The Mathematical Association of Victoria

Trial Written Examination 2016

SPECIALIST MATHEMATICS

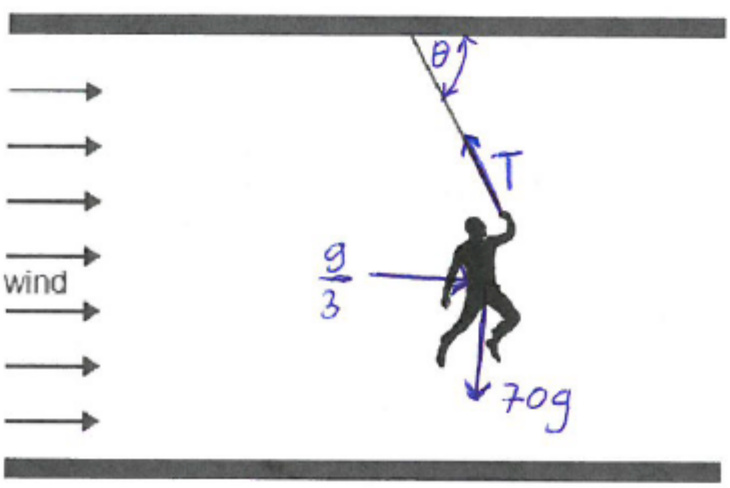
Trial Written Examination 1 - SOLUTIONS

Question 1

Force from wind $= \frac{g}{3}$.

Weight force $= 70g$.

Let T be the tension in the rope and let the rope make an acute angle θ to the roof.



Resolve forces acting on the person in the vertical and horizontal directions.

Note that the tension force resolves into a vertical component of size $T\sin(\theta)$ in the upwards direction and a horizontal component of size $T\cos(\theta)$ to the left.

Take the upwards and right directions as positive.

Vertical direction: $0 = T\sin(\theta) - 70g \quad \Rightarrow T\sin(\theta) = 70g$. .... (1) **[A1]**

Horizontal direction: $0 = \frac{g}{3} - T\cos(\theta) \quad \Rightarrow T\cos(\theta) = \frac{g}{3}$. .... (2) **[A1]**

Solve equations (1) and (2) simultaneously for T.

$$(1)^2 + (2)^2 : \; T^2 \sin^2(\theta) + T^2 \cos^2(\theta) = (70g)^2 + \left(\frac{g}{3}\right)^2$$

$$\Rightarrow T^2\left(\sin^2(\theta) + \cos^2(\theta)\right) = 4900g^2 + \frac{g^2}{9}$$

$$\Rightarrow T^2 = 4900g^2 + \frac{g^2}{9} \qquad = \frac{g^2(9 \times 4900 + 1)}{9} \qquad = \frac{g^2(44101)}{9}.$$

Answer: $\dfrac{g\sqrt{44101}}{3}$. **[A1]**

Question 2

a.

The required vector is $\sqrt{45}\,\hat{\underset{\sim}{a}}$ where $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{\left|\underset{\sim}{a}\right|}$.

$\left|\underset{\sim}{a}\right| = \sqrt{(-1)^2 + (-2\sqrt{3})^2 + (-\sqrt{2})^2} \quad = \sqrt{1+12+2} \quad = \sqrt{15}$.

Therefore the required vector is

$$\frac{\sqrt{45}\left(-\underset{\sim}{i} - 2\sqrt{3}\,\underset{\sim}{j} - \sqrt{2}\,\underset{\sim}{k}\right)}{\sqrt{15}}$$

$$= \sqrt{3}\left(-\underset{\sim}{i} - 2\sqrt{3}\,\underset{\sim}{j} - \sqrt{2}\,\underset{\sim}{k}\right)$$

$$= -\sqrt{3}\,\underset{\sim}{i} - 6\,\underset{\sim}{j} - \sqrt{6}\,\underset{\sim}{k}.$$

Answer: $-\sqrt{3}\,\underset{\sim}{i} - 6\,\underset{\sim}{j} - \sqrt{6}\,\underset{\sim}{k}$. **[A1]**

b. i.

$|\underset{\sim}{b}| = \sqrt{(2)^2 + (-m)^2 + (-3\sqrt{2})^2} \qquad = \sqrt{22 + m^2}$.

Therefore:

$5 = \sqrt{22 + m^2}$ **[M1]**

$\Rightarrow 25 = 22 + m^2$

$\Rightarrow m^2 = 3$.

Answer: $m = \pm\sqrt{3}$. **[A1]**

b. ii.

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\Rightarrow -2 + 2m\sqrt{3} + 6 = 0$. **[M1]**

Answer: $m = \dfrac{-2}{\sqrt{3}}$. **[A1]**

Question 3

$z^3 - (2-i)z^2 = z - 2 + i$

$\Rightarrow z^3 - (2-i)z^2 - z + 2 - i = 0$.

