VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2018

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

SOLNS

+MJ

STUDENT NUMBER | | | | | | | | | Letter | |

SPECIALIST MATHEMATICS

Written examination 1

Tuesday 5 June 2018

Reading time: 2.00 pm to 2.15 pm (15 minutes)
Writing time: 2.15 pm to 3.15 pm (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 11 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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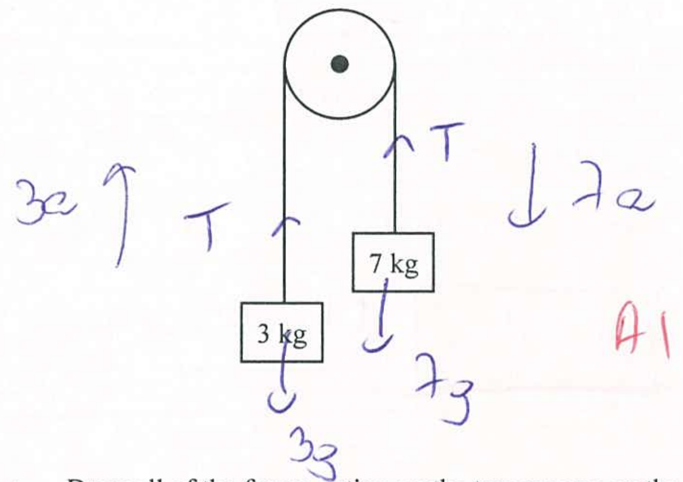
Instructions

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (3 marks)

A light inextensible string hangs over a frictionless pulley connecting masses of 3 kg and 7 kg, as shown below.



A1

a. Draw all of the forces acting on the two masses on the diagram above. 1 mark

b. Calculate the tension in the string. 2 marks

$7a = 7g - T \quad (1)$

$3a = T - 3g \quad (2)$ M1

$10a = 4g$

$a = \frac{4g}{10}$

$a = \frac{2g}{5}$

$T = 3 \cdot \frac{2g}{5} + 3g$

$= \frac{6g}{5} + 3g$

$\boxed{T = \frac{21g}{5}}$ A1

TURN OVER

Question 2 (3 marks)

Let $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + m\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$, where $m \in R$.

Find the value(s) of m such that the magnitude of the vector resolute of $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$ is equal to $\sqrt{14}$.

$|\underset{\sim}{b}| = \sqrt{4+1+9} = \sqrt{14}$. $\quad \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \sqrt{14}$

$\hat{\underset{\sim}{b}} = \frac{1}{14}(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$ M1

$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{1}{\sqrt{14}}(3 \cdot 2 - 2(-1) + 3m)$

M1

$8 + 3m = 14$

$3m = 6$

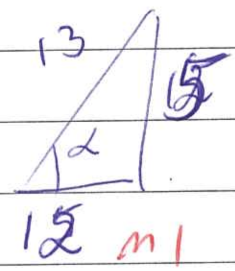
$\boxed{m = 2}$ A1

Question 3 (3 marks) $\quad \alpha + \beta$

Find $\sin(t)$ given that $t = \arccos\left(\frac{12}{13}\right) + \arctan\left(\frac{3}{4}\right)$.

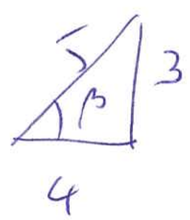
$\sin t = \sin(\alpha + \beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$



$= \frac{5}{13} \cdot \frac{4}{5} + \frac{12}{13} \times \frac{3}{5}$

M1



$= \frac{20 + 36}{65}$

$= \boxed{\frac{56}{65}}$ A1

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Question 4 (4 marks)

Throughout this question, use an integer multiple of standard deviations in calculations.

The standard deviation of all scores on a particular test is 21.0

a. From the results of a random sample of n students, a 95% confidence interval for the mean score for all students was calculated to be (44.7, 51.7).

