

Year 2017

VCE

Specialist Mathematics

Trial Examination 1

Solutions



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Question 1

$$\begin{aligned}
 \text{a. } z &= \sqrt{2} \operatorname{cis}\left(-\frac{2\pi}{3}\right) \\
 &= \sqrt{2} \left(\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \right) \text{ since } \cos(-\theta) = \cos(\theta), \sin(-\theta) = -\sin(\theta) \\
 &= \sqrt{2} \left(\cos\left(\frac{2\pi}{3}\right) - i \sin\left(\frac{2\pi}{3}\right) \right) \\
 &= \sqrt{2} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \quad \text{M1} \\
 &= \frac{\sqrt{2}}{2} (-1 - \sqrt{3}i)
 \end{aligned}$$

by the conjugate root theorem, $\bar{z} = \frac{\sqrt{2}}{2}(-1 + \sqrt{3}i)$ is also a root

$$z + \bar{z} = -\sqrt{2}, \quad z \cdot \bar{z} = \frac{2}{4}(1 - 3i^2) = 2$$

$z^2 - (\text{sum of the roots})z + \text{product of the roots}$

$$= (z^2 + \sqrt{2}z + 2) \text{ is the quadratic factor} \quad \text{A1}$$

$$\begin{aligned}
 \text{b. } f(z) &= z^4 + \sqrt{2}z^3 + 5z^2 + 3\sqrt{2}z + 6 = 0 \\
 &= (z^2 + \sqrt{2}z + 2)(z^2 + 3) = 0 \\
 &= \left(z - \frac{\sqrt{2}}{2}(-1 + \sqrt{3}i) \right) \left(z - \frac{\sqrt{2}}{2}(-1 - \sqrt{3}i) \right) (z - \sqrt{3}i)(z + \sqrt{3}i) = 0 \quad \text{M1} \\
 \text{all the roots are } z &= \frac{\sqrt{2}}{2}(-1 \pm \sqrt{3}i), \pm\sqrt{3}i \quad \text{A1}
 \end{aligned}$$

Question 2

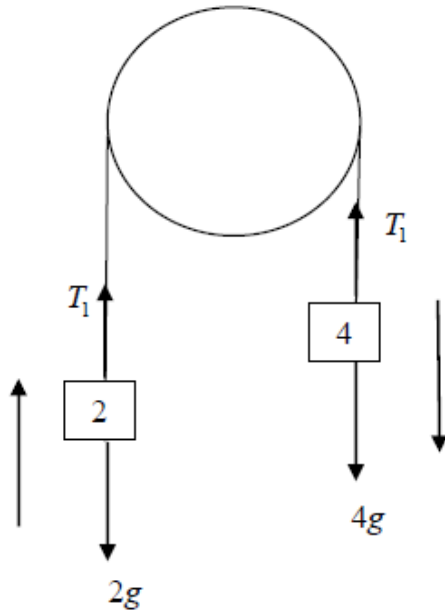
$$95\% \text{ confidence interval is } \mu \pm 2 \times \frac{\mu}{2\sqrt{n}} = \left(\mu \left(1 - \frac{1}{\sqrt{n}} \right), \mu \left(1 + \frac{1}{\sqrt{n}} \right) \right)$$

$$\text{this has a width of } \frac{2\mu}{\sqrt{n}} \leq \frac{\mu}{10} \quad \text{M1}$$

