\tilde{j}$

$|\tilde{r}_B-\tilde{r}_A|^2=(9t-13)^2+(2t-5)^2$,

$\frac{d}{dt}|\tilde{r}_B-\tilde{r}_A|^2=18(9t-13)+4(2t-5)=0$, $t\approx 1.494$

Q4dii When $t\approx 1.494$,

min. distance $=|\tilde{r}_B-\tilde{r}_A|=\sqrt{(9t-13)^2+(2t-5)^2}\approx 2.06$ km

Q5a $t\in[0,5]$, $t=0$, $v=0$, $x=0$

$a=\frac{F}{m}=\frac{50-10t}{2}-g$, $\frac{dv}{dt}=25-9.8-5t$, $\frac{dv}{dt}=\frac{76}{5}-5t$

Q5b $v=\frac{76}{5}t-\frac{5}{2}t^2$, $v(5)=13.5$ ms^{-1}

Q5c $x=\frac{38}{5}t^2-\frac{5}{6}t^3$, $x(5)\approx 85.83$, height ≈ 85.83 m

Q5d $u=13.5$, $a=-9.8$, $v=0$, $s=\frac{v^2-u^2}{2a}\approx 9.30$

Maximum height $\approx 85.83+9.30=95.13$ m

Q5e After the initial 5 s, the time to reach the ground:

$u=13.5$, $a=-9.8$, $s=-85.83$, .: $-85.83=13.5t+\frac{1}{2}(-9.8)t^2$

$t\approx 5.78$

Total time of flight $\approx 5+5.78\approx 10.8$ s

Q6a Population: $\mu=1.1$ mg/L, $\sigma=0.16$ mg/L

Samples: $n=25$,

Mean of $\bar{X}=\mathrm{E}(\bar{X})=\mu=1.1$, $\mathrm{sd}(\bar{X})=\frac{\sigma}{\sqrt{n}}=\frac{0.16}{5}=0.032$

Q6b H_0 : The mean level of pollutant remains the same.

H_1 : The mean level of pollutant has increased.

Q6ci p value $=\Pr\left(Z\ge\frac{1.2-1.1}{0.032}\right)\approx 0.0009$

Q6cii Since $p<0.05$, the sample supports H_1

Q6d $\Pr(\bar{X}>\bar{x}_c \mid \mu=1.1)=0.05$,

$\Pr(\bar{X}<\bar{x}_c \mid \mu=1.1)=\Pr\left(Z<\frac{\bar{x}_c-1.1}{0.032}\right)=0.95$

$\bar{x}_c\approx 1.153$

Q6e $\Pr(\bar{X}<1.163 \mid \mu=1.2)=\Pr\left(Z<\frac{1.163-1.2}{0.032}\right)\approx 0.124$

Please inform mathline@itute.com re conceptual and/or mathematical errors

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