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YEAR 12 *Trial Exam Paper*

2017
SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- fully worked solutions
- mark allocations
- tips on how to approach the exam

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SECTION A – Multiple-choice questions

Question 1

Answer: B

Worked solution

Domain

$$-1 \le bx - \frac{\pi}{2} \le 1$$

$$\frac{\pi}{2} - 1 \le bx \le 1 + \frac{\pi}{2}$$

$$\frac{\pi - 2}{2b} \le x \le \frac{2 + \pi}{2b}$$

$$\left[\frac{\pi - 2}{2b}, \frac{2 + \pi}{2b}\right]$$

Range

$$-\frac{\pi}{2} \le \sin^{-1}\left(bx - \frac{\pi}{2}\right) \le \frac{\pi}{2}$$

$$-\frac{a\pi}{2} \le a\sin^{-1}\left(bx - \frac{\pi}{2}\right) \le \frac{a\pi}{2}$$

$$-\frac{a\pi}{2} + 1 \le a\sin^{-1}\left(bx - \frac{\pi}{2}\right) + 1 \le \frac{a\pi}{2} + 1$$

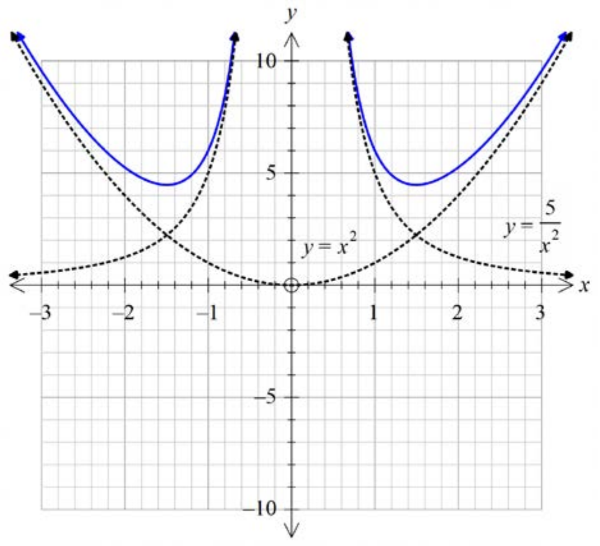
$$\left[\frac{2 - a\pi}{2}, \frac{a\pi + 2}{2}\right]$$

Question 2

Answer: B

Worked solution

The graph of $f(x)=\dfrac{x^4+5}{x^2}$ looks like



The graph has asymptotes of $x=0$, $y=x^2$ and $y=\dfrac{5}{x^2}$, making options A, D and E all true.

The second derivative of $f(x)$ is $f''(x)=2+\dfrac{30}{x^4}$, so $f''(x)>0$ for all *x*, which means $f(x)$ does not have any stationary points of inflection, making option B false.

Tip

- *When given an equation and asked a question relating to the graph, it is helpful to graph the equation on the CAS to visualise the shape of the graph and to help eliminate incorrect multiple-choice options.*

Question 3

Answer: D

Worked solution

The graph shown is a secant function that has been:

- dilated by a factor of 2 parallel to the *y*-axis
- dilated by a factor of $\dfrac{1}{2}$ parallel to the *x*-axis
- translated $\dfrac{\pi}{4}$ units in the positive *x* direction
- translated 1 unit in the negative *y* direction.

Option D is the only function that matches the transformations of the graph.

Question 4

Answer: C

Worked solution

Rearranging the equations to make sec and tan the subjects gives

$\frac{x-1}{5} = \sec(t)$ and $\frac{y+2}{4} = \tan(t)$.

Recall that $\sec^2(t) = 1 + \tan^2(t)$ which, when rearranged, gives

$$\left(\frac{x-1}{5}\right)^2 - \left(\frac{y+2}{4}\right)^2 = 1$$

$$\frac{(x-1)^2}{25} - \frac{(y+2)^2}{16} = 1$$

Tip

- *When trying to find the cartesian equations of parametric equations involving sin and cos or sec or tan or cosec and cot, try to rearrange the equations so that identity formulas, such as* $\sin^2(x) + \cos^2(x) = 1$, *can be applied.*

Question 5

Answer: C

Worked solution

The graph is a circle centred at (–1, 1) with a radius of 2.

Options A and D represent rays, so can be eliminated.

Option B represents a circle centred at (–1, 1) with a radius of 4, so can be eliminated.

Option E represents a straight line, so can be eliminated.

Option C represents a circle centred at (–1, 1) with a radius of 2, so is therefore the correct response.

So the circle can be represented by $\{z : |z - (i-1)| = 2\}$.

Question 6

Answer: D

Worked solution

Let $z = r\,cis\,\theta$ and $2\sqrt{3}+2i = \sqrt{(2\sqrt{3})^2+2^2}\,cis\left(\frac{\pi}{6}\right) = 4\,cis\left(\frac{\pi}{6}\right)$.

So $z^4 = 4\,cis\left(\frac{\pi}{6}\right)$.

Using De Moivre's theorem gives

$z^n = r^n\,cis(n\theta)$

Then

$$z^4 = 4\,cis\left(\frac{\pi}{6}\right)$$

$$z = \sqrt[4]{4}\,cis\left(\frac{1}{4}\left(\frac{\pi}{6}+2k\pi\right)\right),\text{ where } k \in Z$$

$$z = \sqrt{2}\,cis\left(\frac{1}{4}\left(\frac{\pi}{6}+2k\pi\right)\right),\text{ where } k \in Z$$

So $z = \sqrt{2}\,cis\left(-\frac{23\pi}{24}\right), \sqrt{2}\,cis\left(-\frac{11\pi}{24}\right), \sqrt{2}\,cis\left(\frac{\pi}{24}\right)$ and $\sqrt{2}\,cis\left(\frac{13\pi}{24}\right)$.

For the diagram given in the question, only z_1 and z_6 are points that match/are solutions to the equation. Therefore, the answer is option D.

Question 7

Answer: D

Worked solution

The polynomial $P(z)$ factorises as follows

$$\begin{aligned} & z^3+2iz^2+9z+18i \\ &= z^2(z+2i)+9(z+2i) \\ &= (z^2+9)(z+2i) \\ &= (z-3i)(z+3i)(z+2i) \end{aligned}$$

So the solutions to $P(z)=0$ are $z=-2i, -3i, 3i$.

Since all roots of $P(z)$ are complex, $P(z)$ has no real roots and the correct response is option D.

Question 8

Answer: C

Worked solution

Let $u = \log_e(x)$.

Then

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$$

and

$$x = e^{5b} \Rightarrow u = \log_e(e^{5b}) = 5b$$
$$x = e^{a} \Rightarrow u = \log_e(e^{a}) = a$$

Substituting into $\int_{e^a}^{e^{5b}} \frac{[\log_e(x)]^4}{5x}\,dx$ gives

$$\int_a^{5b} \frac{u^4}{5x} x\,du$$
$$= \int_a^{5b} \frac{u^4}{5}\,du$$

Tip

- *Don't forget to change the values of the terminals according to the substitution used.*

Question 9

Answer: E

Worked solution

$x_0 = 2,\ y_0 = 0$ and $h = 0.1$.