$$\Rightarrow \sqrt{n} \geq 20$$

$$\text{the minimum sample size is 400} \quad \text{A1}$$

Question 3



when the 4kg mass moves downwards,
with an acceleration of a , resolving

$$(1) \quad 4g - T_1 = 4a$$

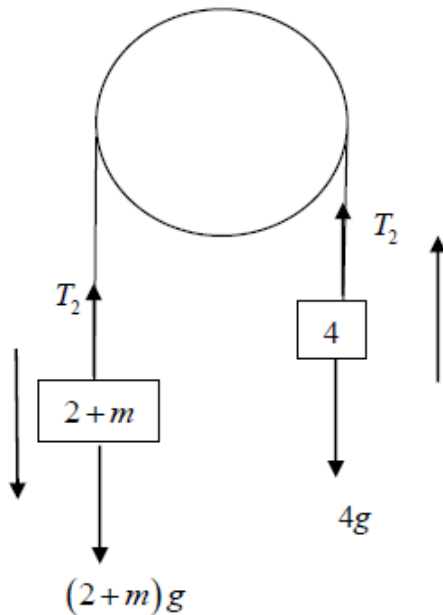
$$(2) \quad T_1 - 2g = 2a$$

adding (1)+(2)

$$2g = 6a$$

$$a = \frac{g}{3}$$

A1



when the $2+m$ kg moves downwards
with an acceleration of $\frac{g}{3}$, resolving

$$(3) \quad (2+m)g - T_2 = (2+m)\frac{g}{3}$$

$$(4) \quad T_2 - 4g = 4 \times \frac{g}{3}$$

adding (3)+(4)

$$(2+m-4)g = \frac{g}{3}(2+m+4)$$

$$3(m-2) = m+6$$

$$3m-6 = m+6$$

$$2m = 12$$

$$m = 6$$

M1

A1

Question 4

$$\log_e(2xy) + \frac{x}{y} = 8 \quad \text{by log laws}$$

$$\log_e(2) + \log_e(x) + \log_e(y) + \frac{x}{y} = 8 \quad \text{using implicit differentiation} \quad \text{M1}$$

$$\frac{1}{x} + \frac{1}{y} \frac{dy}{dx} + \frac{1}{y} - \frac{x}{y^2} \frac{dy}{dx} = 0$$

$$\frac{1}{x} + \frac{1}{y} = \frac{dy}{dx} \left(\frac{x}{y^2} - \frac{1}{y} \right) \quad \text{A1}$$

$$\frac{x+y}{xy} = \frac{dy}{dx} \left(\frac{x-y}{y^2} \right)$$

$$\frac{dy}{dx} = \frac{y(x+y)}{x(x-y)} \quad \text{A1}$$

at the point $\left(2, \frac{1}{4}\right)$

$$\left. \frac{dy}{dx} \right|_{\left(2, \frac{1}{4}\right)} = \frac{\frac{1}{4} \left(2 + \frac{1}{4}\right)}{2 \left(2 - \frac{1}{4}\right)} = \frac{1}{8} \times \frac{9}{7} = \frac{9}{56}$$

gradient of the normal is $-\frac{56}{9}$ A1

Question 5

a. $s = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2} dt$

$$x = t^3 + 5, \quad y = 6t^2 - 1$$

$$\dot{x} = \frac{dx}{dt} = 3t^2 \quad \dot{y} = \frac{dy}{dt} = 12t \quad \text{M1}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9t^4 + 144t^2 = 9t^2(t^2 + 16)$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3t\sqrt{t^2 + 16} \quad \text{since } t \geq 0 \quad \text{A1}$$

$$s = \int_0^3 3t\sqrt{t^2 + 16} dt$$

$$\text{b. } s = \int_0^3 3t\sqrt{t^2+16} dt \quad \text{let } u = t^2 + 16 \quad \frac{du}{dt} = 2t \Rightarrow dt = \frac{1}{2t} du$$

terminals when $t=0 \Rightarrow u=16$ when $t=3 \Rightarrow u=25$

$$s = \int_{16}^{25} 3t\sqrt{u} \times \frac{1}{2t} du \quad \text{M1}$$

$$s = \frac{3}{2} \int_{16}^{25} u^{\frac{1}{2}} du$$

$$= \frac{3}{2} \left[\frac{2}{3} u^{\frac{3}{2}} \right]_{16}^{25}$$

$$= 25^{\frac{3}{2}} - 16^{\frac{3}{2}}$$

$$= (\sqrt{25})^3 - (\sqrt{16})^3$$

$$= 5^3 - 4^3 = 125 - 64$$

$$= 61$$

A1

Question 6

$$\frac{dy}{dx} = \frac{\sqrt{4-y^2}}{4+x^2} \quad \text{and } y(0)=1 \quad \text{separating the variables}$$

$$\int \frac{1}{\sqrt{4-y^2}} dy = \int \frac{1}{4+x^2} dx$$

M1

$$\sin^{-1}\left(\frac{y}{2}\right) = \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