Method 1: Apply the 'pair-pair grouping' technique:

$z^2(z - 2 + i) - 1(z - 2 + i) = 0$ **[M1]**

$\Rightarrow \left(z^2 - 1\right)(z - 2 + i) = 0$. **[M1]**

Apply the Null Factor Law and solve for *z*.

Answer: $z = \pm 1$, $2 - i$. **[A1]**

NOTE: The 'grouping in pairs' technique is often useful for factorising a cubic polynomial when not all the coefficients are real.

Method 2:

By inspection, $z = 1$ is a solution to $z^3 - (2-i)z^2 - z + 2 - i = 0$.

Therefore $z - 1$ is a factor of $z^3 - (2-i)z^2 - z + 2 - i$.

By inspection, $z = -1$ is a solution to $z^3 - (2-i)z^2 - z + 2 - i = 0$.

Therefore $z + 1$ is a factor of $z^3 - (2-i)z^2 - z + 2 - i$.

Therefore $(z-1)(z+1) = z^2 - 1$ is a quadratic factor of $z^3 - (2-i)z^2 - z + 2 - i$. **[M1]**

Option 1:

Therefore $z^3 - (2-i)z^2 - z + 2 - i$ has the factorised form $(z^2 - 1)(z - \alpha)$, $\alpha \in C$.

By comparison with $z^3 - (2-i)z^2 - z + 2 - i$ the constant term in the expansion of $(z^2 - 1)(z - \alpha)$ must equal $2 - i$.

Therefore $\alpha = 2 - i$.

Therefore $z^3 - (2-i)z^2 - z + 2 - i = (z^2 - 1)(z - [2 - i])$. **[M1]**

Apply the Null Factor Law and solve for *z*.

Answer: $z = \pm 1$, $2 - i$. **[A1]**

Option 2:

From polynomial long division it follows that

$z^3 - (2-i)z^2 - z + 2 - i = (z^2 - 1)(z - 2 + i)$: **[M1]**

$$
\begin{array}{r l}
 & z - (2-i) \\
z^2 - 1 \,\big) & z^3 + (-2+i)z^2 - z + 2 - i \\
 & -(z^3 - \quad 0z^2 - z) \\
 & \quad (-2+i)z^2 \quad + 2 - i \\
 & -\big((-2+i)z^2 \quad + 2 - i\big) \\
 & \qquad\qquad 0
\end{array}
$$

Apply the Null Factor Law and solve for z.

Answer: $z = \pm 1$, $2 - i$. **[A1]**

Method 3: *This method is NOT recommended because there is too much work and therefore it takes too long. However, the method is included because instructive and therefore worth understanding.*

By inspection, $z = 1$ is a solution to $z^3 - (2-i)z^2 - z + 2 - i = 0$.

Therefore $z - 1$ is a factor of $z^3 - (2-i)z^2 - z + 2 - i$.

Option 1:

Therefore $z^3 - (2-i)z^2 - z + 2 - i$ has the factorised form $(z-1)(z^2 + \alpha z + \beta)$, $\alpha, \beta \in C$.

By comparison with $z^3 - (2-i)z^2 - z + 2 - i$:

- The constant term in the expansion of $(z-1)(z^2 + \alpha z + \beta)$ must equal $2 - i$.
Therefore $\beta = -(2 - i)$.

- The coefficient of z in the expansion of $(z-1)(z^2 + \alpha z + \beta)$ must equal -1.

Therefore $\beta - \alpha = -1$.

Substitute $\beta = -(2 - i)$:

$-(2 - i) - \alpha = -1$

$\Rightarrow \alpha = -1 + i$.

Therefore $z^3 - (2-i)z^2 - z + 2 - i = (z-1)\big(z^2 + (-1+i)z - (2-i)\big)$. **[M1]**

Option 2:

From polynomial long division it follows that

$$z^3 - (2-i)z^2 - z + 2 - i = (z-1)\left(z^2 + (-1+i)z - (2-i)\right):$$ **[M1]**

$$\begin{array}{r}
z^2 + (-1+i)z + (-2+i) \\
z-1 \overline{\big)\, z^3 + (-2+i)z^2 - z + 2 - i} \\
-(z^3 - \quad z^2) \qquad\qquad\qquad \\
\hline
(-1+i)z^2 - \quad z \quad + 2 - i \\
-\left((-1+i)z^2 - (-1+i)z\right) \qquad \\
\hline
(-2+i)z + 2 - i \\
-\left((-2+i)z - (-2+i)\right) \\
\hline
0
\end{array}$$

The remaining solutions are therefore found by solving $z^2 + (-1+i)z - 2 + i = 0$.
(If it is now seen by inspection that $z = -1$ is a solution, the quadratic can be factorised and solved in a way similar to **Option 1**).