Using Euler's method gives

$x_1 = 2 + 0.1 = 2.1$ and $y_1 = 0 + 0.1(2^3 - 2\times 2) = 0.4$

$x_2 = 2.1 + 0.1 = 2.2$ and $y_2 = 0.4 + 0.1(2.1^3 - 2\times 2.1) = 0.9061$

$y_3 = 0.9061 + 0.1(2.2^3 - 2\times 2.2) = 1.5309$

Question 10

Answer: D

Worked solution

$$
\begin{aligned}
a &= v\frac{dv}{dx} \\
&= \sin(x)\frac{d(\sin(x))}{dx} \\
&= \sin(x)\cos(x)
\end{aligned}
$$

Rearranging the double angle formula $\sin(2x) = 2\sin(x)\cos(x)$ gives

$$a = \frac{1}{2}\sin(2x)$$

Alternatively,

$$
\begin{aligned}
a &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}\sin^2(x)\right) \\
&= \sin(x)\cos(x) \\
&= \frac{1}{2}\sin(2x)
\end{aligned}
$$

Question 11

Answer: B

Worked solution

Option A does not give a gradient of 0 when $x = 0$, so can be eliminated.

Option C is the differential equation of a cubic, so can be eliminated.

Option D is the differential equation for a square root function and doesn’t always give the correct direction for y in the first and fourth quadrants. Hence, option D can be eliminated.

Option E is the differential equation for a hyperbola. It also doesn’t give the correct directions of the gradients. Hence, option E can be eliminated.

Option B is the differential equation for a circle and also has the correct directions of the gradients, so is the correct answer.

Question 12

Answer: C

Worked solution

$$4\underset{\sim}{a}-2\underset{\sim}{b}=(4\times-6-2\times-4)\underset{\sim}{i}+(4\times3-2\times5)\underset{\sim}{j}+(4\times-2-2\times-3)\underset{\sim}{k}$$

$$=-16\underset{\sim}{i}+2\underset{\sim}{j}-2\underset{\sim}{k}$$

So $|4\underset{\sim}{a}-2\underset{\sim}{b}|=\sqrt{(-16)^2+(2)^2+(-2)^2}$

$$=\sqrt{264}$$

$$=2\sqrt{66}$$

Question 13

Answer: B

Worked solution

The scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is given by

$\underset{\sim}{a}.\hat{\underset{\sim}{b}}$, where $\hat{\underset{\sim}{b}}=\dfrac{\underset{\sim}{b}}{|\underset{\sim}{b}|}=\dfrac{1}{3}(2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k})$

So $\underset{\sim}{a}.\hat{\underset{\sim}{b}}=\dfrac{1}{3}(3\underset{\sim}{i}+m\underset{\sim}{j}-3\underset{\sim}{k}).(2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k})=5$

$$=\frac{1}{3}(3+2m)=5$$

$$\Rightarrow m=6$$

Question 14

Answer: B

Worked solution

Velocity vectors of the two particles are $\dot{\underset{\sim}{r}}=2\underset{\sim}{i}+-4\underset{\sim}{j}$ and $\dot{\underset{\sim}{s}}=6t\underset{\sim}{i}+3\underset{\sim}{j}$.

Two vectors are perpendicular when their dot products are equal to zero.

So $\dot{\underset{\sim}{r}}.\dot{\underset{\sim}{s}}=2.6t+-4.3=0$

$$=12t-12=0$$

$$\Rightarrow t=1$$

Question 15

Answer: A

Worked solution

Newton's Second Law for the 10 kg mass gives

$10g - T_1 = 10a$

Newton's Second Law for the 6 kg mass gives

$T_2 - 6g = 6a$

Adding the equations and recalling that for pulleys $T_1 = T_2$, gives

$$\begin{aligned} 10g - T_1 + T_2 - 6g &= 10a + 6a \\ 4g &= 16a \\ \Rightarrow a &= \frac{g}{4}\ \text{ms}^{-2} \end{aligned}$$

Question 16

Answer: B

Worked solution

Take moving to the left as the positive direction of the tennis ball.

Change in momentum is given by $\Delta \underset{\sim}{\text{p}} = m(\underset{\sim}{\text{v}}_2 - \underset{\sim}{\text{v}}_1)$

Where $\underset{\sim}{\text{v}}_1 = -40\ \text{m s}^{-1}$ and $\underset{\sim}{\text{v}}_2 = 35\ \text{m s}^{-1}$

So $$\begin{aligned} \Delta \underset{\sim}{\text{p}} &= 0.2(35 - -40) \\ &= 0.2 \times 75 \\ &= 15\ \text{kg ms}^{-1} \end{aligned}$$

Question 17

Answer: C

Worked solution

$$\begin{aligned}\mu = E(Z) &= E(2X - 3Y)\\ &= 2E(X) - 3E(Y)\\ &= 2\times 9 - 3\times 6\\ &= 0\end{aligned}$$

$$\mathrm{Var}(X) = \sigma_X^2 = 5^2 = 25$$

$$\mathrm{Var}(Y) = \sigma_Y^2 = 2^2 = 4$$

$$\begin{aligned}\mathrm{Var}(Z) &= 2^2\,\mathrm{Var}(X) + 3^2\,\mathrm{Var}(Y)\\ &= 2^2\times 25 + 3^2\times 4\\ &= 136\end{aligned}$$

$$\sigma = \sqrt{\mathrm{Var}(Z)} = \sqrt{136}$$

$$\sigma = 2\sqrt{34}$$

Question 18

Answer: E

Worked solution

A type I error occurs when the null hypothesis is rejected when true, which eliminates option A as the null hypothesis is true. This means that a type I error cannot occur.

The *p* value is less than the significance level, which eliminates option C.

When the *p* value is less than the significance level, then the null hypothesis is rejected, which means option B is incorrect.

Since the alternative hypothesis is $\mathrm{H}_1 \mu \neq 4$, a two-tailed test needs to be conducted. So option D is incorrect.

Therefore, a type II error would occur if the null hypothesis is false and not rejected, which is the correct response.

Question 19

Answer: B

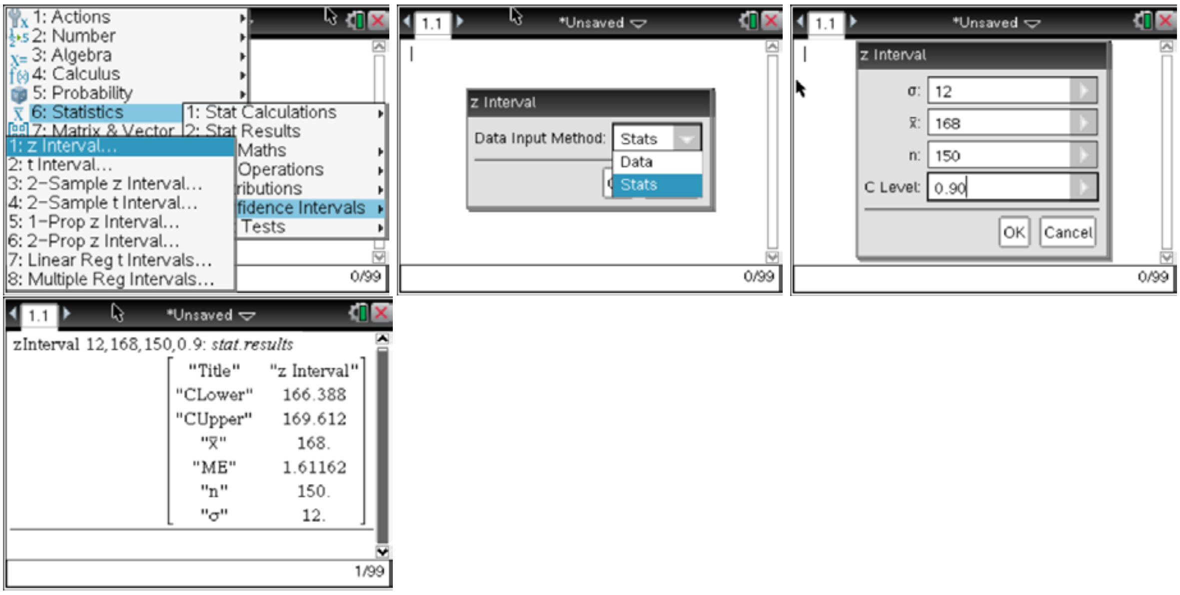
Worked solution

A 90% confidence interval is closest to

$$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right) = \left(168 - 1.645\frac{12}{\sqrt{150}}, 168 + 1.645\frac{12}{\sqrt{150}}\right)$$

$$= (166.388, 169.612)$$

Using the CAS



Tip

- *To avoid rounding errors, always use more accuracy during a calculation than is required in the final answer.*

Question 20

Answer: A

Worked solution

$$\mu_{\bar{x}} = \mu = 65$$

$$sd(\bar{x}) = \frac{\sigma}{\sqrt{n}} = \frac{2.5}{\sqrt{10}}$$

$$\Pr(\bar{X} < 63.5) = \Pr\left(Z < \frac{63.5 - 65}{\frac{2.5}{\sqrt{10}}}\right) = 0.02889$$

$$= 0.0289$$

SECTION B

Question 1a.i.

