insight
▶innovative ▶engaging ▶evolving

YEAR 12 *Trial Exam Paper*

2017
SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- fully worked solutions
- mark allocations
- tips on how to approach the exam

THIS PAGE IS BLANK

Question 1

Worked solution

$$z^4 - (1+i)z^3 = z^2 - z - iz$$
$$\Rightarrow z^3\left(z-(1+i)\right) = z\left(z-1-i\right)$$
$$\Rightarrow z^3\left(z-(1+i)-z\left(z-(1+i)\right)\right) = 0$$
$$z\left(z-(1+i)\right)\left(z^2-1\right) = 0$$
$$\therefore z = 0, 1+i, \pm 1$$

Mark allocation: 3 marks

- 1 mark for correctly factorising each side of the equation
- 1 mark for setting the equation equal to zero and factorising
- 1 mark for four correct solutions

Question 2

Worked solution

$(x+1)e^{v}-x^{2}-2x-3=0$

When $x=1$,

$2e^{v}-6=0$

$\Rightarrow e^{v}=3$

$\Rightarrow v=\log_{e}3$

Differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}\left((x+1)e^{v}-x^{2}-2x-3\right)=\frac{d}{dx}(0)$$

$$e^{v}+(x+1)e^{v}\frac{dv}{dx}-2x-2=0$$

$$(x+1)e^{v}\frac{dv}{dx}=2x+2-e^{v}$$

$$\Rightarrow \frac{dv}{dx}=\frac{2x+2-e^{v}}{(x+1)e^{v}}$$

Substitute $x=1$ and $v=\log_{e}3$

$$\Rightarrow \frac{dv}{dx}=\frac{2+2-3}{6}=\frac{1}{6}$$

$$\text{Accelerate}=v\frac{dv}{dx}$$

$\therefore$ the acceleration when $x=1$ is $\frac{1}{6}\log_{e}3\ \text{ms}^{-2}$.

Mark allocation: 4 marks

- 1 mark for the correct value of v when $x = 1$
- 1 mark for correctly differentiating both sides of the equation with respect to x
- 1 mark for the correct evaluation of $\frac{dv}{dx}$
- 1 mark for the correct answer

Alternative method for finding $\frac{dv}{dx}$

$(x+1)e^v - x^2 - 2x - 3 = 0$

$\Rightarrow e^v = \dfrac{x^2+2x+3}{x+1}$

$\Rightarrow v = \log_e\left(\dfrac{x^2+2x+3}{x+1}\right)$

$$\text{Let } u = \frac{x^2+2x+3}{x+1} \Rightarrow \frac{du}{dx} = \frac{(2x+2)(x+1)-(x^2+2x+3)}{(x+1)^2}$$
$$= \frac{2x^2+4x+2-x^2-2x-3}{(x+1)^2}$$
$$= \frac{x^2+2x-1}{(x+1)^2}$$

$v = \log_e u$

$\Rightarrow \dfrac{dv}{du} = \dfrac{1}{u} = \dfrac{x+1}{x^2+2x+3}$

$$\therefore \frac{dv}{dx} = \frac{x^2+2x-1}{(x+1)^2} \times \frac{x+1}{x^2+2x+3}$$
$$= \frac{x^2+2x-1}{(x+1)(x^2+2x+3)}$$

When $x = 1$,

$\dfrac{dv}{dx} = \dfrac{2}{2\times 6} = \dfrac{1}{6}$ and $v = \log_e 3$

Acceleration $= v\dfrac{dv}{dx}$

$\therefore$ the acceleration when $x = 1$ is $\dfrac{1}{6}\log_e 3 \text{ ms}^{-2}$.

Tip

- *It is easier to obtain $v\frac{dv}{dx}$ using implicit differentiation, rather than explicitly expressing v as a function of x and then differentiating.*

Question 3a.

