# Year 2017

# VCE

# Specialist Mathematics

# Trial Examination 1

# Solutions

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**Question 1**

**a.** $z = \sqrt{2}\,\text{cis}\left(-\frac{2\pi}{3}\right)$

$= \sqrt{2}\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$ since $\cos(-\theta) = \cos(\theta)$ , $\sin(-\theta) = -\sin(\theta)$

$= \sqrt{2}\left(\cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)\right)$

$= \sqrt{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ M1

$= \frac{\sqrt{2}}{2}\left(-1 - \sqrt{3}\,i\right)$

by the conjugate root theorem, $\bar{z} = \frac{\sqrt{2}}{2}\left(-1 + \sqrt{3}\,i\right)$ is also a root

$z + \bar{z} = -\sqrt{2}$ , $z.\bar{z} = \frac{2}{4}\left(1 - 3i^2\right) = 2$

$z^2 - (\text{sum of the roots})z + \text{product of the roots}$

$= \left(z^2 + \sqrt{2}\,z + 2\right)$ is the quadratic factor A1

**b.** $f(z) = z^4 + \sqrt{2}\,z^3 + 5z^2 + 3\sqrt{2}\,z + 6 = 0$

$= \left(z^2 + \sqrt{2}\,z + 2\right)\left(z^2 + 3\right) = 0$

$= \left(z - \frac{\sqrt{2}}{2}\left(-1 + \sqrt{3}\,i\right)\right)\left(z - \frac{\sqrt{2}}{2}\left(-1 - \sqrt{3}\,i\right)\right)\left(z - \sqrt{3}\,i\right)\left(z + \sqrt{3}\,i\right) = 0$ M1

all the roots are $z = \frac{\sqrt{2}}{2}\left(-1 \pm \sqrt{3}\,i\right), \pm\sqrt{3}\,i$ A1

**Question 2**

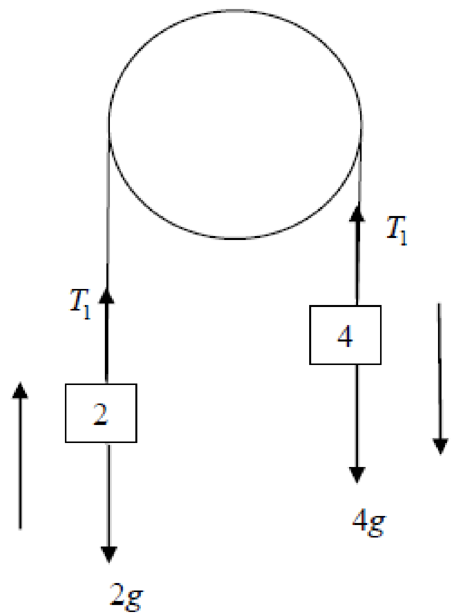
95% confidence interval is $\mu \pm 2 \times \frac{\mu}{2\sqrt{n}} = \left(\mu\left(1 - \frac{1}{\sqrt{n}}\right), \mu\left(1 + \frac{1}{\sqrt{n}}\right)\right)$

this has a width of $\frac{2\mu}{\sqrt{n}} \le \frac{\mu}{10}$ M1

$\Rightarrow \sqrt{n} \ge 20$

the minimum sample size is 400 A1

**Question 3**




when the 4kg mass moves downwards, with an acceleration of *a*, resolving

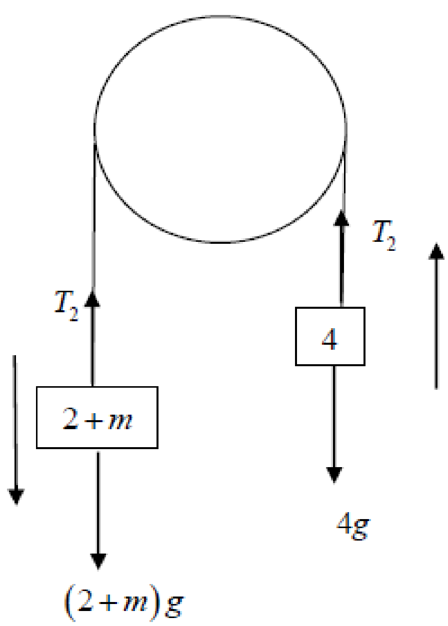
$(1) \quad 4g - T_1 = 4a$

$(2) \quad T_1 - 2g = 2a$

adding $(1)+(2)$

$2g = 6a$

$a = \frac{g}{3}$ A1




when the $2+m$ kg moves downwards with an acceleration of $\frac{g}{3}$, resolving