Worked solution

$$\frac{dy}{dx} = 1 - \frac{4}{(x-1)^2}$$

Turning points occur when $\frac{dy}{dx} = 0$.

$$1 - \frac{4}{(x-1)^2} = 0$$

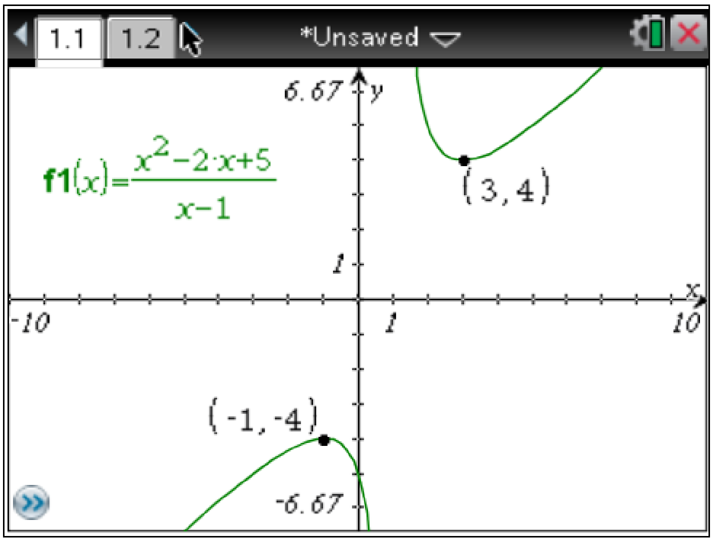
$\Rightarrow x = -1$ and $x = 3$

$f(-1) = -4$

$f(3) = 4$

So the coordinates of the turning points are (–1, –4) and (3, 4).

Or, using the CAS:



Mark allocation: 2 marks

- 1 mark for finding the turning point (3, 4)
- 1 mark for finding the turning point (–1, –4)

Tip

- *Using the CAS to sketch the graph and find the turning points is a good way of checking the accuracy of the solution, as well as any graphs sketched in subsequent questions.*

Question 1a.ii.

Worked solution

$$\frac{d^2y}{dx^2} = \frac{8}{(x-1)^3}$$

When $x = 3$:

$$\frac{d^2y}{dx^2} = \frac{8}{(3-1)^3}$$

$$= 1$$

As $\frac{d^2y}{dx^2} > 0$, when $x = 3$, (3, 4) is a local minimum.

Mark allocation: 2 marks

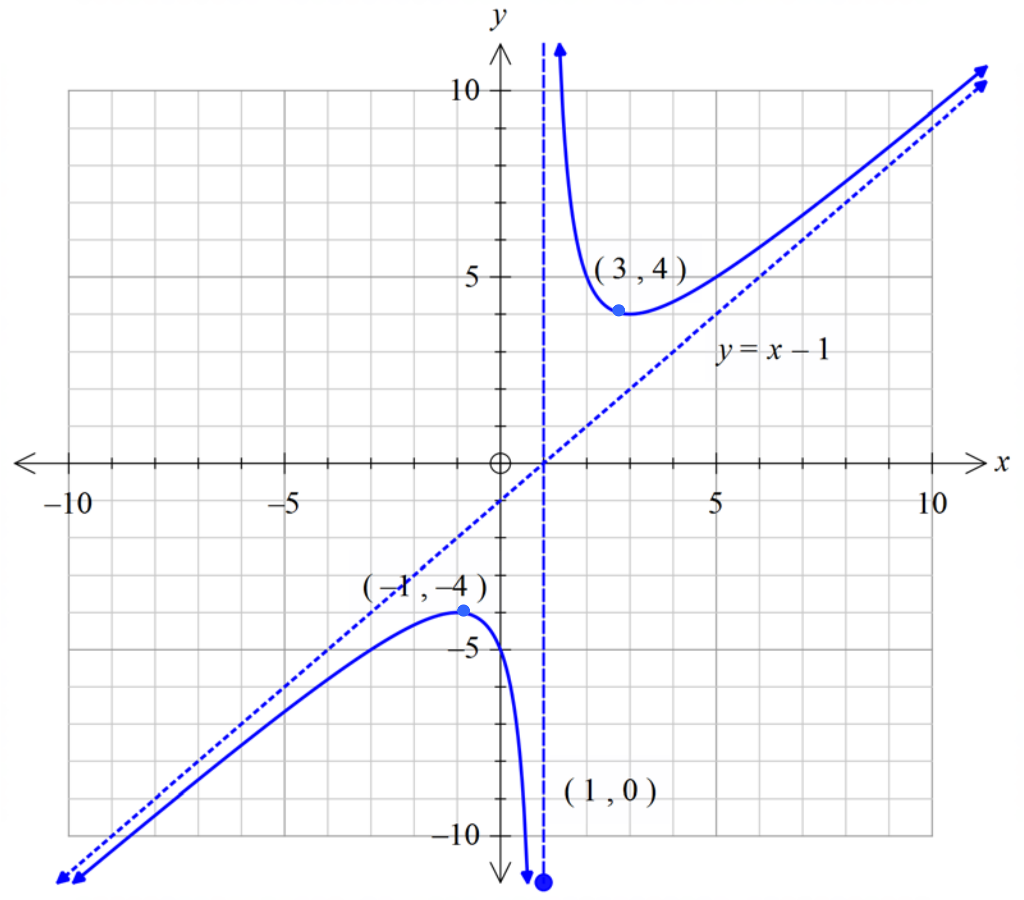
- 1 mark for using the second derivative at $x = 3$ to determine the type of turning point
- 1 mark for concluding that (3, 4) is a local minimum

Tip

- *In 'show that' questions, even though the result may be known or obvious, some form of working must be shown that proves/justifies the result: for example, showing use of the second derivative to determine the type of turning point.*

Question 1b.

Worked solution



Mark allocation: 3 marks

- 1 mark for an accurately shaped graph that goes through the point (0, –5)
- 1 mark for accurately showing and labelling the asymptotes
- 1 mark for accurately labelling the turning points

Question 1c.

Worked solution

This question is asking for the arc length.