Worked solution

$$\mu = \overline{x} \pm z \times \frac{s}{\sqrt{n}}$$

$$\mu = 160 \pm 1.645 \times \frac{10}{\sqrt{100}}$$

$$\mu = 160 \pm 1.645 = 160 \pm 1.6$$

$$158.4 \le \mu \le 161.6$$

Mark allocation: 2 marks

- 1 mark for correctly substituting the values into $\mu = \overline{x} \pm z \times \frac{s}{\sqrt{n}}$
- 1 mark for the correct answer

Question 3b.i.

Worked solution

$H_0 : \mu = 157$

$H_1 : \mu > 157$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3b.ii.

Worked solution

$$p = \Pr\left(z > \frac{159 - 157}{\frac{6}{\sqrt{36}}} \right) = \Pr(z > 2) \approx \Pr(z > 1.96)$$

$\Rightarrow p = 0.025$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 3b.iii.

Worked solution

$\therefore$ Reject the null hypothesis as $p = 0.025 < 0.05$

Mark allocation: 1 mark

- 1 mark for the correct reason

Question 4

Worked solution

$$\int_0^{\frac{\pi}{3}} (\tan^2 x + \tan^4 x)dx$$

$$= \int_0^{\frac{\pi}{3}} \tan^2 x(1 + \tan^2 x)dx$$

$$= \int_0^{\frac{\pi}{3}} \tan^2 x(\sec^2 x)dx$$

Let $u = \tan x$

$$\Rightarrow \frac{du}{dx} = \sec^2 x$$

$$x = \frac{\pi}{3} \Rightarrow u = \sqrt{3}$$

$$x = 0 \Rightarrow u = 0$$

$$\therefore \int_0^{\frac{\pi}{3}} (\tan^2 x + \tan^4 x)dx = \int_0^{\sqrt{3}} u^2 du$$

$$= \left[\frac{1}{3}u^3\right]_0^{\sqrt{3}}$$

$$= \frac{1}{3} \times 3\sqrt{3} - 0$$

$$= \sqrt{3}$$

Mark allocation: 3 marks

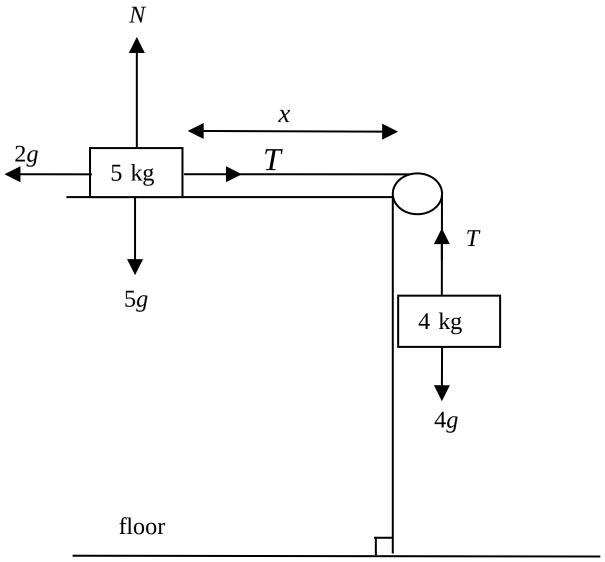
- 1 mark for correctly setting up the integrand for a substitution of $u = \tan x$
- 1 mark for obtaining a correct integrand with respect to u
- 1 mark for the correct answer

Tip

- *Use the substitution* $u = \tan(x)$ *as its derivative,* $\sec^2(x)$ *is a factor of the integrand.*