A1

$$\text{when } y=1, x=0 \quad \sin^{-1}\left(\frac{1}{2}\right) = c = \frac{\pi}{6}$$

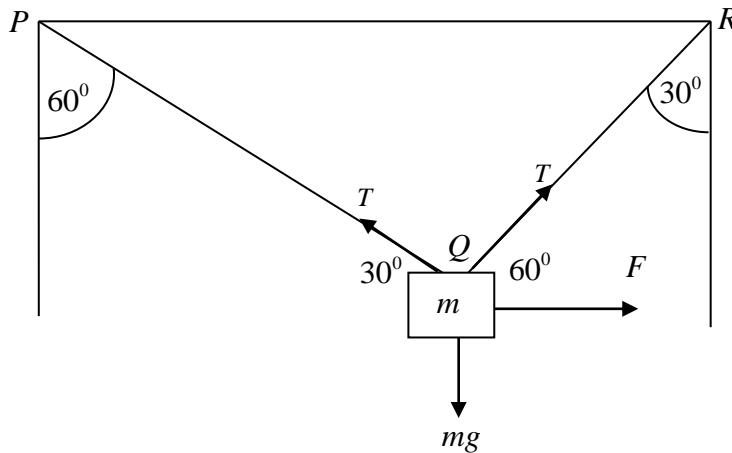
$$\sin^{-1}\left(\frac{y}{2}\right) = \frac{\pi}{6} + \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right)$$

$$y = 2 \sin\left(\frac{\pi}{6} + \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right)\right)$$

A1

Question 7

a.



A1

b.

Let T be the tension in the strings, all the forces are in newtons.resolving horizontally (1) $F + T \cos(60^\circ) - T \cos(30^\circ) = 0$ resolving vertically (2) $T \sin(60^\circ) + T \sin(30^\circ) - mg = 0$

A1

$$(1) \quad F = T(\cos(30^\circ) - \cos(60^\circ)) \Rightarrow F = T\left(\frac{\sqrt{3}-1}{2}\right) \Rightarrow T = \frac{2F}{\sqrt{3}-1}$$

$$(2) \quad mg = T(\sin(60^\circ) + \sin(30^\circ)) \Rightarrow mg = T\left(\frac{\sqrt{3}+1}{2}\right) \Rightarrow T = \frac{2mg}{\sqrt{3}+1}$$

M1

eliminating T , gives $T = \frac{2F}{\sqrt{3}-1} = \frac{2mg}{\sqrt{3}+1} \Rightarrow F = \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)mg$

$$F = \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)mg \times \left(\frac{\sqrt{3}-1}{\sqrt{3}-1}\right) = \left(\frac{3-2\sqrt{3}+1}{3-1}\right)mg$$

$$F = (2-\sqrt{3})mg \quad a=2, b=3$$

A1

Question 8

a.i. Let $y = \sec(kx) = \frac{1}{\cos(kx)} = (\cos(kx))^{-1}$ using the chain rule

$$\frac{dy}{dx} = -1 \times -k \sin(kx) \times (\cos(kx))^{-2} = \frac{k \sin(kx)}{\cos^2(kx)} = \frac{k \sin(kx)}{\cos(kx)} \times \frac{1}{\cos(kx)}$$