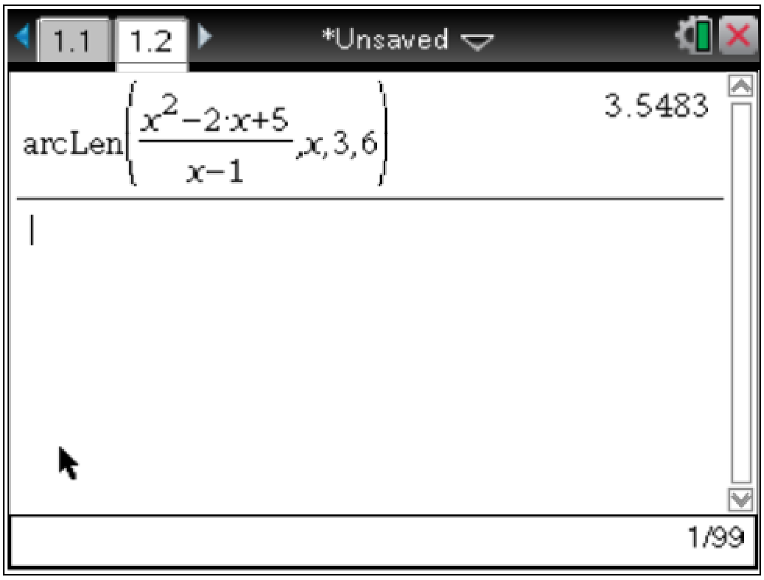
$$\int_{x_1}^{x_2} \sqrt{1+(f'(x))^2}\,dx$$

$$=\int_3^6 \sqrt{1+\left(1-\frac{4}{(x-1)^2}\right)^2}\,dx$$

$$=3.5483$$

$$=3.55$$

Or, using the CAS:



Mark allocation: 1 mark

- 1 mark for correct arc length of 3.55

Tip

- *Using the CAS to find the arc length of a curve is a much more efficient and accurate method than using the formula and calculating by hand.*

Question 2a.

Worked solution

In polar form $u = r\,\text{cis}(\theta)$.

So $r = \sqrt{2^2 + 2^2}$

$r = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{2}{2}\right)$

$\theta = \frac{\pi}{4}$

So $u = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$.

Mark allocation: 2 marks

- 1 mark for showing calculations to find $r = 2\sqrt{2}$ and $\theta = \frac{\pi}{4}$
- 1 mark for stating $u = 2\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$

Question 2b.

Worked solution

If $z = x + iy$, then

$$\begin{aligned} |\bar{z} + z| &= 4 \\ |x - iy + x + iy| &= 4 \\ |2x| &= 4 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for showing calculations that lead to $|2x| = 4$
- 1 mark for line equation $x = 2$

Question 2c.

Worked solution

If $z = x + iy$, then

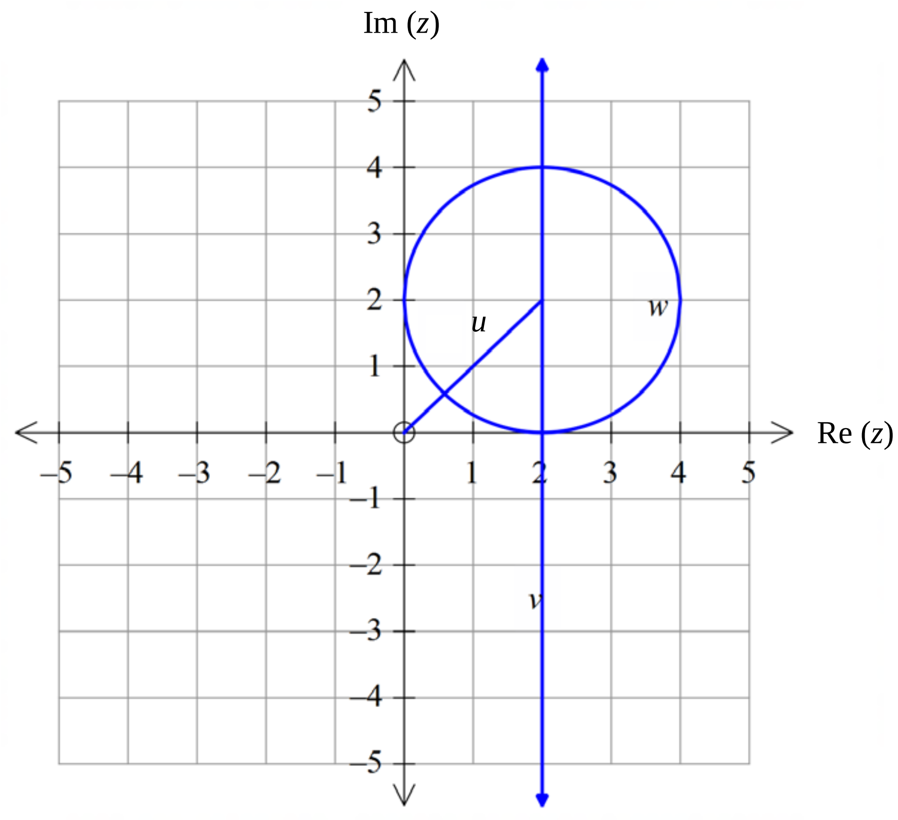
$$\begin{aligned}
(z-2-2i)(\overline{z}-2+2i)-4&=0\\
(x+iy-2-2i)(x-iy-2+2i)-4&=0\\
\left[(x-2)+i(y-2)\right]\left[(x-2)-i(y-2)\right]-4&=0\\
(x-2)^2-i(x-2)(y-2)+i(x-2)(y-2)-i^2(y-2)^2-4&=0\\
(x-2)^2+(y-2)^2-4&=0\\
\Rightarrow (x-2)^2+(y-2)^2&=4
\end{aligned}$$

Mark allocation: 3 marks

- 1 mark for showing $\left[(x-2)+i(y-2)\right]\left[(x-2)-i(y-2)\right]-4=0$
- 1 mark for expanding using difference of two squares
- 1 mark for the cartesian equation $(x-2)^2+(y-2)^2=4$

Question 2d.

Worked solution



Mark allocation: 3 marks

- 1 mark for accurately sketching the line from (0, 0) to u
- 1 mark for accurately sketching $v = \{z : |\overline{z} + z| = 4\}$
- 1 mark for accurately sketching $w = \{z : (z-2-2i)(\overline{z}-2+2i)-4=0\}$

Question 2e.

Worked solution

The area of a sector is given by $A = \frac{1}{2}r^2\theta$.

For the sector, if the circle $\theta = \frac{\pi}{4}$, then the area of the sector is

$$\begin{aligned} A &= \frac{1}{2} \times 2^2 \times \frac{\pi}{4} \\ &= \frac{\pi}{2} \text{ units}^2 \end{aligned}$$

Mark allocation: 2 marks

- 1 mark for using the formula $A = \frac{1}{2}r^2\theta$
- 1 mark for the answer $\frac{\pi}{2}$ units^2

Question 3a.i.

Worked solution

$$\frac{d}{dx}\left(\sqrt{1-x^2}+x\sin^{-1}(x)\right)$$
$$=\frac{d}{dx}\left(\sqrt{1-x^2}\right)+\frac{d}{dx}\left(x\sin^{-1}(x)\right)$$
$$=-\frac{2x}{2\sqrt{1-x^2}}+x\frac{d(\sin^{-1}(x))}{dx}+\frac{d(x)}{dx}\sin^{-1}(x)$$
$$=-\frac{x}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}+\sin^{-1}(x)$$
$$=\sin^{-1}(x)$$

This is an inverse sine function that exists for $x \in [-1,1]$.

Mark allocation: 2 marks

- 1 mark for the domain $x \in [-1,1]$
- 1 mark for determining $\sin^{-1}(x)$

Question 3a.ii.