Calculate the mean score and the size of this random sample. 2 marks

$\frac{51.7 - 44.7}{2} = \frac{7}{2} = 3.5$

mean $= 44.7 + 3.5$

mean $= 48.2$ A1

$\frac{42}{3.5} = \sqrt{n}$

$12 = \sqrt{n}$

$n = 144$ A1

b. Determine the size of another random sample for which the endpoints of the 95% confidence interval for the population mean of the particular test would be 1.0 either side of the sample mean. 2 marks

$\frac{2 \times 21}{\sqrt{n}} = 1$ M1

$\sqrt{n} = 42$

$\boxed{n = 1764}$ A1

Question 5 (4 marks)

Evaluate $\displaystyle\int_1^{2\sqrt{3}-1} \left(\frac{1}{x^2+2x+5}\right) dx.$

$(x+1)^2 - 1 + 5 = (x+1)^2 + 4$

$u = x+1$

$x = 1, \quad u = 2$

$x = 2\sqrt{3}-1, \quad u = 2\sqrt{3}$

$= \displaystyle\int_2^{2\sqrt{3}} \frac{1}{u^2+4}\, du$ M1 $= \frac{1}{2}\left[\tan^{-1}\left(\frac{u}{2}\right)\right]_2^{2\sqrt{3}}$ A1

$= \frac{1}{2}\left(\tan^{-1}\frac{2\sqrt{3}}{2} - \tan^{-1}(1)\right)$

$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{1}{2} \times \frac{\pi}{12} = \boxed{\frac{\pi}{24}}$ A1

M1

Question 6 (4 marks)

Given that $y = (x-1)e^{2x}$ is a solution to the differential equation $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} = y$, find the values of a and b, where a and b are real constants.

$y = x - 1\, e^{2x}$; $\dfrac{dy}{dx} = e^{2x} + 2e^{2x}(x-1)$

$= e^{2x}(1 + 2x - 2) = e^{2x}(2x-1)$

A1

$\dfrac{d^2y}{dx^2} = 2e^{2x}(2x-1) + 2e^{2x} = 2e^{2x}(2x \cancel{-1} \cancel{+1})$

$= 4x\, e^{2x}$ A1

$a \cdot 4x \cancel{e^{2x}} + b \cancel{e^{2x}}(2x-1) = (x-1)\cancel{e^{2x}}$

$4xa + 2bx - b = x - 1$ $\boxed{a = \frac{1}{4}}$

$\boxed{b = 1}$ A1 $4a + 2b = 1$ A1

$4a + 2 = 1$

$a = -\frac{1}{4}$

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Question 7 (4 marks)

a. Find $\frac{d}{dx}\left((1-x^2)^{\frac{1}{2}}\right)$. 2 marks

$= \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$ M1

$= -\frac{x}{\sqrt{1-x^2}}$ A1

b. Hence, find the length of the curve specified by $y=\sqrt{1-x^2}$ from $x=\frac{1}{2}$ to $x=\frac{\sqrt{3}}{2}$. Give your answer in the form $k\pi$, $k \in R$. 2 marks

$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{1+\frac{x^2}{1-x^2}}\,dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx$

$= \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{\frac{1}{1-x^2}}\,dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ A1

$= \left[\sin^{-1}x\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3} - \frac{\pi}{6} = \boxed{\frac{\pi}{3}}$ A1