A1

$$\frac{d}{dx} [\sec(kx)] = k \tan(kx) \sec(kx)$$

ii Let $y = \log_e(\tan(kx) + \sec(kx)) = \log_e(u)$ where $u = \tan(kx) + \sec(kx)$

$$\frac{dy}{du} = \frac{1}{u} \quad \text{using the chain rule}$$

$$\frac{du}{dx} = k \sec^2(kx) + k \sec(kx) \tan(kx) = k \sec(kx) (\tan(kx) + \sec(kx))$$

A1

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\tan(kx) + \sec(kx)} \times k \sec(kx) (\tan(kx) + \sec(kx)) = k \sec(kx)$$

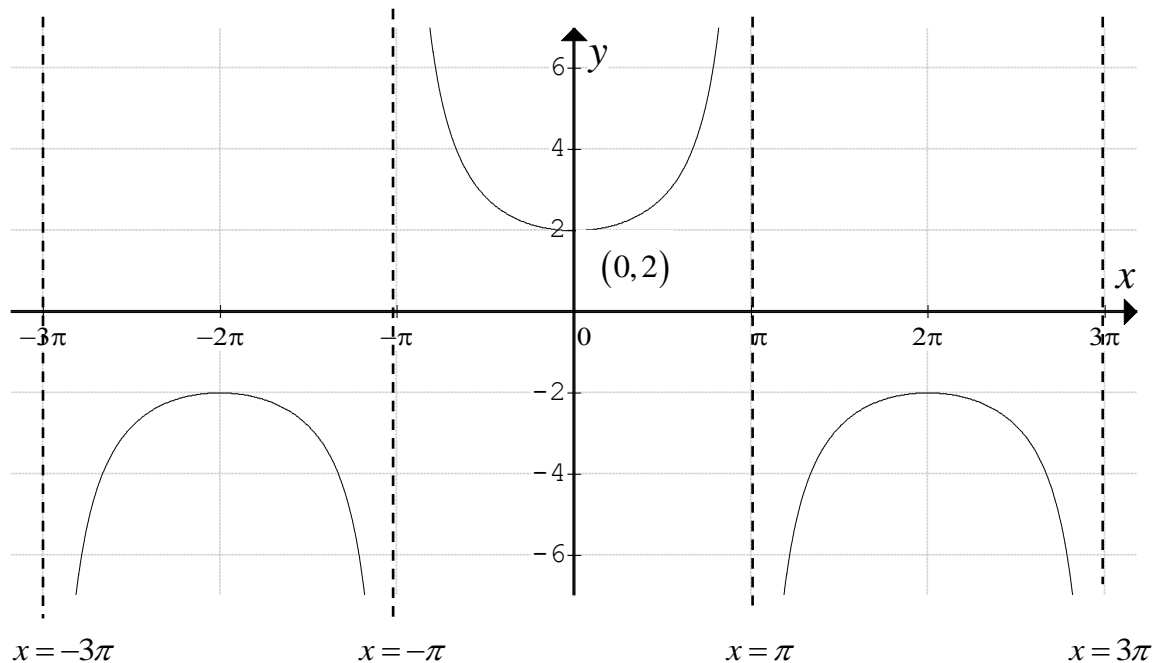
$$\text{so that } \frac{d}{dx} [\log_e(\tan(kx) + \sec(kx))] = k \sec(kx)$$

b. $y = 2 \sec\left(\frac{x}{2}\right) = \frac{2}{\cos\left(\frac{x}{2}\right)}$ when $x=0$, $y=2$, y intercept $(0,2)$

The graph has vertical asymptotes when $\cos\left(\frac{x}{2}\right) = 0 \Rightarrow \frac{x}{2} = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$

so that $x = \pm\pi, \pm 3\pi$ are the vertical asymptotes.

