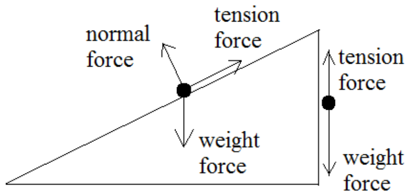
2018 VCAA Specialist Mathematics Exam 1 Solutions

Q1a



Q1b $5g-8g\sin 30^\circ=(5+8)a$, $a=\dfrac{g}{13}$ ms^{-2} upward along the slope.

Q2a $1+i=\sqrt{1^2+1^2}\,\text{cis}(\tan^{-1}1)=\sqrt{2}\,\text{cis}\left(\dfrac{\pi}{4}\right)$

Q2b $\dfrac{\left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^{10}}{\left(\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)\right)^{12}}=\dfrac{2^{10}\text{cis}\left(-\frac{5\pi}{3}\right)}{2^6\text{cis}(3\pi)}=16\text{cis}\left(-\dfrac{2\pi}{3}\right)$

$=16\cos\left(-\dfrac{2\pi}{3}\right)+16i\sin\left(-\dfrac{2\pi}{3}\right)=-8-8\sqrt{3}\,i$

Q3 Implicit differentiation:

$4x\sin(y)+2x^2\cos(y)\dfrac{dy}{dx}+y+x\dfrac{dy}{dx}=0$

$2x^2\cos(y)\dfrac{dy}{dx}+x\dfrac{dy}{dx}=-4x\sin(y)-y$, $\dfrac{dy}{dx}=\dfrac{-4x\sin(y)-y}{2x^2\cos(y)+x}$

At $\left(\dfrac{\pi}{6},\dfrac{\pi}{6}\right)$, gradient of the curve

$=\dfrac{dy}{dx}=\dfrac{-4\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right)-\frac{\pi}{6}}{2\left(\frac{\pi}{6}\right)^2\cos\left(\frac{\pi}{6}\right)+\frac{\pi}{6}}=\dfrac{-18}{\pi\sqrt{3}+6}$

Q4 X: $\text{E}(X)=2$, $\text{Var}(X)=2$; Y: $\text{E}(Y)=2$, $\text{Var}(Y)=4$

$\text{E}(aX+bY)=a\text{E}(X)+b\text{E}(Y)=2a+2b=10$

$\text{Var}(aX+bY)=a^2\text{Var}(X)+b^2\text{Var}(Y)=2a^2+4b^2=44$

$\therefore$ $a+b=5$ and $a^2+2b^2=22$

$\therefore$ $a^2+2(5-a)^2=22$, $3a^2-20a+28=0$, $(3a-14)(a-2)=0$

$\therefore$ $a=2$ and $b=3$ (a and b are integers)

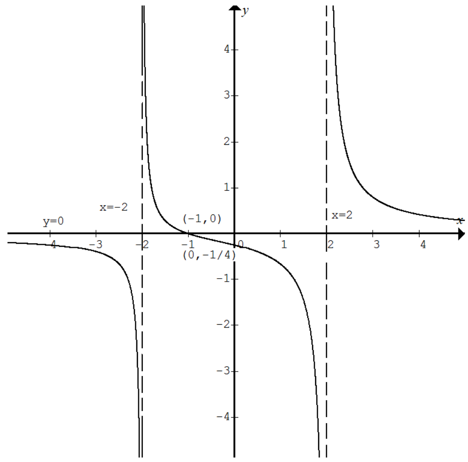
Q5 $f(x)=\dfrac{x+1}{(x-2)(x+2)}$

Asymptotes: $x=\pm 2$, $y=0$, x-intercept: $x=-1$, y-intercept: $y=-\dfrac{1}{4}$

As $x\to\infty$, $y\to 0^+$; as $x\to-\infty$, $y\to 0^-$

As $x\to-2$ from the left, $y\to-\infty$; as $x\to-2$ from the right, $y\to\infty$

As $x\to 2$ from the left, $y\to-\infty$; as $x\to 2$ from the right, $y\to\infty$



Q6 $\tilde{v}(t)=\dfrac{d}{dt}\tilde{r}=\cos(t)\,\tilde{i}-\sin(t)\,\tilde{j}+2t\,\tilde{k}$