$(3) \quad (2+m)g - T_2 = (2+m)\frac{g}{3}$

$(4) \quad T_2 - 4g = 4 \times \frac{g}{3}$

adding $(3)+(4)$ M1

$(2+m-4)g = \frac{g}{3}(2+m+4)$

$3(m-2) = m+6$

$3m-6 = m+6$

$2m = 12$

$m = 6$ A1

**Question 4**

$\log_e(2xy)+\frac{x}{y}=8$ by log laws

$\log_e(2)+\log_e(x)+\log_e(y)+\frac{x}{y}=8$ using implicit differentiation M1

$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}+\frac{1}{y}-\frac{x}{y^2}\frac{dy}{dx}=0$

$\frac{1}{x}+\frac{1}{y}=\frac{dy}{dx}\left(\frac{x}{y^2}-\frac{1}{y}\right)$ A1

$\frac{x+y}{xy}=\frac{dy}{dx}\left(\frac{x-y}{y^2}\right)$

$\frac{dy}{dx}=\frac{y(x+y)}{x(x-y)}$ A1

at the point $\left(2,\frac{1}{4}\right)$

$$\left.\frac{dy}{dx}\right|_{\left(2,\frac{1}{4}\right)}=\frac{\frac{1}{4}\left(2+\frac{1}{4}\right)}{2\left(2-\frac{1}{4}\right)}=\frac{1}{8}\times\frac{9}{7}=\frac{9}{56}$$

gradient of the normal is $-\frac{56}{9}$ A1

**Question 5**

**a.** $s=\int_{t_1}^{t_2}\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt$

$x=t^3+5$, $y=6t^2-1$

$\dot{x}=\frac{dx}{dt}=3t^2$ $\dot{y}=\frac{dy}{dt}=12t$ M1

$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=9t^4+144t^2=9t^2\left(t^2+16\right)$

$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=3t\sqrt{t^2+16}$ since $t\geq 0$ A1

$s=\int_0^3 3t\sqrt{t^2+16}\,dt$

**b.** $s=\int_0^3 3t\sqrt{t^2+16}\,dt$ let $u=t^2+16$ $\frac{du}{dt}=2t \Rightarrow dt=\frac{1}{2t}\,du$

terminals when $t=0 \Rightarrow u=16$ when $t=3 \Rightarrow u=25$

$s=\int_{16}^{25} 3t\sqrt{u}\times\frac{1}{2t}\,du$ M1

$s=\frac{3}{2}\int_{16}^{25} u^{\frac{1}{2}}\,du$

$=\frac{3}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{16}^{25}$

$=25^{\frac{3}{2}}-16^{\frac{3}{2}}$

$=\left(\sqrt{25}\right)^3-\left(\sqrt{16}\right)^3$

$=5^3-4^3=125-64$

$=61$ A1

**Question 6**

$\frac{dy}{dx}=\frac{\sqrt{4-y^2}}{4+x^2}$ and $y(0)=1$ separating the variables

$\int\frac{1}{\sqrt{4-y^2}}\,dy=\int\frac{1}{4+x^2}\,dx$ M1

$\sin^{-1}\left(\frac{y}{2}\right)=\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+c$ A1

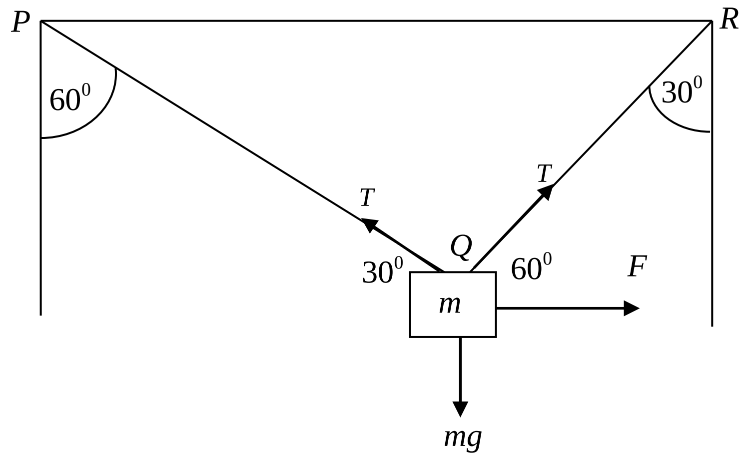
when $y=1$ , $x=0$ $\sin^{-1}\left(\frac{1}{2}\right)=c=\frac{\pi}{6}$