Question 5

Worked solution



$R = 4g - T + T - 2g = 9a$

$\Rightarrow 9a = 2g$

$\Rightarrow a = \frac{2g}{9}$

$a = \frac{d^2 s}{dt^2} = \frac{2g}{9}$

$\Rightarrow v = \frac{ds}{dt} = \frac{2g}{9}t + c$

$v = 0,\ t = 0$

$\Rightarrow c = 0$

$\therefore \frac{ds}{dt} = \frac{2g}{9}t$

$s = \frac{g}{9}t^2 + d$

$s = 0,\ t = 0$

$\Rightarrow d = 0$

$\therefore s = \frac{g}{9}t^2$

$t = 1 \Rightarrow s = \frac{g}{9}$

$\therefore x = \frac{g}{9}$

Alternative method

$R = 4g - T + T - 2g = 9a$

$\Rightarrow 9a = 2g$

$\Rightarrow a = \dfrac{2g}{9}$

Using the constant acceleration formula:

$\Rightarrow x = \dfrac{1}{2} \times \dfrac{2g}{9} \times 1^2$

$\therefore x = \dfrac{g}{9}$

Mark allocation: 3 marks

- 1 mark for the correct acceleration
- 1 mark for correctly obtaining the displacement as a function of time
- 1 mark for the correct answer

Note: constant acceleration formulae are NOT covered by the VCAA Study Design for VCE Specialist Mathematics Units 3 & 4. If the incorrect answer is obtained using these formulae, no working marks can be awarded.

Tip

- *Resolving the total of the forces acting in the direction of the intended motion of the total of the masses is an efficient way to solve some dynamics problems.*

Question 6

Worked solution

$$\frac{dy}{dx} = 3y\sqrt{x}$$

Use the separation of variables technique to antidifferentiate.

$$\Rightarrow \int \frac{1}{y}dy = \int 3\sqrt{x}dx$$

$$\log_e|y| = 2x^{\frac{3}{2}} + c$$

$y = e$ when $x = 1$

$$\Rightarrow \log_e e = 1 = 2 + c$$

$$\Rightarrow c = -1$$

$$\log_e|y| = 2x^{\frac{3}{2}} - 1$$

$$|y| = e^{2x^{\frac{3}{2}} - 1}$$

$$\therefore y = \pm e^{2x\sqrt{x} - 1}$$

$$\therefore y = e^{2x\sqrt{x} - 1} \text{ as } y > 0$$

Mark allocation: 3 marks

- 1 mark for correctly integrating using separation of variables
- 1 mark for correctly evaluating the constant of antidifferentiation
- 1 mark for the correct answer

Question 7a.

Worked solution

$x - 2 = \sin(2t)$

$y = 2\sin^2(t) = 2 \times \dfrac{1-\cos(2t)}{2} = 1 - \cos(2t)$

$\Rightarrow 1 - y = \cos(2t)$

$\therefore (x-2)^2 + (1-y)^2 = \sin^2(2t) + \cos^2(2t) = 1$

or $(x-2)^2 + (y-1)^2 = 1$

Mark allocation: 2 marks

- 1 mark for correctly expressing y in terms of $\cos(2t)$
- 1 mark for the correct equation

Tip

- *Use the double angle formula to express y in terms of* $\cos(2t)$ *so that the cartesian equation is easily found.*

Question 7b.

Worked solution

$$\frac{dx}{dt} = 2\cos(2t)$$

$$\frac{dy}{dt} = 2\sin(2t)$$

Length = $\displaystyle\int_0^{\frac{\pi}{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$$= \int_0^{\frac{\pi}{3}} \sqrt{\left(2\cos(2t)\right)^2 + \left(2\sin(2t)\right)^2}\,dt$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{4\cos^2(2t) + 4\sin^2(2t)}\,dt$$

$$= \int_0^{\frac{\pi}{3}} \sqrt{4}\,dt = \int_0^{\frac{\pi}{3}} 2\,dt$$

$$= \left[2t\right]_0^{\frac{\pi}{3}}$$

$$= \frac{2\pi}{3} - 0$$

$= \dfrac{2\pi}{3}$ units

Mark allocation: 2 marks

- 1 mark for correctly setting up the integrand representing the curve length
- 1 mark for the correct answer

Question 8a.