A1



$$\begin{aligned} \text{c. } A &= \int_0^{\frac{\pi}{2}} 2 \sec\left(\frac{x}{2}\right) dx && \text{using a. with } k = \frac{1}{2} \\ A &= \left[4 \log_e \left(\tan\left(\frac{x}{2}\right) + \sec\left(\frac{x}{2}\right) \right) \right]_0^{\frac{\pi}{2}} && \text{A1} \\ &= 4 \log_e \left(\tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right) \right) - 4 \log_e (\tan(0) + \sec(0)) \\ &= 4 \log_e (1 + \sqrt{2}) \text{ units}^2 && \text{A1} \end{aligned}$$

Question 9

$$\text{Let } \underline{a} = 2\underline{i} - 3\underline{j} + \underline{k}, \quad \underline{b} = \underline{i} + 2\underline{j} + 4\underline{k}, \quad \underline{c} = x\underline{i} + y\underline{j} + z\underline{k}$$

$$\text{Since } \underline{a} \text{ is perpendicular to } \underline{c} \quad \underline{a} \cdot \underline{c} = 0 \Rightarrow (1) \quad 2x - 3y + z = 0$$

$$\text{Since } \underline{b} \text{ is perpendicular to } \underline{c} \quad \underline{b} \cdot \underline{c} = 0 \Rightarrow (2) \quad x + 2y + 4z = 0 \quad \text{A1}$$

$$\text{Since } \underline{c} \text{ is a unit vector } \quad |\underline{c}| = 1 \Rightarrow x^2 + y^2 + z^2 = 1$$

$$(1) \quad 2x - 3y + z = 0$$

$$2 \times (2) \quad 2x + 4y + 8z = 0 \quad \text{subtracting gives } 7y + 7z = 0 \Rightarrow z = -y$$

$$4 \times (1) \quad 8x - 12y + 4z = 0 \quad \text{M1}$$

$$(2) \quad x + 2y + 4z = 0 \quad \text{subtracting gives } 7x - 14y = 0 \Rightarrow x = 2y$$

$$\text{substituting into (3)} \quad 4y^2 + y^2 + y^2 = 6y^2 = 1 \Rightarrow y = \pm \frac{1}{\sqrt{6}}$$

$$\hat{\underline{c}} = \frac{\pm 1}{\sqrt{6}} (2\underline{i} + \underline{j} - \underline{k}) \quad \text{A1}$$

Question 10

$$\text{a. } \left. \frac{dB}{dt} \right|_{B=60} = \frac{60}{600}(200-60) = 14 \text{ gm/month}$$

$$\left. \frac{dB}{dt} \right|_{B=120} = \frac{120}{600}(200-120) = 16 \text{ gm/month}$$

Since when $B = 120$ the rate is larger,
the bird is gaining weight faster when $B = 120$

A1

$$\text{b. } \frac{dB}{dt} = \frac{B}{600}(200-B) \quad \text{separating the variables}$$

$$\int \frac{600}{B(200-B)} dB = t \quad \text{using partial fractions}$$

$$\frac{600}{B(200-B)} = \frac{A}{B} + \frac{D}{200-B} = \frac{A(200-B) + DB}{B(200-B)} = \frac{B(D-A) + 200A}{B(200-B)}$$

M1

$$\Rightarrow 200A = 600 \quad D - A = 0 \Rightarrow A = D = 3$$

$$3 \int \left(\frac{1}{B} + \frac{1}{200-B} \right) dB = t$$

$$\frac{t}{3} = \log_e(|B|) - \log_e(|200-B|) + c$$

A1

since $20 \leq B < 200$ the moduli are not needed

Now when $t = 0$, $B = 20$

$$0 = \log_e 20 - \log_e(200-20) + c \Rightarrow$$

$$c = \log_e(180) - \log_e(20) = \log_e\left(\frac{180}{20}\right) = \log_e(9)$$

A1

$$\frac{t}{3} = \log_e B - \log_e(200-B) + \log_e(9)$$

$$\frac{t}{3} = \log_e\left(\frac{9B}{200-B}\right)$$

$$e^{\frac{t}{3}} = \frac{9B}{200-B} \Rightarrow \frac{200-B}{9B} = e^{-\frac{t}{3}}$$