From the quadratic formula:

$$z = \frac{1 - i \pm \sqrt{(-1+i)^2 - 4(-2+i)}}{2} \quad = \frac{1 - i \pm \sqrt{-2i + 8 - 4i}}{2}$$

$$= \frac{1 - i \pm \sqrt{8 - 6i}}{2}.$$

Finding the square roots of $8 - 6i$:

Let $\sqrt{8-6i} = \alpha + \beta i$ where $\alpha, \beta \in R$.

Therefore

$$8 - 6i = (\alpha + \beta i)^2 = \alpha^2 - \beta^2 + 2\alpha\beta i.$$

Equate real and imaginary parts of each side:

Real parts:

$$8 = \alpha^2 - \beta^2. \qquad \ldots. (1)$$

Imaginary parts:

$$-6 = 2\alpha\beta \Rightarrow -3 = \alpha\beta \Rightarrow \beta = -\frac{3}{\alpha}. \qquad \ldots. (2)$$

Solve equations (1) and (2) simultaneously. Substitute (2) into (1):

$$8 = \alpha^2 - \left(\frac{-3}{\alpha}\right)^2$$

$\Rightarrow 8\alpha^2 = \alpha^4 - 9 \quad \Rightarrow \alpha^4 - 8\alpha^2 - 9 = 0$

$\Rightarrow (\alpha^2 - 9)(\alpha^2 + 1) = 0$

$\Rightarrow \alpha = \pm 3$. **[M1]**

(The non-real solutions $\alpha = \pm i$ are rejected because $\alpha \in R$).

Substitute $\alpha = \pm 3$ into equation (2):

$\alpha = 3$: $\beta = -1$.

$\alpha = -3$: $\beta = 1$.

Therefore

$$z = \frac{1 - i \pm (3 - i)}{2} \qquad = \frac{4 - 2i}{2}, \ \frac{-2}{2} \qquad = 2 - i, \ -1.$$

Answer: $z = \pm 1, \ 2 - i$. **[A1]**

NOTE: The pair $\alpha = -3$ and $\beta = 1$ obviously give the same answers.

Question 4

$\frac{dy}{dx} = \frac{y-1}{x+1}$

$\Rightarrow \int \frac{1}{y-1}\,dy = \int \frac{1}{x+1}\,dx$ **[M1]**

$\Rightarrow \log_e |y-1| = \log_e |x+1| + C$

$\Rightarrow e^{\log_e |y-1|} = e^{\log_e |x+1| + C} \quad = e^{\log_e |x+1|} e^{C}$

$\Rightarrow |y-1| = e^{C}\,|x+1|$

$\Rightarrow y-1 = A(x+1)$. **[A1]**

NOTE: The modulus signs can be removed because A is a new arbitrary constant and can be positive or negative.

Substitute $y = \frac{1}{2}$ and $x = -2$:

$\frac{1}{2} - 1 = A(-2+1)$

$\Rightarrow A = \frac{1}{2}$.

Therefore $y - 1 = \frac{1}{2}(x+1)$.

Answer: $y = \frac{x}{2} + \frac{3}{2}$. **[A1]**

Question 5

The given relation $2v^2 - 3vx - 2x^2 = 1$ suggests using the form $a = v\dfrac{dv}{dx}$ for acceleration.

NOTE 1: The form $a = v\dfrac{dv}{dx}$ is obtained from the VCAA detachable formula sheet.

Therefore the given relation $2v^2 - 3vx - 2x^2 = 1$ must be used to get the positive value of v and the corresponding value of $\dfrac{dv}{dx}$ when $x = \dfrac{1}{5}$.

To get the value of v, substitute $x = \dfrac{1}{5}$ into $2v^2 - 3vx - 2x^2 = 1$:

$$2v^2 - \frac{3}{5}v - \frac{2}{25} = 1 \quad \Rightarrow 50v^2 - 15v - 2 = 25 \quad \Rightarrow 50v^2 - 15v - 27 = 0.$$

Solve for $v > 0$:

$$(10v - 9)(5v + 3) = 0$$

$$\Rightarrow v = \frac{9}{10}.$$ **[A1]**

Use implicit differentiation to differentiate $2v^2 - 3vx - 2x^2 = 1$ with respect to x:

$$\underbrace{4v\frac{dv}{dx}}_{\text{From chain rule}} \underbrace{- 3v - 3\frac{dv}{dx}x}_{\text{From product rule}} - 4x = 0.$$ **[M1]**

Substitute $x = \dfrac{1}{5}$ and $v = \dfrac{9}{10}$ and solve for $\dfrac{dv}{dx}$:

$$4\left(\frac{9}{10}\right)\frac{dv}{dx} - 3\left(\frac{9}{10}\right) - 3\frac{dv}{dx}\left(\frac{1}{5}\right) - 4\left(\frac{1}{5}\right) = 0 \quad \Rightarrow \frac{18}{5}\frac{dv}{dx} - \frac{27}{10} - \frac{3}{5}\frac{dv}{dx} - \frac{4}{5} = 0$$

$$\Rightarrow \frac{dv}{dx} = \frac{7}{6}.$$ **[H1]**

Consequential on value of v.