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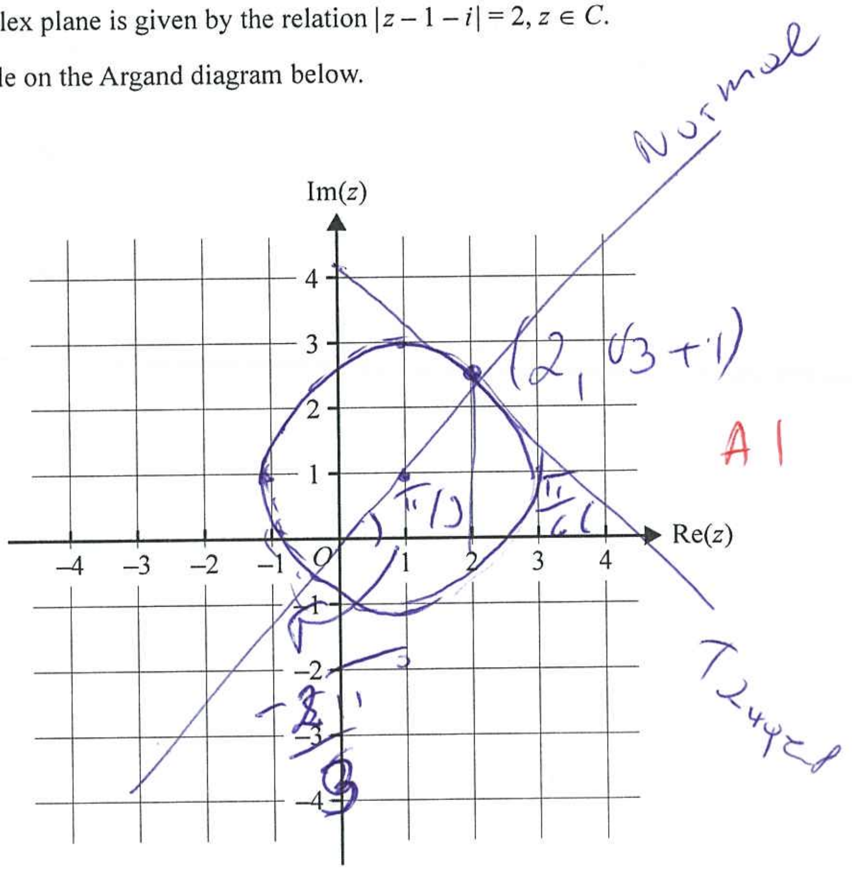
Question 8 (6 marks)

A circle in the complex plane is given by the relation $|z - 1 - i| = 2, z \in C$.

a. Sketch the circle on the Argand diagram below. 1 mark



b. **i.** Write the equation of the circle in the form $(x - a)^2 + (y - b)^2 = c$ and show that the gradient of a tangent to the circle can be expressed as $\frac{dy}{dx} = \frac{1-x}{y-1}$. 2 marks

$(x-1)^2 + (y-1)^2 = 4$ A1

$2(x-1) + 2(y-1)\frac{dy}{dx} = 0$ M1

$\frac{dy}{dx} = -\frac{x-1}{y-1} = \frac{1-x}{y-1}$

ii. Find the gradient of the tangent to the circle where $x = 2$ in the first quadrant of the complex plane. 1 mark

$x = 2,$

$1 + (y-1)^2 = 4$

$(y-1)^2 = 3$

$y - 1 = \sqrt{3}$ (1st Q)

$y = \sqrt{3} + 1$

$\frac{dy}{dx} = \frac{1-2}{\sqrt{3}+1-1} = \boxed{-\frac{1}{\sqrt{3}}}$ A1

c. Find the equations of all rays that are perpendicular to the circle in the form $\text{Arg}(z) = \alpha$. 2 marks

$\alpha = \frac{\pi}{3}, \ \frac{-2\pi}{3}$

A1 A1

Question 9 (9 marks)

a. **i.** Given that $\cot(2\theta) = a$, show that $\tan^2(\theta) + 2a\tan(\theta) - 1 = 0$. 2 marks

$$\frac{1-\tan^2\theta}{2\tan\theta} = a \quad M1$$

$$1 - \tan^2\theta = 2a\tan\theta$$

$$1 - \tan^2\theta - 2a\tan\theta = 0 \quad A1$$

$$\tan^2\theta + 2a\tan\theta - 1 = 0$$

ii. Show that $\tan(\theta) = -a \pm \sqrt{a^2+1}$. 1 mark

A1

$$\tan\theta = \frac{-2a \pm \sqrt{4a^2+4}}{2} = \frac{-2a \pm 2\sqrt{a^2+1}}{2}$$