Worked solution

$\underset{\sim}{b} \,.\, \underset{\sim}{c} = (2\underset{\sim}{i} + \underset{\sim}{j} + n\underset{\sim}{k}).\,(4\underset{\sim}{i} + 3\underset{\sim}{j} + 6\underset{\sim}{k})$

$\Rightarrow \underset{\sim}{b} \,.\, \underset{\sim}{c} = 8 + 3 + 6n = 0$

$\Rightarrow 6n = -11$

$\therefore n = \dfrac{-11}{6}$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 8b.

Worked solution

$$\cos\theta = \frac{(2\underset{\sim}{i} + \underset{\sim}{j} + n\underset{\sim}{k}).(2\underset{\sim}{i} + n\underset{\sim}{k})}{\sqrt{5+n^2} \times \sqrt{4+n^2}}$$

$$\Rightarrow \cos\theta = \frac{4+n^2}{\sqrt{5+n^2} \times \sqrt{4+n^2}} = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \frac{\sqrt{4+n^2}}{\sqrt{5+n^2}} = \frac{3}{\sqrt{10}}$$

$$\sqrt{10(4+n^2)} = 3\sqrt{5+n^2}$$

$$\Rightarrow 40 + 10n^2 = 45 + 9n^2$$

$$\Rightarrow n^2 = 5$$

$$\therefore n = \pm\sqrt{5}$$

Mark allocation: 2 marks

- 1 mark for the correct equation for $\cos(\theta)$ in terms of n
- 1 mark for the correct answer

Tip

- *The angle between a vector and a plane is the angle between the vector and the component of the vector in the direction of the plane only.*

Question 8c.

Worked solution

Vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent if:

$\underset{\sim}{b} = m\underset{\sim}{a} + p\underset{\sim}{c}$ where $m, p \in R$ and not both zero.

$\Rightarrow \underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + n\underset{\sim}{k} = 3m\underset{\sim}{i} + 2m\underset{\sim}{j} + 2m\underset{\sim}{k} + 4p\underset{\sim}{i} + 3p\underset{\sim}{j} + 6p\underset{\sim}{k}$

$\Rightarrow 3m + 4p = 2$

$2m + 3p = 1$

$2m + 6p = n$

$3m + 4p = 2 \Rightarrow 6m + 8p = 4$

$2m + 3p = 1 \Rightarrow 6m + 9p = 3$

$\Rightarrow p = -1$

$\Rightarrow m = 2$

$\Rightarrow n = -2$

$\therefore$ vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly independent if $n \in R \setminus \{-2\}$

Mark allocation: 2 marks

- 1 mark for finding correct values of m and p where $\underset{\sim}{b} = m\underset{\sim}{a} + p\underset{\sim}{c}$
- 1 mark for the correct answer