$\sin^{-1}\left(\frac{y}{2}\right)=\frac{\pi}{6}+\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)$

$y=2\sin\left(\frac{\pi}{6}+\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right)$ A1

**Question 7**

**a.**




A1

**b.**

Let $T$ be the tension in the strings, all the forces are in newtons.

resolving horizontally (1) $F + T\cos(60^0) - T\cos(30^0) = 0$

resolving vertically (2) $T\sin(60^0) + T\sin(30^0) - mg = 0$ A1

(1) $F = T\left(\cos(30^0) - \cos(60^0)\right) \Rightarrow F = T\left(\dfrac{\sqrt{3}-1}{2}\right) \Rightarrow T = \dfrac{2F}{\sqrt{3}-1}$

(2) $mg = T\left(\sin(60^0) + \sin(30^0)\right) \Rightarrow mg = T\left(\dfrac{\sqrt{3}+1}{2}\right) \Rightarrow T = \dfrac{2mg}{\sqrt{3}+1}$ M1

eliminating $T$, gives $T = \dfrac{2F}{\sqrt{3}-1} = \dfrac{2mg}{\sqrt{3}+1} \Rightarrow F = \left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)mg$

$F = \left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)mg \times \left(\dfrac{\sqrt{3}-1}{\sqrt{3}-1}\right) = \left(\dfrac{3-2\sqrt{3}+1}{3-1}\right)mg$

$F = \left(2-\sqrt{3}\right)mg \quad a = 2 \ , \ b = 3$ A1

**Question 8**

**a.i.** Let $y=\sec(kx)=\dfrac{1}{\cos(kx)}=(\cos(kx))^{-1}$ using the chain rule

$$\frac{dy}{dx}=-1\times-k\sin(kx)\times(\cos(kx))^{-2}=\frac{k\sin(kx)}{\cos^2(kx)}=\frac{k\sin(kx)}{\cos(kx)}\times\frac{1}{\cos(kx)}$$ A1

$$\frac{d}{dx}\left[\sec(kx)\right]=k\tan(kx)\sec(kx)$$

**ii** Let $y=\log_e(\tan(kx)+\sec(kx))=\log_e(u)$ where $u=\tan(kx)+\sec(kx)$

$\dfrac{dy}{du}=\dfrac{1}{u}$ using the chain rule

$$\frac{du}{dx}=k\sec^2(kx)+k\sec(kx)\tan(kx)=k\sec(kx)(\tan(kx)+\sec(kx))$$ A1

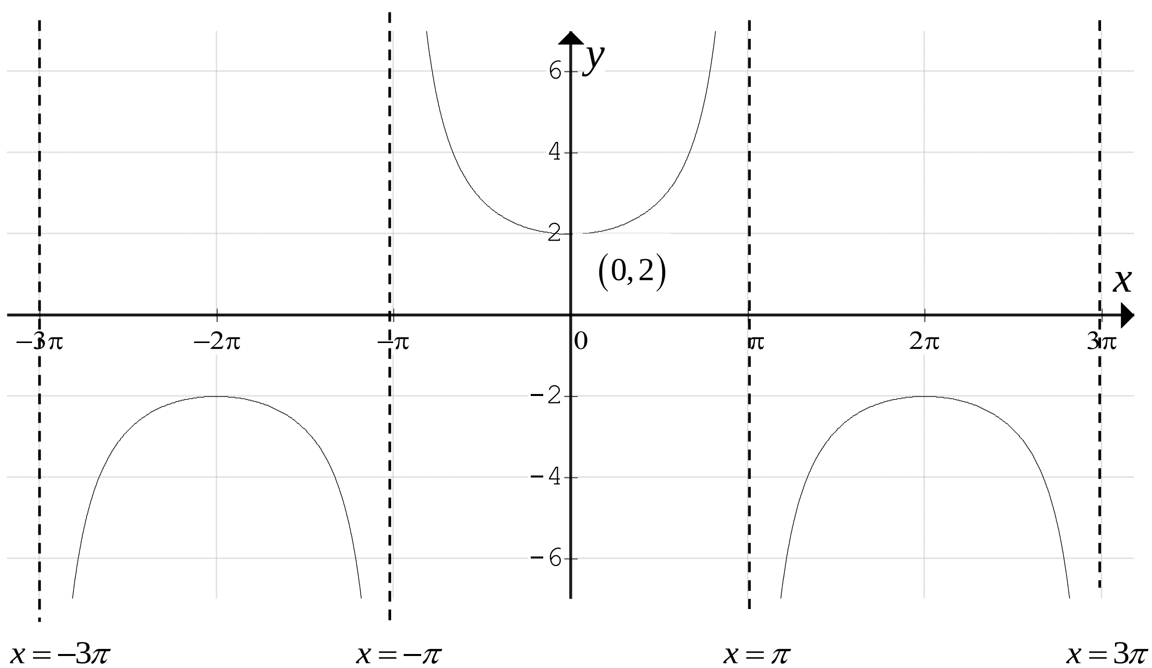
$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{\tan(kx)+\sec(kx)}\times k\sec(kx)(\sec(kx)+\tan(kx))=k\sec(kx)$$