$$= -a \pm \sqrt{a^2+1}$$

iii. Hence, show that $\tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$, given that $\cot(2\theta) = \sqrt{3}$, where $\theta \in (0, \pi)$. 1 mark

$$\tan 2\theta = \frac{1}{\sqrt{3}}$$

$$2\theta = \frac{\pi}{6}, \quad \theta = \frac{\pi}{12} \quad A1$$

$$\tan\left(\frac{\pi}{12}\right) = -\sqrt{3} + \sqrt{3+1} = -\sqrt{3} + 2 = 2 - \sqrt{3}$$

as $\frac{\pi}{12}$ is in the 1st quad

b. Find the gradient of the tangent to the curve $y = \tan(\theta)$ at $\theta = \frac{\pi}{12}$. 2 marks

$$\frac{dy}{d\theta} = \sec^2\theta \quad A1$$

$$\theta = \frac{\pi}{12}, \quad \frac{dy}{d\theta} = \sec^2\left(\frac{\pi}{12}\right)$$

$$= 1 + \tan^2\left(\frac{\pi}{12}\right)$$

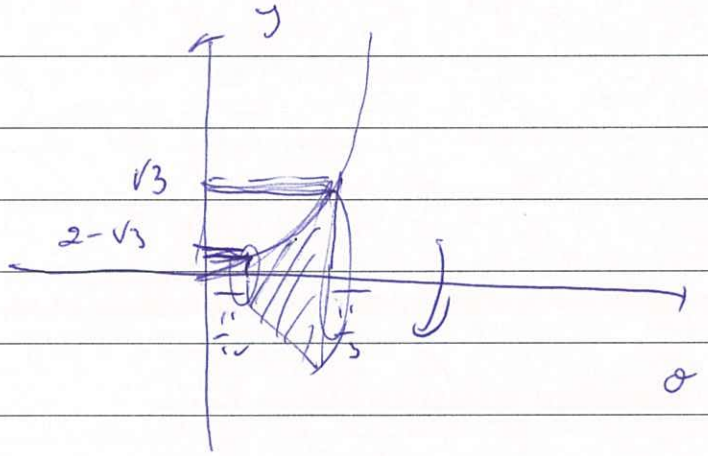
$$= 1 + 4 - 4\sqrt{3} + 3$$

$$= \boxed{8 - 4\sqrt{3}} \quad A1$$

Question 9 – continued

c. A solid of revolution is formed by rotating the region between the graph of $y = \tan(\theta)$, the horizontal axis, and the lines $\theta = \frac{\pi}{12}$ and $\theta = \frac{\pi}{3}$ about the horizontal axis.

Find the volume of the solid of revolution. 3 marks



$$V = \pi \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} \tan^2\theta \, d\theta$$ A1

$$= \pi \int_{\frac{\pi}{12}}^{\frac{\pi}{3}} (\sec^2\theta - 1) \, d\theta$$

$$= \pi \left[\tan\theta - \theta \right]_{\frac{\pi}{12}}^{\frac{\pi}{3}}$$ A1

$$= \pi \left[\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{12} - \frac{\pi}{12}\right) \right]$$

$$= \pi \left[\left(\sqrt{3} - \frac{\pi}{3}\right) - 2 + \sqrt{3} + \frac{\pi}{12} \right]$$ A1

END OF QUESTION AND ANSWER BOOK

$$= \pi \left(\sqrt{3} - \frac{\pi}{3} - 2 + \sqrt{3} + \frac{\pi}{12} \right) = \boxed{\pi \left(2\sqrt{3} - 2 - \frac{\pi}{4} \right)}$$

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FORMULA SHEET

Instructions

This formula sheet is provided for your reference. A question and answer book is provided with this formula sheet.

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