NOTE 2: A less efficient approach is to first solve for $\dfrac{dv}{dx}$ in terms of v and x using algebra and then to substitute $x = \dfrac{1}{5}$ and $v = \dfrac{9}{10}$.

NOTE 3: A very inefficient and NOT recommended approach is to solve $2v^2 - 3vx - 2x^2 = 1$ for v (use the quadratic formula with $a = 2$, $b = -3x$ and $c = -2x^2$) and then differentiate with respect to x.

Substitute $v = \frac{9}{10}$ and $\frac{dv}{dx} = \frac{7}{6}$ into $a = v\frac{dv}{dx}$:

$a = \left(\frac{9}{10}\right)\left(\frac{7}{6}\right) \quad = \frac{21}{20}\text{ ms}^{-2}$.

Answer: $\frac{21}{20}\text{ ms}^{-2}$. **[H1]**

Consequential on value of v and value of $\frac{dv}{dx}$.

NOTE 4: A VERY inefficient and NOT recommended approach is to solve $2v^2 - 3vx - 2x^2 = 1$ for v (use the quadratic formula with $a = 2$, $b = -3x$ and $c = -2x^2$), squaring to get v^2 and then using $a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$.

Question 6

a.

$\underset{\sim}{r} = t^2\underset{\sim}{i} + \frac{4t^2}{5}\sqrt{t}\,\underset{\sim}{j}$

$= t^2\underset{\sim}{i} + \frac{4}{5}t^{5/2}\underset{\sim}{j}$

$\Rightarrow \underset{\sim}{v} = \dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = 2t\underset{\sim}{i} + 2t^{3/2}\underset{\sim}{j}$

$\Rightarrow \underset{\sim}{a} = \ddot{\underset{\sim}{r}} = \frac{d\underset{\sim}{v}}{dt} = 2\underset{\sim}{i} + 3t^{1/2}\underset{\sim}{j}.$

Equate the given acceleration $2\underset{\sim}{i} + 2\underset{\sim}{j}$ with $2\underset{\sim}{i} + 3t^{1/2}\underset{\sim}{j}$:

Equate $\underset{\sim}{j}$-components: $2 = 3t^{1/2}$

$\Rightarrow t = \frac{4}{9}.$ **[A1]**

Substitute $t = \frac{4}{9}$ into $\underset{\sim}{v} = 2t\underset{\sim}{i} + 2t^{3/2}\underset{\sim}{j}$ and simplify:

Answer: $\underset{\sim}{v} = \frac{8}{9}\underset{\sim}{i} + \frac{16}{27}\underset{\sim}{j}.$ **[H1]**

Consequential on value of *t*.

b.

The length D of the path followed by the body from $t = 0$ to $t = 3$ is required.

The path is defined by the parametric equations $x = t^2$ and $y = \frac{4}{5}t^{5/2}$.

Method 1: Parametric approach.

Arclength formula (refer to VCAA detachable formula sheet):

$$\text{Arclength} = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Therefore:

$$D = \int_{t=0}^{t=3} \sqrt{(2t)^2 + \left(2t^{3/2}\right)^2}\, dt$$

$$= \int_{t=0}^{t=3} \sqrt{4t^2 + 4t^3}\, dt$$ **[M1]**

$$= \int_{t=0}^{t=3} \sqrt{4t^2(1+t)}\, dt$$

$$= \int_{t=0}^{t=3} 2t\sqrt{1+t}\, dt$$

since $\sqrt{4t^2} = |2t| = 2t$ because $t \geq 0$.

Substitute $u = t + 1$:

$$D = 2\int_{u=1}^{u=4} (u-1)\sqrt{u}\, du$$

$$= 2\int_{u=1}^{u=4} u^{3/2} - u^{1/2}\, du$$ **[M1]**

$$= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4$$

$$= 2\left\{\left(\frac{2}{5}\times 4^{5/2} - \frac{2}{3}\times 4^{3/2}\right) - \left(\frac{2}{5}\times 1^{5/2} - \frac{2}{3}\times 1^{3/2}\right)\right\}$$

$$= 2\left\{\left(\frac{2}{5}\times 4^{2}\times 4^{1/2} - \frac{2}{3}\times 4\times 4^{1/2}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right\}$$

$$= 2\left\{\left(\frac{64}{5} - \frac{16}{3}\right) - \left(\frac{2}{5} - \frac{2}{3}\right)\right\}$$

$$= 2\left(\frac{62}{5} - \frac{14}{3}\right) \qquad = 2\left(\frac{186-70}{15}\right).$$

Answer: $\frac{232}{15}$ metres. **[A1]**

Units are not required.