$$200 - B = 9Be^{-\frac{t}{3}}$$

$$200 = B\left(1 + 9e^{-\frac{t}{3}}\right)$$

$$B(t) = \frac{200}{1 + 9e^{-\frac{t}{3}}}$$

M1

c. $\frac{dB}{dt} = \frac{1}{600}(200B - B^2)$
 $\frac{d^2B}{dt^2} = \frac{1}{600}(200 - 2B)\frac{dB}{dt} = \frac{1}{300}(100 - B)\frac{dB}{dt} = 0$ for inflexion points $\frac{d^2B}{dt^2} = 0$
 $\Rightarrow B = 100$

when $B(t) = 100 = \frac{200}{1 + 9e^{-\frac{t}{3}}} \Rightarrow 1 + 9e^{-\frac{t}{3}} = 2$ M1

$9e^{-\frac{t}{3}} = 1 \Rightarrow e^{\frac{t}{3}} = 9$

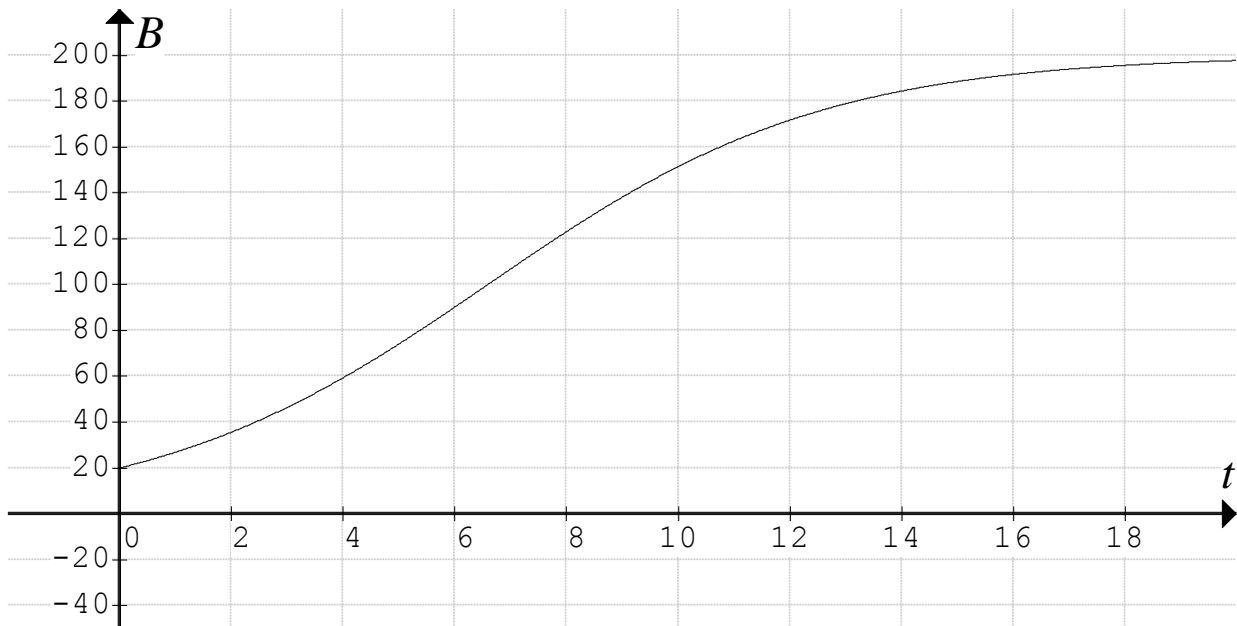
$t = 3\log_e(9)$

point of inflexion $(3\log_e(9), 100)$

the curve is concave up for $0 \leq t < 3\log_e(9)$ A1

d. Since $B(0) = 20$ and as $t \rightarrow \infty$, $B \rightarrow 200$, the horizontal asymptote is 200

G1



END OF SUGGESTED SOLUTIONS