Worked solution

If $\int \sin^{-1}(x)\,dx = \sqrt{1-x^2}+x\sin^{-1}(x),$ then

$$\int \sin^{-1}\left(\frac{x}{2}\right)dx = \sqrt{4-x^2}+x\sin^{-1}\left(\frac{x}{2}\right).$$

Therefore

$$\int 2\sin^{-1}\left(\frac{x}{2}\right)dx = 2\left(\sqrt{4-x^2}+x\sin^{-1}\left(\frac{x}{2}\right)\right)$$

Mark allocation: 2 marks

- 1 mark for using the result in **part a.** to find an antiderivative
- 1 mark for $2\left(\sqrt{4-x^2}+x\sin^{-1}\left(\frac{x}{2}\right)\right)$

Question 3b.i.

Worked solution

When rotated about the *y*-axis, volume is given by

$$V = \int_a^b \pi x^2 dy$$

Rearranging to make *x* the subject gives

$$x = 2\sin\left(y - \frac{\pi}{2}\right) + 4,$$

when $x = 2 \Rightarrow y = 0$, and when $x = 6 \Rightarrow y = \pi$.

So the volume of the solid generated by rotating the curve about the *y*-axis is

$$V = \int_0^\pi \pi\left(2\sin\left(y - \frac{\pi}{2}\right) + 4\right)^2 dy$$

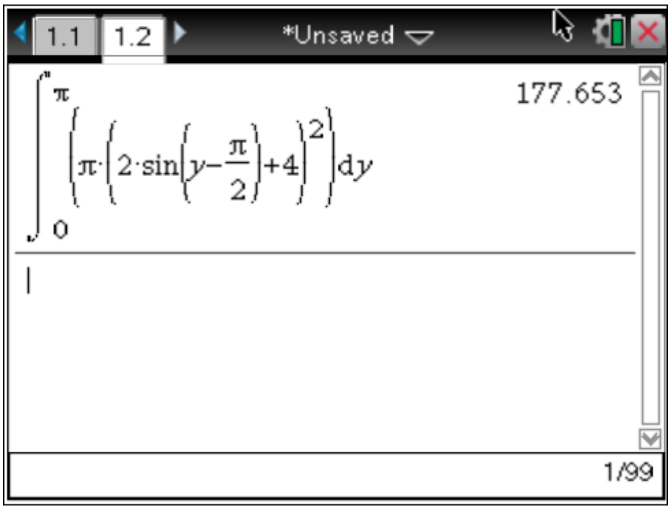
Mark allocation: 1 mark

- 1 mark for showing the integral $V = \int_0^\pi \pi\left(2\sin\left(y - \frac{\pi}{2}\right) + 4\right)^2 dy$

Question 3b.ii.

Worked solution

$$\begin{aligned} V &= 177.653 \\ &= 177.65 \text{ cm}^3 \end{aligned}$$



Mark allocation: 1 mark

- 1 mark for correctly evaluating 177.65 cm^3

Question 3c.

Worked solution

The differential equation describing the change of height is $\dfrac{dy}{dt} = 3t$.

Using substitution to find $\dfrac{dy}{dx}$:

Let $u = \dfrac{x}{2} - 2 \Rightarrow \dfrac{du}{dx} = \dfrac{1}{2}$

$$y = \sin^{-1}(u) + \frac{\pi}{2}$$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1-\left(\frac{x}{2}-2\right)^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{2}.\frac{1}{\sqrt{1-\left(\frac{x}{2}-2\right)^2}}$$

$$= \frac{1}{2\sqrt{1-\left(\frac{x}{2}-2\right)^2}}$$

So, a differential equation relating the change in radius to the change in time is

$$\frac{dx}{dt} = \frac{dx}{dy}.\frac{dy}{dt}$$

$$= 2\sqrt{1-\left(\frac{x}{2}-2\right)^2}.3t$$

$$= 6t\sqrt{1-\left(\frac{x}{2}-2\right)^2}$$

$$= 3t\sqrt{4-(x-4)^2} \quad \text{or} \quad 6t\sqrt{1-\left(\frac{x}{2}-2\right)^2}$$

Mark allocation: 3 marks

- 1 mark for recognising $\frac{dy}{dt} = 3t$
- 1 mark for using substitution to find $\frac{dy}{dx} = \frac{1}{2\sqrt{1-\left(\frac{x}{2}-2\right)^2}}$
- 1 mark for $\frac{dx}{dt} = 6t\sqrt{1-\left(\frac{x}{2}-2\right)^2}$ or $\frac{dx}{dt} = 3t\sqrt{4-(x-4)^2}$

Tip

- *When a value is given in a question and has a rate of change unit, this value is often a differential equation that is to be applied in solving the problem. For instance 3t cm/s is the differential equation* $\frac{dy}{dt} = 3t.$ *Wording such as 'changes with respect to' also indicate a differential equation.*

Question 3d.

Worked solution

$$\frac{dx}{dt}=3t\sqrt{4-(x-4)^2}$$

$$\Rightarrow \int \frac{1}{\sqrt{4-(x-4)^2}}.dx=\int 3t.dt$$

$$\sin^{-1}\left(\frac{x-4}{2}\right)=\frac{3t^2}{2}+c$$

Applying the conditions $x=2, t=0$

$$\sin^{-1}(-1)=c$$

$$\Rightarrow c=-\frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\frac{x-4}{2}\right)=\frac{3t^2-\pi}{2}$$

$$\frac{x-4}{2}=\sin\left(\frac{3t^2-\pi}{2}\right)$$

$$\Rightarrow x=4+2\sin\left(\frac{3t^2-\pi}{2}\right)$$

Alternatively,

From **part c.**

$$\frac{dy}{dt}=3t$$

$$\Rightarrow y=\frac{3t^2}{2}+c$$

Applying initial conditions $y=0$ when $t=0$

$$\Rightarrow y=\frac{3t^2}{2}$$

From **part b.** it is known that $x=2\sin\left(y-\frac{\pi}{2}\right)+4.$

So substituting $y=\frac{3t^2}{2}$ gives

$$x=4+2\sin\left(\frac{3t^2-\pi}{2}\right)$$

Mark allocation: 3 marks

- 1 mark for the integration by separating the variables to get $\sin^{-1}\left(\dfrac{x-4}{2}\right)=\dfrac{3t^2}{2}+c$

  or for integrating $\dfrac{dy}{dt}=3t$ to get $y=\dfrac{3t^2}{2}+c$
- 1 mark for finding $c=-\dfrac{\pi}{2}$ or for applying initial conditions to get $y=\dfrac{3t^2}{2}$
- 1 mark for $x=4+2\sin\left(\dfrac{3t^2-\pi}{2}\right)$

Question 4a.

Worked solution

Let $x = 2 - 3\cos(t)$ and $y = 1 - 2\sin(t)$.

Hence, $\cos(t) = \dfrac{x-2}{-3}$ and $\sin(t) = \dfrac{y-1}{-2}$.

Recall that $\cos^2(t) + \sin^2(t) = 1$.

Then

$$\left(\frac{x-2}{-3}\right)^2 + \left(\frac{y-1}{-2}\right)^2 = 1$$

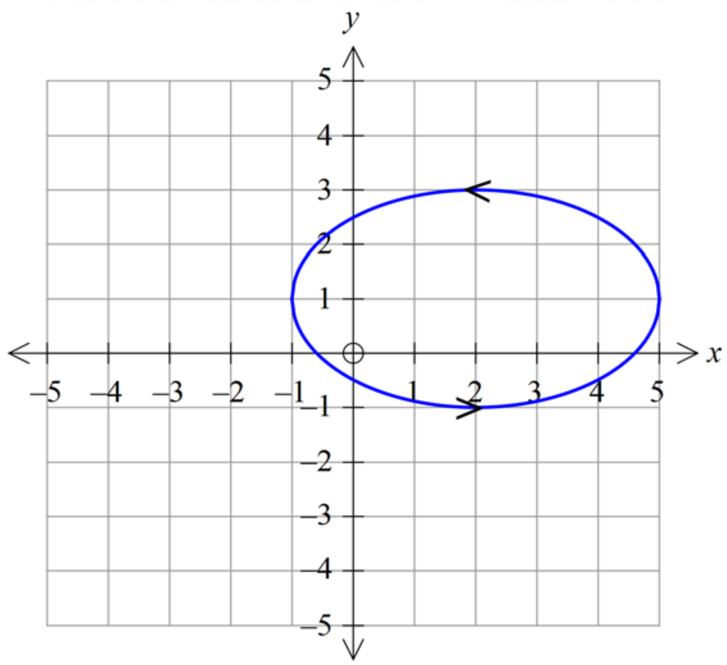
$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1$$

Mark allocation: 2 marks

- 1 mark for $\cos(t) = \dfrac{x-2}{-3}$ and $\sin(t) = \dfrac{y-1}{-2}$
- 1 mark for $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$

Question 4b.