Method 2: Cartesian approach.

This method is NOT recommended because there is more work and therefore it takes longer. However, the method is included because instructive and therefore worth understanding.

The path is defined by the parametric equations

$x = t^2 \Rightarrow \sqrt{x} = t$. .... (1)

$y = \frac{4}{5} t^{5/2}$. .... (2)

Substitute equation (1) into equation (2):

$$y = \frac{4}{5}\left(x^{1/2}\right)^{5/2} = \frac{4}{5} x^{5/4}.$$

The length D of the path followed by the body from $t = 0 \Rightarrow x = 0$ to $t = 3 \Rightarrow x = 9$ is required.
Arclength formula (refer to the VCAA detachable formula sheet):

$$\text{Arclength} = \int_{x_1}^{x_2} \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

Therefore:

$$D = \int_{x=0}^{x=9} \sqrt{1 + \left(x^{1/4}\right)^2}\, dx$$

$$= \int_{x=0}^{x=9} \sqrt{1 + x^{1/2}}\, dx.$$ **[M1]**

Substitute $u = 1 + x^{1/2}$:

$$\frac{du}{dx} = \frac{1}{2} x^{-1/2} \Rightarrow dx = 2x^{1/2} du = 2(u-1) du.$$

$x = 0 \Rightarrow u = 1$.

$x = 9 \Rightarrow u = 4$.

Therefore:

$$D = \int_{u=1}^{u=4} 2\sqrt{u}\,(u-1)\, du$$

and the solution continues in the same way as Method 1.

Question 7

a.

X and Y are independent random variables. Therefore:

$\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) \quad = 3^2 + 4^2$

$= 25$. **[A1]**

$\text{sd}(X - Y) = \sqrt{\text{Var}(X - Y)}$

$= \sqrt{25} \quad = 5$.

Answer: 5 minutes. **[A1]**

Consequential on variance.

b. i.

H_0: $\mu = 33$ minutes

H_1: $\mu < 33$ minutes

[A1]

b. ii.

$p = \Pr\left(\overline{X} \le 31 \mid H_0\right)$

where $\overline{X} \sim \text{Norm}\left(\mu = 33, \sigma = \dfrac{4}{\sqrt{25}} = \dfrac{4}{5}\right)$ under H_0

$= \Pr(Z \le a)$

where $a = \dfrac{\bar{x} - \mu}{\sigma} = \dfrac{31 - 33}{\frac{4}{5}} = \dfrac{-2}{\frac{4}{5}} = -2.5$.

Answer: $a = -2.5$. **[A1]**

Conclusion 1:

$\Pr(Z \le -2.5) < \Pr(Z < -2) \approx 0.025 < 0.05$.

Therefore H_0 is rejected at the 0.05 level of significance.

Both statements: **[A1]**

Conclusion 2:

$z_{0.025} \approx -2$ therefore $a < -z_{0.025} < -z_{0.05}$.

Therefore H_0 is rejected at the 0.05 level of significance.

Both statements: **[A1]**

NOTE: Students are expected to know that $\Pr(Z < -2) = \Pr(Z \ge 2) \approx 0.025$.

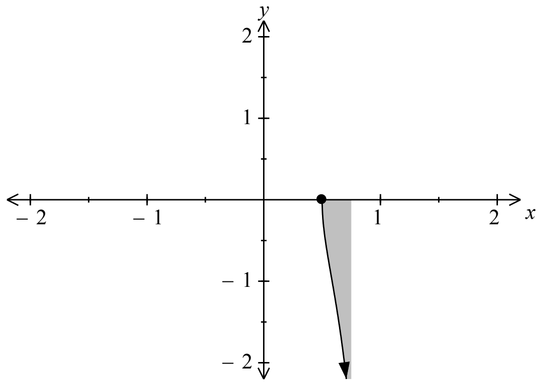
Question 8

The line $y = 0$ is the x-axis.

$y = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

A rough graph of $y = \frac{\sqrt{2x-1}}{x-1}$ over the relevant interval $\frac{1}{2} \le x \le \frac{3}{4}$ should be drawn and the required area shaded.

- Vertical asymptote: $x = 1$.
- Domain: $2x - 1 \ge 0 \Rightarrow x \ge \frac{1}{2}$, $x \ne 1$.
- Shape: $y \le 0$ and strictly decreasing for $\frac{1}{2} \le x \le \frac{3}{4}$.



NOTE: Although drawing a graph of $y = \frac{\sqrt{2x-1}}{x-1}$ over its maximal domain of $x \ge \frac{1}{2}$ is challenging, drawing a graph over the restricted domain $\frac{1}{2} \le x \le \frac{3}{4}$ is straightforward.