so that $\dfrac{d}{dx}\left[\log_e(\tan(kx)+\sec(kx))\right]=k\sec(kx)$

**b.** $y=2\sec\left(\dfrac{x}{2}\right)=\dfrac{2}{\cos\left(\dfrac{x}{2}\right)}$ when $x=0$ , $y=2$ , $y$ intercept $(0,2)$

The graph has vertical asymptotes when $\cos\left(\dfrac{x}{2}\right)=0 \Rightarrow \dfrac{x}{2}=\pm\dfrac{\pi}{2},\pm\dfrac{3\pi}{2}$

so that $x=\pm\pi,\pm3\pi$ are the vertical asymptotes. A1



**c.** $A = \int_0^{\frac{\pi}{2}} 2\sec\left(\frac{x}{2}\right)dx$ using **a.** with $k = \frac{1}{2}$

$A = \left[ 4\log_e\left( \tan\left(\frac{x}{2}\right) + \sec\left(\frac{x}{2}\right)\right)\right]_0^{\frac{\pi}{2}}$ A1

$= 4\log_e\left( \tan\left(\frac{\pi}{4}\right) + \sec\left(\frac{\pi}{4}\right)\right) - 4\log_e\left(\tan(0) + \sec(0)\right)$

$= 4\log_e\left(1+\sqrt{2}\right)$ units$^2$ A1

**Question 9**

Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ , $\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$ , $\underset{\sim}{c} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

Since $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{c}$ $\underset{\sim}{a}.\underset{\sim}{c} = 0 \Rightarrow (1)\ 2x - 3y + z = 0$

Since $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{c}$ $\underset{\sim}{b}.\underset{\sim}{c} = 0 \Rightarrow (2)\ x + 2y + 4z = 0$ A1

Since $\underset{\sim}{c}$ is a unit vector $|\underset{\sim}{c}| = 1 \Rightarrow x^2 + y^2 + z^2 = 1$

$(1)\ 2x - 3y + z = 0$

$2\times(2)\ 2x + 4y + 8z = 0$ subtracting gives $7y + 7z = 0 \Rightarrow z = -y$

$4\times(1)\ 8x - 12y + 4z = 0$ M1

$(2)\ x + 2y + 4z = 0$ subtracting gives $7x - 14y = 0 \Rightarrow x = 2y$

substituting into $(3)$ $4y^2 + y^2 + y^2 = 6y^2 = 1 \Rightarrow y = \pm\frac{1}{\sqrt{6}}$

$\hat{\underset{\sim}{c}} = \frac{\pm 1}{\sqrt{6}}\left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$ A1

**Question 10**

**a.** $\left.\frac{dB}{dt}\right|_{B=60} = \frac{60}{600}(200-60) = 14$ gm/month

$\left.\frac{dB}{dt}\right|_{B=120} = \frac{120}{600}(200-120) = 16$ gm/month

Since when $B = 120$ the rate is larger,
the bird is gaining weight faster when $B = 120$ A1

**b.** $\frac{dB}{dt} = \frac{B}{600}(200-B)$ separating the variables

$\int \frac{600}{B(200-B)}\,dB = t$ using partial fractions

$\frac{600}{B(200-B)} = \frac{A}{B} + \frac{D}{200-B} = \frac{A(200-B)+DB}{B(200-B)} = \frac{B(D-A)+200A}{B(200-B)}$ M1

$\Rightarrow 200A = 600 \quad D - A = 0 \quad \Rightarrow A = D = 3$

$3\int\left(\frac{1}{B} + \frac{1}{200-B}\right)dB = t$

$\frac{t}{3} = \log_e(|B|) - \log_e(|200-B|) + c$ A1

since $20 \le B < 200$ the moduli are not needed
Now when $t = 0$ , $B = 20$

$0 = \log_e 20 - \log_e(200-20) + c \quad \Rightarrow$

$c = \log_e(180) - \log_e(20) = \log_e\left(\frac{180}{20}\right) = \log_e(9)$ A1