Worked solution



When $t = 0$

$x = -1,\ y = 1$

When $t = \frac{\pi}{2}$

$x = 2,\ y = -1$

When $t = \pi$

$x = 5,\ y = 1$

Mark allocation: 2 marks

- 1 mark for accurately drawing the ellipse
- 1 mark for arrows showing an anticlockwise direction

Tip

- *Testing different values of t in the parametric equations to find the cartesian coordinates is a useful way of determining the direction of motion.*

Question 4c.

Worked solution

The $\underset{\sim}{i}$ component of the position vector of the walker and the dog is the same for all time, so it just needs to be shown that the $\underset{\sim}{j}$ component of the position vector of the walker and the dog is the same at $t = \frac{5\pi}{6}$.

Walker:

When $t = \frac{5\pi}{6}$, the $\underset{\sim}{j}$ component is

$$1 - 2\sin\left(\frac{5\pi}{6}\right)\underset{\sim}{j} = 1 - 1\,\underset{\sim}{j} = 0\,\underset{\sim}{j}$$

Dog:

When $t = \frac{5\pi}{6}$, the $\underset{\sim}{j}$ component is

$$\frac{24 \times 5\pi}{5\pi \times 6} - 4\,\underset{\sim}{j} = 4 - 4\,\underset{\sim}{j} = 0\,\underset{\sim}{j}$$

Since the $\underset{\sim}{i}$ and $\underset{\sim}{j}$ components of $\underset{\sim}{r}_W$ and $\underset{\sim}{r}_D$ are the same at $t = \frac{5\pi}{6}$, then the walker and the dog collide at $t = \frac{5\pi}{6}$.

Mark allocation: 2 marks

- 1 mark for equating the $\underset{\sim}{j}$ components of the position vectors at $t = \frac{5\pi}{6}$
- 1 mark for a concluding statement that the walker and the dog collide because they have the same position vector at $t = \frac{5\pi}{6}$

Question 4d.

Worked solution

The velocity vectors can be found by differentiating the position vectors

$\dot{\underset{\sim}{r}}_W = 3\sin(t)\underset{\sim}{i} - 2\cos(t)\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}_D = 3\sin(t)\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j}$

When $t = \frac{5\pi}{6}$

$\dot{\underset{\sim}{r}}_W = 3\sin\left(\frac{5\pi}{6}\right)\underset{\sim}{i} - 2\cos\left(\frac{5\pi}{6}\right)\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}_D = 3\sin\left(\frac{5\pi}{6}\right)\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}_W = \frac{3}{2}\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}$ and $\dot{\underset{\sim}{r}}_D = \frac{3}{2}\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j}$

The angle between the velocity vectors would be given by

$$\theta = \cos^{-1}\left(\frac{\dot{\underset{\sim}{r}}_W . \dot{\underset{\sim}{r}}_D}{|\dot{\underset{\sim}{r}}_W||\dot{\underset{\sim}{r}}_D|}\right)$$

$$= \cos^{-1}\left(\frac{(\frac{3}{2}\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j}).(\frac{3}{2}\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j})}{\sqrt{\left(\frac{3}{2}\right)^2 + \left(-\sqrt{3}\right)^2}\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{24}{5\pi}\right)^2}}\right)$$

$$= \cos^{-1}\left(\frac{\frac{9}{4} + \frac{24\sqrt{3}}{5\pi}}{\sqrt{\left(\frac{3}{2}\right)^2 + \left(\sqrt{3}\right)^2}\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{24}{5\pi}\right)^2}}\right)$$

$$= 3.579946343$$

$$= 3.6°$$

Mark allocation: 3 marks

- 1 mark for correctly differentiating the position vectors
- 1 mark for using $\theta = \cos^{-1}\left(\frac{\dot{\underset{\sim}{r}}_W . \dot{\underset{\sim}{r}}_D}{|\dot{\underset{\sim}{r}}_W||\dot{\underset{\sim}{r}}_D|}\right)$ at $t = \frac{5\pi}{6}$ to find the angle between the velocity vectors
- 1 mark for a correct angle of $\theta = 3.6°$

Question 4e.

Worked solution

Speed of the dog is given by

$\text{Speed} = |\dot{\underset{\sim}{r}}_D|$

So at $t = \frac{5\pi}{6}$

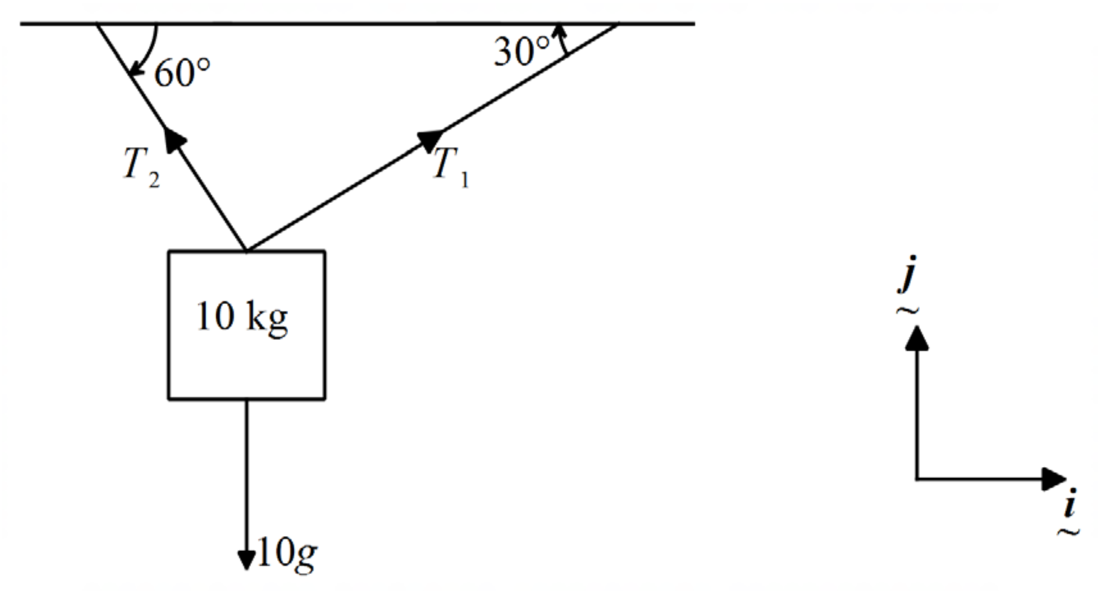
$$\begin{aligned}\text{speed} &= |\dot{\underset{\sim}{r}}_D| \\ &= \left|\frac{3}{2}\underset{\sim}{i} + \frac{24}{5\pi}\underset{\sim}{j}\right| \\ &= \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{24}{5\pi}\right)^2} \\ &= 2.14113 \\ &= 2.14 \text{ ms}^{-1}\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for showing $\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{24}{5\pi}\right)^2}$
- 1 mark for the correct answer 2.14 ms^{-1}

Question 5a.