Required volume:

$$V = \pi \int_{\frac{1}{2}}^{\frac{3}{4}} y^2 \, dx$$

$$= \pi \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{2x-1}{(x-1)^2} \, dx.$$ **[M1]**

There are three methods for calculating the integral.

Method 1: Substitution.

Substitute $u = x - 1$:

$$\frac{du}{dx} = 1 \Rightarrow dx = du$$

$$x = \frac{1}{2} \Rightarrow u = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$x = \frac{3}{4} \Rightarrow u = \frac{3}{4} - 1 = -\frac{1}{4}$$

Therefore

$$V = \pi \int_{-\frac{1}{2}}^{-\frac{1}{4}} \frac{2(u+1) - 1}{u^2} \, du$$

$$= \pi \int_{-\frac{1}{2}}^{-\frac{1}{4}} \frac{2u - 1}{u^2} \, du$$ **[M1]**

$$= \pi \int_{-\frac{1}{2}}^{-\frac{1}{4}} \frac{2}{u} - \frac{1}{u^2} \, du$$

$$= \pi \left[2\log_e |u| - \frac{1}{u} \right]_{-\frac{1}{2}}^{-\frac{1}{4}}$$

$$= \pi \left\{ \left(2\log_e\left(\frac{1}{4}\right) + 4 \right) - \left(2\log_e\left(\frac{1}{2}\right) + 2 \right) \right\}$$

$$= 2\pi \left(\log_e\left(\frac{1}{2}\right) + 1 \right).$$

Answer: $2\pi\left(1 - \log_e(2)\right)$ cubic units. **[A1]**

Accept all correct forms of answer.

Method 2: Partial fraction decomposition.

$$\frac{2x-1}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \qquad = \frac{A(x-1)+B}{(x-1)^2}$$

$\Rightarrow 2x-1 = A(x-1)+B$ for all values of x.

There are two options for finding the values of A and B:

Option 1: Substitute convenient values of x into $2x-1 = A(x-1)+B$. Substitute $x=1$: $1=B$. Substitute $x=2$: $3=A+B$. Substitute $B=1$ and solve for A: $A=2$.	**Option 2:** Use simultaneous equations. Expand and group like terms: $2x-1 = Ax - A + B$. Equate coefficients of powers of x: $2 = A$. …. (1) Equate constant terms: $-1 = -A + B$. …. (2) Solve equations (1) and (2) simultaneously: $A=2$ and $B=1$.

Therefore $\dfrac{2x-1}{(x-1)^2} = \dfrac{2}{x-1} + \dfrac{1}{(x-1)^2}$.

Therefore:

$$V = \pi\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{2}{x-1} + \frac{1}{(x-1)^2}\,dx$$ **[M1]**

$$= \pi\left[2\log_e|x-1| - \frac{1}{x-1}\right]_{\frac{1}{2}}^{\frac{3}{4}}$$

$$= \pi\left\{\left(2\log_e\left|\frac{3}{4}-1\right| - \frac{1}{\frac{3}{4}-1}\right) - \left(2\log_e\left|\frac{1}{2}-1\right| - \frac{1}{\frac{1}{2}-1}\right)\right\} \qquad = \pi\left\{\left(2\log_e\left(\frac{1}{4}\right)+4\right) - \left(2\log_e\left(\frac{1}{2}\right)+2\right)\right\}$$

$$= 2\pi\left(\log_e\left(\frac{1}{2}\right)+1\right).$$

Answer: $2\pi\left(1-\log_e(2)\right)$ cubic units. **[A1]**

Accept all correct forms of answer.

Method 3: Algebraic re-arrangement.

$$\frac{2x-1}{(x-1)^2} = \frac{(2x-2)+1}{(x-1)^2} = \frac{2(x-1)+1}{(x-1)^2} = \frac{2(x-1)}{(x-1)^2} + \frac{1}{(x-1)^2}$$

$$= \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Therefore:

$$V = \pi \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{2}{x-1} + \frac{1}{(x-1)^2}\, dx$$ **[M1]**

and the solution continues in the same way as **Method 2**.

Answer: $2\pi(1-\log_e(2))$ cubic units. **[A1]**

Question 9

a.

Vertical asymptotes:

Solve $2x^2 - 1 = 0$: $x = \pm\frac{1}{\sqrt{2}}$.