$\frac{t}{3} = \log_e B - \log_e(200-B) + \log_e(9)$

$\frac{t}{3} = \log_e\left(\frac{9B}{200-B}\right)$

$e^{\frac{t}{3}} = \frac{9B}{200-B} \quad \Rightarrow \quad \frac{200-B}{9B} = e^{-\frac{t}{3}}$

$200 - B = 9Be^{-\frac{t}{3}}$

$200 = B\left(1 + 9e^{-\frac{t}{3}}\right)$

$B(t) = \frac{200}{1 + 9e^{-\frac{t}{3}}}$ M1

**c.** $\dfrac{dB}{dt}=\dfrac{1}{600}\left(200B-B^2\right)$

$\dfrac{d^2B}{dt^2}=\dfrac{1}{600}(200-2B)\dfrac{dB}{dt}=\dfrac{1}{300}(100-B)\dfrac{dB}{dt}=0$ for inflexion points $\dfrac{d^2B}{dt^2}=0$

$\Rightarrow B=100$

when $B(t)=100=\dfrac{200}{1+9e^{-\frac{t}{3}}} \quad \Rightarrow 1+9e^{-\frac{t}{3}}=2$ M1

$9e^{-\frac{t}{3}}=1 \quad \Rightarrow e^{\frac{t}{3}}=9$

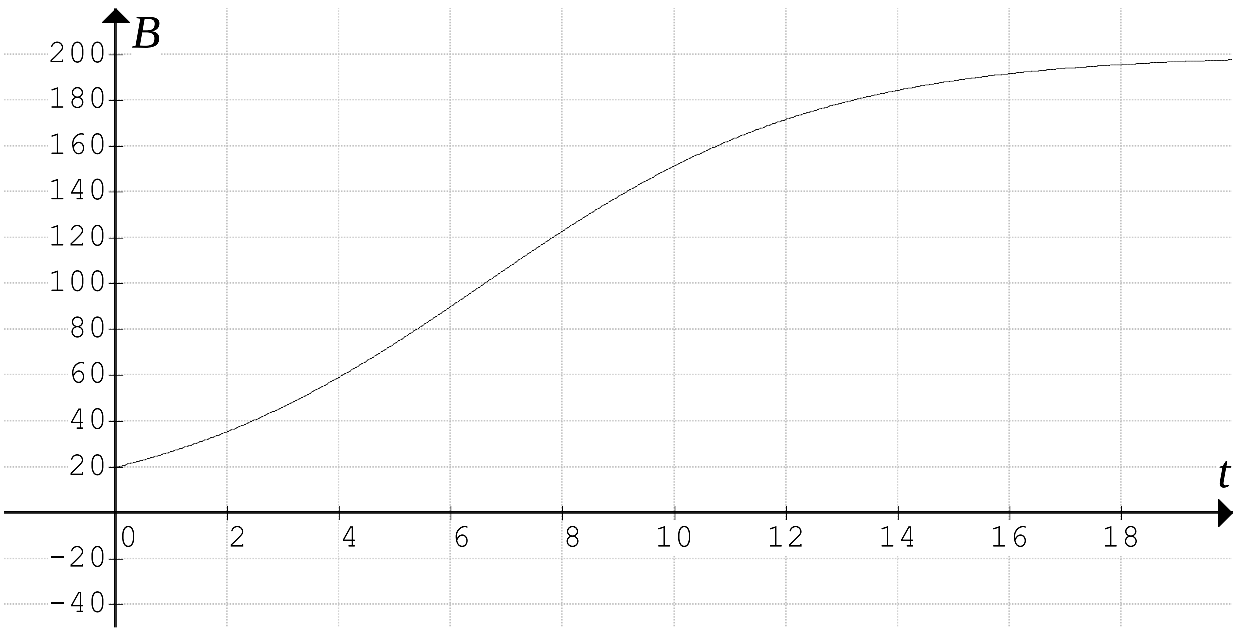
$t=3\log_e(9)$

point of inflexion $\left(3\log_e(9),100\right)$

the curve is concave up for $0\le t<3\log_e(9)$ A1

**d.** Since $B(0)=20$ and as $t\to\infty$ , $B\to 200$, the horizontal asymptote is 200

G1




**END OF SUGGESTED SOLUTIONS**