Tip

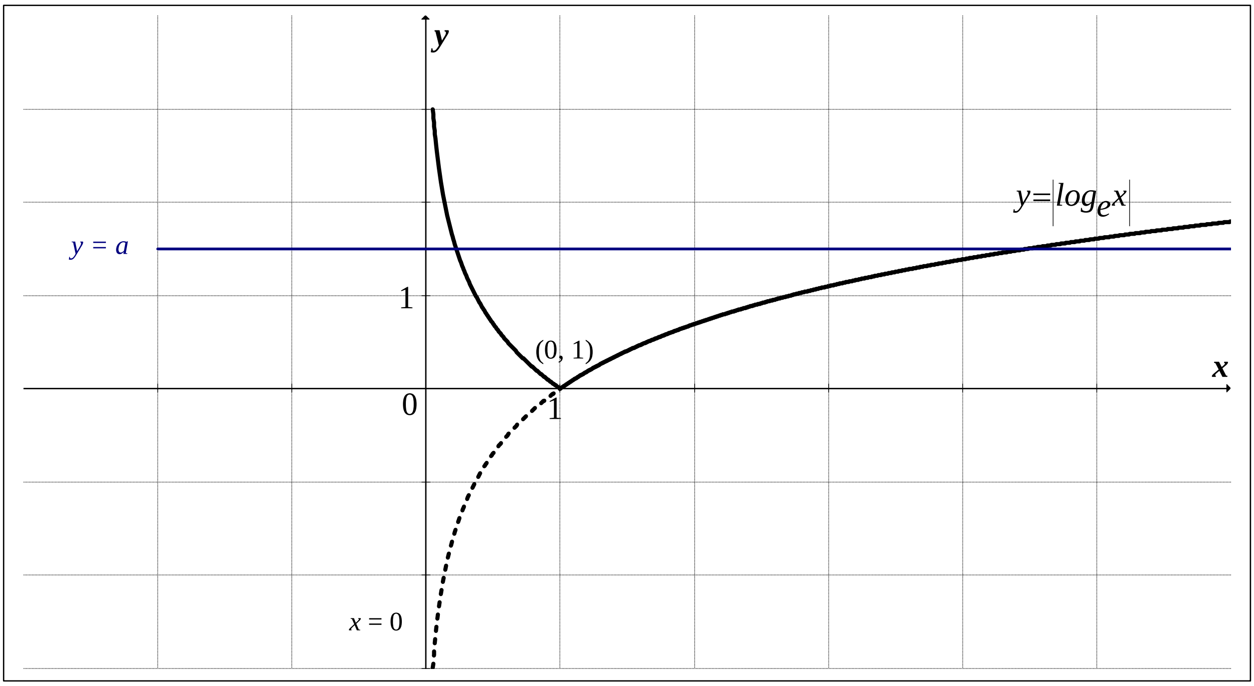
- *Determine the value of n for which the vectors are linearly dependent first.*

Question 9a.

Worked solution

Sketch the graph of $y = \log_e(x)$.

Reflect the part of the graph of $y = \log_e(x)$ that is below the *x*-axis through the *x*-axis (i.e. for $0 < x < 1$).



Mark allocation: 1 mark

- 1 mark for the correctly labelled graph

Question 9b.

Worked solution

For $x \geq 1, y = \log_e(x)$

$\Rightarrow x = e^y$ or $x^2 = e^{2y}$

For $0 \leq x < 1, y = -\log_e(x)$

$\Rightarrow -y = \log_e(x)$

$\Rightarrow x = e^{-y}$ or $x^2 = e^{-2y}$

Using volume of revolution about y-axis, $V = \pi\int_a^b x^2 dy$

Volume $= \pi\int_0^a (e^{2y} - e^{-2y}).dy$ ($y = a$ is indicated on the graph)

$\Rightarrow \pi\left[\frac{1}{2}e^{2y} + \frac{1}{2}e^{-2y}\right]_0^a = 2\pi$

$\Rightarrow \frac{1}{2}e^{2a} + \frac{1}{2}e^{-2a} - \left(\frac{1}{2} + \frac{1}{2}\right) = 2$

$\Rightarrow e^{2a} + e^{-2a} - 2 = 4$

$\therefore e^{2a} + e^{-2a} = 6$

Mark allocation: 3 marks

- 1 mark for correctly expressing x as a function of y for $x \geq 1$ and also for $0 < x < 1$
- 1 mark for the correct evaluation of the volume
- 1 mark for obtaining the correct result

Question 10a.