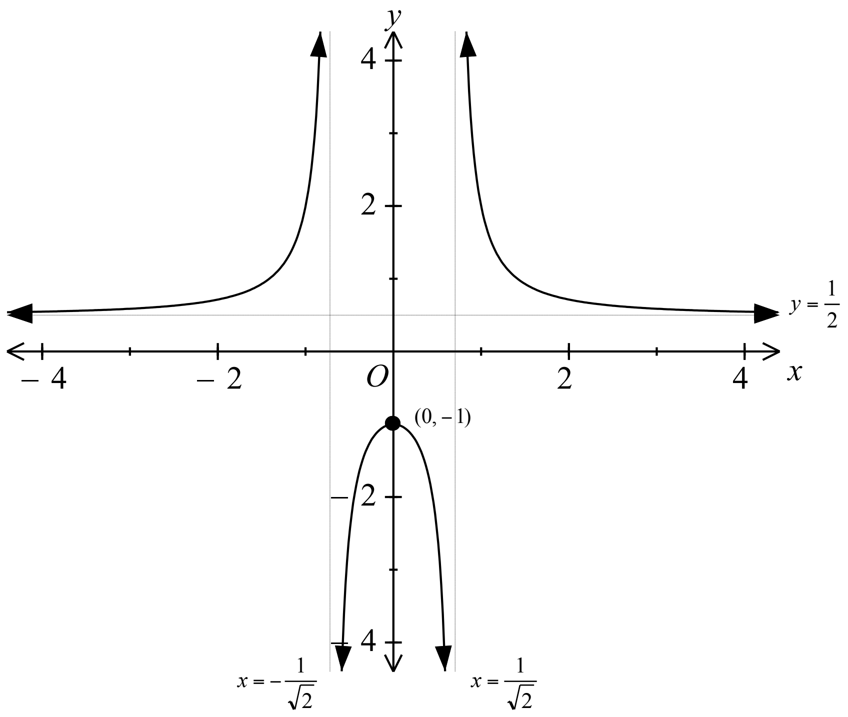
Oblique asymptotes:

From polynomial long division or algebraic re-arrangement: $f(x) = \frac{\frac{3}{2}}{2x^2 - 1} + \frac{1}{2}$.

Consider the limits $x \to +\infty$ and $x \to -\infty$ of $f(x)$: $\lim_{x \to \pm\infty}\left(\frac{\frac{3}{2}}{2x^2 - 1} + \frac{1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}$.

Answer: $x = \pm\frac{1}{\sqrt{2}}$, $y = \frac{1}{2}$. **[A1]**

b.



Correct shape: **[A1]**

Coordinates $(0, -1)$ **of turning point:** **[A1]**

Correctly labelled asymptotes (equations found in part a.): **[H1]**

Consequential on equations found in **part a.** provided three asymptotes were found.

Working:

Shape:

- $f(x) = \underbrace{\dfrac{\frac{3}{2}}{2x^2 - 1}}_{\text{Reciprocal quadratic function}} + \dfrac{1}{2}$.
- Consider the graph of $g(x) = \dfrac{2x^2 - 1}{\frac{3}{2}} = \dfrac{4}{3}x^2 - \dfrac{2}{3}$ and draw the reciprocal quadratic graph.
- Translate the graph $y = g(x)$ up by $\dfrac{1}{2}$.

Coordinates of turning point:

Method 1:

- Consider the turning point of $g(x) = \dfrac{2x^2 - 1}{\frac{3}{2}} = \dfrac{4}{3}x^2 - \dfrac{2}{3}$: $\left(0,\ -\dfrac{2}{3}\right)$.
- Take the reciprocal of the y-coordinate of this turning point: $\left(0,\ -\dfrac{3}{2}\right)$.
- Translate $\left(0, -\dfrac{3}{2}\right)$ up by $\dfrac{1}{2}$: $(0, -1)$.

Method 2: Use calculus.

Option 1: Differentiate $f(x)=\dfrac{x^2+1}{2x^2-1}$ using the quotient rule.

$$f'(x)=\frac{2x(2x^2-1)-4x(x^2+1)}{(2x^2-1)^2}=\frac{-6x}{(2x^2-1)^2}.$$

$$f'(x)=0 \quad \Rightarrow x=0.$$

$$f(0)=\frac{1}{-1}=-1.$$

Therefore there is a turning point at $(0,-1)$.

Option 2: Differentiate $f(x)=\dfrac{\frac{3}{2}}{2x^2-1}+\dfrac{1}{2}=\dfrac{3}{2}(2x^2-1)^{-1}+\dfrac{1}{2}$ using the chain rule (see **part a.** solution).

$$f'(x)=-\frac{3}{2}(2x^2-1)^{-2}(4x)=\frac{-6x}{(2x^2-1)^2}.$$

$$f'(x)=0 \quad \Rightarrow x=0.$$

$$f(0)=\frac{1}{-1}=-1.$$

Therefore there is a turning point at $(0,-1)$.

c.

$g(f(x)) = \pi - \cos^{-1}\left(\dfrac{x^2+1}{2x^2-1}\right)$.

Therefore the solution to the inequation

$$-1 \le \frac{x^2+1}{2x^2-1} \le 1$$

is required.