Worked solution



Resolving forces in the horizontal direction

$$T_1\cos(30°) - T_2\cos(60°) = 0$$
$$\frac{T_1\sqrt{3}}{2} - \frac{T_2}{2} = 0$$
$$\Rightarrow T_2 = \sqrt{3}\,T_1$$

Resolving forces in the vertical direction

$$T_1\sin(30°) + T_2\sin(60°) - 10g = 0$$
$$\frac{T_1}{2} + \frac{T_2\sqrt{3}}{2} - 10g = 0$$
$$\frac{T_1}{2} + \frac{\sqrt{3}\times\sqrt{3}\,T_1}{2} - 10g = 0$$
$$\frac{3T_1}{2} + \frac{T_1}{2} - 10g = 0$$
$$2T_1 = 10g$$
$$T_1 = 5g \text{ N}$$

So $T_2 = \sqrt{3}\,T_1$

$T_2 = 5g\sqrt{3}$ N

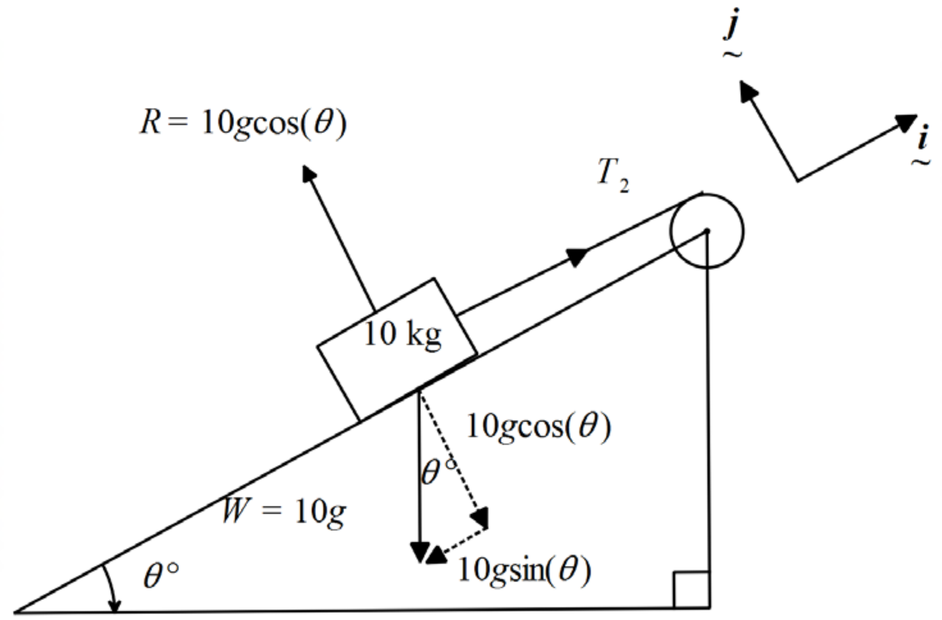
Mark allocation: 2 marks

- 1 mark for $T_2 = \sqrt{3}\,T_1$ or similar
- 1 mark for $T_2 = 5g\sqrt{3}$ N and $T_1 = 5g$ N

Question 5b.

Worked solution

A free body diagram for the forces acting on the mass when resolved into its components is



Since the mass is on the verge of accelerating according to Newton's Second Law, the sum of the forces is 0.

In the $\underset{\sim}{j}$ direction the forces are balanced, so the forces in the $\underset{\sim}{i}$ direction must be balanced and θ found.

$$T_2 - 10g\sin(\theta) = 0$$

$$5g\sqrt{3} - 10g\sin(\theta) = 0$$

$$\sin(\theta) = \frac{\sqrt{3}}{2}$$

$$\theta = 60°$$

Mark allocation: 2 marks

- 1 mark for balancing the forces in the $\underset{\sim}{i}$ direction
- 1 mark for $\theta = 60°$

Tips

- *Drawing a free body diagram is a good way of visualising the forces acting on the body. There have been instances on examiners' reports where the examiners have mentioned that those students who drew a diagram were more successful than those who did not.*
- *Wording such as 'on the verge of accelerating' means that the forces acting on the mass are all balanced and the net force is 0.*

Question 5c.i.

Worked solution

The mass will accelerate up the plane when $10g\sin(\theta) < T_2$, so $\theta < 60°$.

Hence, the range of angles that the mass will accelerate up the plane is $0° < \theta < 60°$.

Mark allocation: 1 mark

- 1 mark for $0° < \theta < 60°$

Explanatory note

The lower boundary condition is not included because an angle of $0°$ would give a horizontal plane.

Question 5c.ii.

Worked solution

The mass will accelerate up the plane when $10g\sin(\theta) > T_2$, so $\theta > 60°$.

Hence, the range of angles that the mass will accelerate up the plane is $60° < \theta < 90°$.

Mark allocation: 1 mark

- 1 mark for $60° < \theta < 90°$

Explanatory note

The upper boundary condition is not included because an angle of 90° would give a vertical plane.

Question 5d.

Worked solution

Using Newton's second law to find the acceleration

$$T_2 - 10g\sin(10°) = ma$$

$$5g\sqrt{3} - 10g\sin(10°) = 10a$$

$$a = \frac{5g\sqrt{3} - 10g\sin(10°)}{10}$$

Acceleration can be expressed as

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Which means that

$$\frac{1}{2}v^2 = \int \frac{5g\sqrt{3} - 10g\sin(10°)}{10}\,dx$$

$$\frac{1}{2}v^2 = \frac{5g\sqrt{3} - 10g\sin(10°)}{10}x + c$$

Applying the initial condition that the block accelerates from rest gives

$$\frac{1}{2}v^2 = \frac{5g\sqrt{3} - 10g\sin(10°)}{10}x$$

So the speed of the block at the end of the ramp when $x = 2.5$

$$v = \sqrt{2 \times \frac{5g\sqrt{3} - 10g\sin(10°)}{10} \times 2.5}$$

$$v = 5.82464$$

$$v = 6 \text{ ms}^{-1}$$

Alternative solution

From the information given, it can be seen that the kinematic formula to find the velocity is $v^2 = u^2 + 2as$, where $u = 0$ and $s = 2.5$ m. The acceleration can be found using Newton's Second Law.

$$T_2 - 10g\sin(10°) = ma$$
$$5g\sqrt{3} - 10g\sin(10°) = 10a$$
$$a = \frac{5g\sqrt{3} - 10g\sin(10°)}{10}$$

So the velocity at the top of the inclined plane, to the nearest integer, is

$$\begin{aligned} v &= \sqrt{u^2 + 2as} \\ &= \sqrt{0^2 + 2 \times \frac{5g\sqrt{3} - 10g\sin(10°)}{10} \times 2.5} \\ &= 5.8246 \\ v &= 6 \text{ ms}^{-1} \end{aligned}$$

Note: constant acceleration formulae are NOT covered by the VCAA Study Design for VCE Specialist Mathematics Units 3 & 4. If the incorrect answer is obtained using these formulae, no working marks can be awarded.