$\tilde{v}\left(\dfrac{\pi}{2}\right)=\cos\left(\dfrac{\pi}{2}\right)\tilde{i}-\sin\left(\dfrac{\pi}{2}\right)\tilde{j}+2\left(\dfrac{\pi}{2}\right)\tilde{k}=-\tilde{j}+\pi\,\tilde{k}$

$\tilde{v}(\pi)=\cos(\pi)\,\tilde{i}-\sin(\pi)\,\tilde{j}+2(\pi)\tilde{k}=-\tilde{i}+2\pi\,\tilde{k}$

$\Delta\tilde{p}=2\left(\tilde{v}(\pi)-\tilde{v}\left(\dfrac{\pi}{2}\right)\right)=-2\tilde{i}+2\tilde{j}+2\pi\,\tilde{k}$ kg ms^{-1}

Q7 $\dfrac{1-\tan^2(x)}{2\tan(x)}+\dfrac{\tan(x)}{2}=\dfrac{a}{\tan(x)}$, $\tan(x)\neq 0$

$\dfrac{1-\tan^2(x)+\tan^2(x)}{2\tan(x)}=\dfrac{2a}{2\tan(x)}$, $\therefore$ $2a=1$, $a=\dfrac{1}{2}$

Q8a The volume increases at a rate of 2L per minute.

At time t, $V=16+2t$, concentration $=\dfrac{Q}{16+2t}$ kg per L,

rate of flow of solution $=-3$ L per minute

$\therefore$ $\dfrac{dQ}{dt}=-\dfrac{3Q}{16+2t}$

Q8b $\displaystyle\int_{0.5}^{Q}\frac{1}{Q}\,dQ=\int_0^t\frac{-3}{16+2t}\,dt$,

$[\log_e Q]_{0.5}^{Q}=\left[-\dfrac{3}{2}\log_e(16+2t)\right]_0^t$

$\log_e(2Q)=-\dfrac{3}{2}\log_e\dfrac{16+2t}{16}$, $\log_e(2Q)=\log_e\left(\dfrac{16+2t}{16}\right)^{-\frac{3}{2}}$

$\log_e(2Q)=\log_e\left(\dfrac{16}{16+2t}\right)^{\frac{3}{2}}$, $2Q=\left(\dfrac{16}{16+2t}\right)^{\frac{3}{2}}$, $Q=\dfrac{32}{(16+2t)^{\frac{3}{2}}}$

Q9a $x=\sec(t)$, $y=\frac{1}{\sqrt{2}}\tan(t)$

$1+\tan^2(t)=\sec^2(t)$, $1+\left(\sqrt{2}\,y\right)^2=x^2$, $x^2-2y^2=1$

Q9b $x^2-2(x-1)^2=1$, $x^2-2\left(x^2-2x+1\right)=1$, $x^2-4x+3=0$
$(x-3)(x-1)=0$, .: $x=1,\ 3$

Q9c Volume $=\int_1^3 \pi y^2\,dx=\int_1^3 \pi\left(\frac{x^2-1}{2}-(x-1)^2\right)dx$

$=\int_1^3 \pi\left(\frac{x^2-1}{2}-(x-1)^2\right)dx=\int_1^3 \pi\left(\frac{-x^2+4x-3}{2}\right)dx$

$=\left[\frac{\pi}{2}\left(\frac{-x^3}{3}+2x^2-3x\right)\right]_1^3=\frac{\pi}{2}\left(-9+18-9+\frac{1}{3}-2+3\right)=\frac{2\pi}{3}$

Q10 $x(t)=\frac{t^3}{3}$, $x'(t)=t^2$

$y(t)=\sin^{-1}(t)+t\sqrt{1-t^2}$,

$y'(t)=\frac{1}{\sqrt{1-t^2}}+\sqrt{1-t^2}-\frac{t^2}{\sqrt{1-t^2}}=2\sqrt{1-t^2}$

$d=\int_0^{\frac{3}{4}}\sqrt{t^4+4\left(1-t^2\right)}\,dt=\int_0^{\frac{3}{4}}\sqrt{4-4t^2+t^4}\,dt$

$=\int_0^{\frac{3}{4}}\sqrt{\left(2-t^2\right)^2}\,dt=\int_0^{\frac{3}{4}}\left(2-t^2\right)dt$

Note: $0<t<1$, .: $2-t^2>0$ and $t^2-2<0$
.: $a=-1$, $b=0$ and $c=2$

Please inform mathline@itute.com re conceptual and/or mathematical errors.