A graphical approach using the graph from **part a.** can be used:

• Solve $\dfrac{x^2+1}{2x^2-1} = 1$:

$x^2 + 1 = 2x^2 - 1$

$\Rightarrow x^2 - 2 = 0$

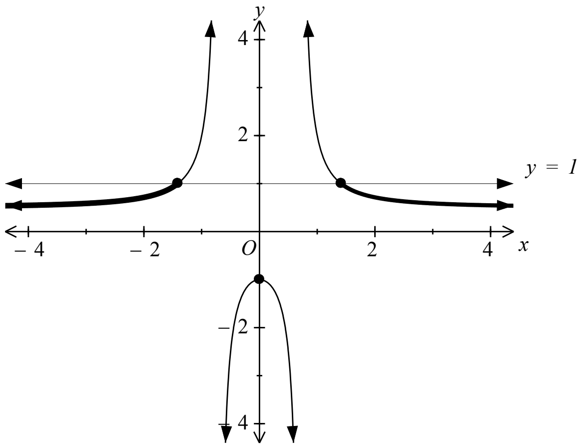
$\Rightarrow x = \pm\sqrt{2}$. **[A1]**

• Solve $\dfrac{x^2+1}{2x^2-1} = -1$:

From the known coordinates of the turning point it follows that $x = 0$. **[A1]**

It can therefore be seen from the graph in **part a.** that the solution to $-1 \le \dfrac{x^2+1}{2x^2-1} \le 1$ is

$x \le -\sqrt{2} \cup x \ge \sqrt{2} \cup x = 0$:



Answer: $\left(-\infty, -\sqrt{2}\right] \cup \left[\sqrt{2}, +\infty\right) \cup \{0\}$. **[A1]**

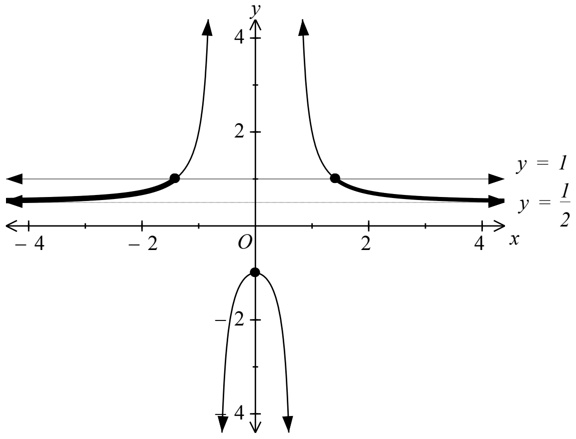
Accept all alternative valid notations.

d.

$f(x)=\dfrac{x^2+1}{2x^2-1}$ and so $g(f(x))=\pi-\cos^{-1}\left(\dfrac{x^2+1}{2x^2-1}\right)$.

• Consider the values of $f(x)$ for $x\in\left[\sqrt{2},+\infty\right)$.

From the graph in **part a.** it can be seen that $f(x)=\dfrac{x^2+1}{2x^2-1}$ is strictly decreasing for $x\in\left[\sqrt{2},+\infty\right)$:



Therefore $\dfrac{1}{2}<f(x)\le 1$ for $x\in\left[\sqrt{2},+\infty\right)$.

$g\left(\dfrac{1}{2}\right)=\pi-\cos^{-1}\left(\dfrac{1}{2}\right)=\pi-\dfrac{\pi}{3}=\dfrac{2\pi}{3}$.

$g(1)=\pi-\cos^{-1}(1)=\pi-0=\pi$.

Therefore $\dfrac{2\pi}{3}<g(f(x))\le\pi$ for $x\in\left[\sqrt{2},+\infty\right)$. **[H1]**

Consequential on answer to **part c.**.

• Consider the value of $f(x)$ for $x\in\left(-\infty,-\sqrt{2}\right]$.

Option 1: From the graph in **part a.** it can be seen that $f(x)=\dfrac{x^2+1}{2x^2-1}$ is strictly increasing for $x\in\left(-\infty,-\sqrt{2}\right]$.

Therefore $\dfrac{1}{2}<f(x)\le 1$ for $x\in\left(-\infty,-\sqrt{2}\right]$.

It follows from the previous calculations that $\dfrac{2\pi}{3}<g(f(x))\le\pi$ for $x\in\left(-\infty,-\sqrt{2}\right]$.

Option 2: Since $f(x)$ is an even function, it follows from symmetry that $\dfrac{2\pi}{3}<g(f(x))\le\pi$.

• Consider the value of $f(x)$ for $x = 0$.

$f(0) = -1$.

$g(-1) = \pi - \cos^{-1}(-1) = 0$.

Answer: $\left(\frac{2\pi}{3}, \pi\right] \cup \{0\}$. **[A1]**