Worked solution

arcsin(ax) = arccos(bx), where $-1 \le ax \le 1$ and $-1 \le bx \le 1$

$\Rightarrow \sin(\arcsin(ax)) = \sin(\arccos(bx))$

$\Rightarrow ax = \sin(\arccos(bx))$

Let u = arcos(bx)

$\Rightarrow \cos u = bx$

$\therefore \sin u = \sqrt{1-\cos^2 u} = \sqrt{1-b^2x^2}$

$\Rightarrow ax = \sqrt{1-b^2x^2}$

$\Rightarrow a^2x^2 = 1-b^2x^2$

$\Rightarrow x^2\left(a^2+b^2\right) = 1$

$\Rightarrow x^2 = \dfrac{1}{a^2+b^2}$

$\therefore x = \dfrac{1}{\sqrt{a^2+b^2}}$

Mark allocation: 2 marks

- 1 mark for correctly simplifying $\sin(\arccos(bx))$
- 1 mark for the correct answer

Question 10b.

Worked solution

$$\frac{d}{dx}\left(x.\arccos(bx) - \frac{\sqrt{1-b^2x^2}}{b}\right) = \arccos(bx) - x.\frac{b}{\sqrt{1-b^2x^2}} - \frac{1}{2}.\frac{1}{b}.(-2b^2x)\left(1-b^2x^2\right)^{\frac{-1}{2}}$$

$= \arccos(bx) - \dfrac{bx}{\sqrt{1-b^2x^2}} + \dfrac{bx}{\sqrt{1-b^2x^2}}$

$= \arccos(bx)$

Mark allocation: 1 mark

- 1 mark for correctly differentiating using both the product rule and the chain rule

Question 10c.

Worked solution

$y=\arcsin(2x)$ and $y=\arccos(x)$ intersect at:

$x=\dfrac{1}{\sqrt{a^2+b^2}}$, where a = 2 and b = 1

$\Rightarrow x=\dfrac{1}{\sqrt{2^2+1^2}}=\dfrac{1}{\sqrt{5}}$

$\Rightarrow$ Area required $=\displaystyle\int_0^{\frac{1}{\sqrt{5}}}[\arccos(x)-\arcsin(2x)]dx$

Area $=\left[x.\arccos(x)-\sqrt{1-x^2}-\left(x.\arcsin(2x)+\dfrac{\sqrt{1-4x^2}}{2}\right)\right]_0^{\frac{1}{\sqrt{5}}}$

Area $=\left[x.\arccos(x)-\sqrt{1-x^2}-x.\arcsin(2x)-\dfrac{\sqrt{1-4x^2}}{2}\right]_0^{\frac{1}{\sqrt{5}}}$

Area $=\left(\dfrac{1}{\sqrt{5}}\arccos\left(\dfrac{1}{\sqrt{5}}\right)-\sqrt{\dfrac{4}{5}}-\dfrac{1}{\sqrt{5}}\arcsin\left(\dfrac{2}{\sqrt{5}}\right)-\dfrac{1}{2}\times\sqrt{\dfrac{1}{5}}\right)-\left(-1-\dfrac{1}{2}\right)$

But $\left(\dfrac{1}{\sqrt{5}}\right)^2+\left(\dfrac{2}{\sqrt{5}}\right)^2=\dfrac{5}{5}=1$

$\therefore\ \arccos\left(\dfrac{1}{\sqrt{5}}\right)=\arcsin\left(\dfrac{2}{\sqrt{5}}\right)$

$\Rightarrow$ Area $=\left(\dfrac{1}{\sqrt{5}}\arccos\left(\dfrac{1}{\sqrt{5}}\right)-\dfrac{2}{\sqrt{5}}-\dfrac{1}{\sqrt{5}}\arccos\left(\dfrac{1}{\sqrt{5}}\right)-\dfrac{1}{2\sqrt{5}}\right)+\dfrac{3}{2}$

$\therefore$ Area = $\dfrac{3}{2}-\dfrac{5}{2\sqrt{5}}=\dfrac{3-\sqrt{5}}{2}$ square units.

Mark allocation: 3 marks

- 1 mark for setting up the correct integrand which represents the required area
- 1 mark for the correct antiderivative
- 1 mark for the correct answer

END OF WORKED SOLUTIONS

THIS PAGE IS BLANK