Mark allocation: 2 marks

- 1 mark for finding the acceleration of the mass as $a = \dfrac{5g\sqrt{3} - 10g\sin(10°)}{10}$
- 1 mark for $v = 6 \text{ ms}^{-1}$

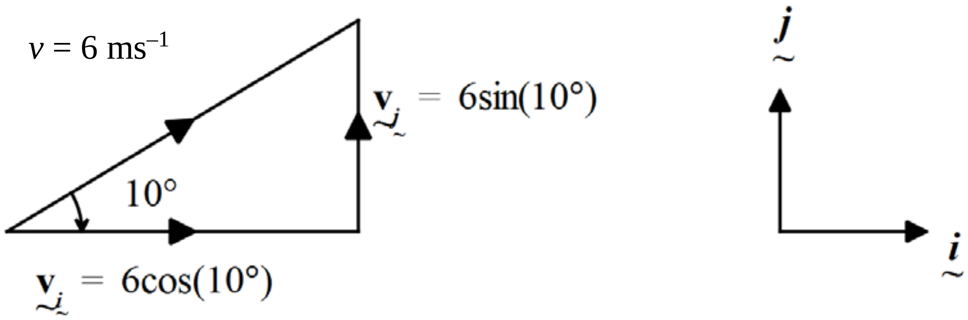
Tip

- *Leaving values in exact form, such as the acceleration in this question, and evaluating it in the final calculation leads to a more accurate result.*

Question 5e.

Worked solution

Since the mass follows a projectile path, it can be broken up into its horizontal and vertical components, as shown in the diagram below.



The horizontal component of the velocity will remain constant but the vertical component will change. By finding the time taken for the vertical component of the velocity to reach zero and doubling that time, it will give the time at which the mass is above the height of the inclined plane.

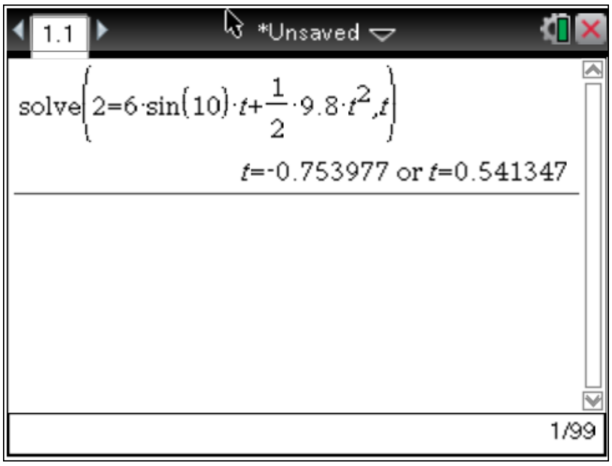
Assuming that the direction of gravity is negative, then

$$\begin{aligned} t &= \frac{v-u}{a} \\ &= \frac{0-6\sin(10°)}{-9.8} \\ &= 0.106 \end{aligned}$$

So the time the mass spends above the ground is 0.212 seconds.

The remaining time it takes for the mass to hit the ground can be found by solving the equation $s = ut + \frac{1}{2}at^2$ for t, where the acceleration is now positive and $u = 6\sin(10°)$.

So, $2 = 6\sin(10°) \times t + \frac{1}{2} \times 9.8 \times t^2$.



Therefore, after rejecting the negative time, the remaining time taken for the mass to hit the ground is 0.541 seconds.

Hence, the time taken for the mass to hit the ground is

$0.212 + 0.541 = 0.753 \approx 0.75$ seconds

Alternative solution

Using vectors and assigning downward acceleration as positive gives an acceleration of

$\underset{\sim}{a} = -9.8\,\underset{\sim}{j}$

Integrating to get the velocity gives

$\underset{\sim}{v} = -9.8t\,\underset{\sim}{j} + \underset{\sim}{c}$

Applying initial conditions gives

$\underset{\sim}{v} = 6\cos(10°)\,\underset{\sim}{i} + (6\sin(10°) - 9.8t)\,\underset{\sim}{j}$

Integrating to get the position vector gives

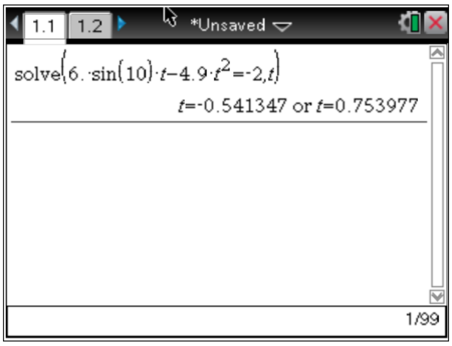
$\underset{\sim}{r} = 6\cos(10°)t\,\underset{\sim}{i} + (6\sin(10°)t - 4.9t^2)\,\underset{\sim}{j} + \underset{\sim}{c}$

Assigning the top of the ramp to be the origin gives a position vector of

$\underset{\sim}{r} = 6\cos(10°)t\,\underset{\sim}{i} + (6\sin(10°)t - 4.9t^2)\,\underset{\sim}{j}$

For the mass to hit the ground, the $\underset{\sim}{j}$ component will be –2. So equate the $\underset{\sim}{j}$ component to –2 and solve for *t*.

$6\sin(10°)t - 4.9t^2 = -2$



Rejecting the negative time gives $t = 0.75$ seconds.

Mark allocation: 3 marks

- 1 mark for calculating the time the mass spends above the top of the inclined plane
- 1 mark for calculating the time the mass spends below the top of the inclined plane
- 1 mark for giving the correct time taken for the mass to hit the ground as 0.75 seconds

OR

- 1 mark for determining the position vector
- 1 mark for equating the position vector to –2
- 1 mark for giving the correct time taken for the mass to hit the ground as 0.75 seconds

Question 6a.

Worked solution

$$\begin{aligned}\mathrm{SD}(\bar{X}) &= \frac{\sigma}{\sqrt{n}}\\ &= \frac{1.5}{\sqrt{60}}\\ &= \frac{\sqrt{15}}{20}\end{aligned}$$

Mark allocation: 1 mark

- 1 mark for $\mathrm{SD}(\bar{X}) = \dfrac{\sqrt{15}}{20}$

Question 6b.

Worked solution

$H_0 : \mu = 9.5$ and $H_1 : \mu \neq 9.5$.

Mark allocation: 2 marks

- 1 mark for $H_0 : \mu = 9.5$
- 1 mark for $H_1 : \mu \neq 9.5$

Tip

- *Looking for words like ‘increase’, ‘decrease’ and ‘not’ in the question’s description will help determine the alternative hypothesis and whether a one-tailed test or two-tailed test must be conducted in subsequent questions.*

Question 6c.i.

Worked solution

Because of the alternative hypothesis, $H_1 : \mu \neq 9.5,$ this is a two-tailed test. So the *p* value must be calculated accordingly.

So

$$
\begin{aligned}
p \text{ value} &= 2 \times \Pr(\bar{X} \le 9 \mid \mu = 9.5) \\
&= 2 \times \Pr\left(Z \le \frac{9-9.5}{\frac{\sqrt{15}}{20}} \right) \\
&= 2 \times \Pr(Z \le -2.58199) \\
&= 2 \times 0.004912 \\
&= 0.009824 \\
&= 0.0098
\end{aligned}
$$

Mark allocation: 2 marks

- 1 mark for $2 \times \Pr\left(Z \le \frac{9-9.5}{\frac{\sqrt{15}}{20}} \right)$
- 1 mark for *p* value = 0.0098

Explanatory note

Because a two-tailed test is non-directional, both tails must be considered in the calculation of the *p* value. So the *p* value is double the value of a single-tailed test.

Question 6c.ii.

Worked solution

Since the *p* value found for this sample is less than $\alpha = 0.05,$ the null hypothesis, H_0, is rejected and the alternative hypothesis, H_1, is accepted. So the sample selected supports the sports shoe store's belief.

Mark allocation: 1 mark

- 1 mark for stating that H_0 is rejected in favour of H_1 and that the sample supports the sports shoe store's belief

END OF WORKED SOLUTIONS