Instructions

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (4 marks)

Solve the differential equation $\frac{dy}{dx} = \frac{2ye^{2x}}{1+e^{2x}}$ given that $y(0) = \pi$.

$$\int \frac{1}{y}\,dy = \int \frac{2e^{2x}}{1+e^{2x}} \quad \text{M1}$$

Note that

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$$

$$\int_{\pi}^{y} \frac{1}{v}\,dv = \int_0^x \frac{2e^{2u}}{1+e^{2u}}\,du$$

$$\Big[\log_e|v|\Big]_{\pi}^{y} \;\text{A1} = \Big[\log_e|1+e^{2u}|\Big]_0^x \;\text{A1}$$

$$\log_e\left(\frac{y}{\pi}\right) = \log_e\left(\frac{1+e^{2x}}{2}\right) \qquad \therefore\ \underline{y = \frac{\pi}{2}(1+e^{2x})} \;\text{A1}$$

Question 2 (3 marks)

Find all values of x for which $|x-4| = \frac{x}{2} + 7$.

Case 1: $x - 4 = \frac{x}{2} + 7$, $\frac{x}{2} = 11$, $\underline{x = 22}$ A1

M1

Case 2: $4 - x = \frac{x}{2} + 7$, $\frac{3x}{2} = -3$, $\underline{x = -2}$ A1

Question 3 (3 marks)

A machine produces chocolate in the form of a continuous cylinder of radius 0.5 cm. Smaller cylindrical pieces are cut parallel to its end, as shown in the diagram below.

The lengths of the pieces vary with a mean of 3 cm and a standard deviation of 0.1 cm.

a. Find the expected volume of a piece of chocolate in cm^3. 1 mark

$$V = \pi r^2 \cdot h$$

$$V = \pi \times 0.5^2 \times 3$$

$$= \frac{3\pi}{4} \ (cm^3) \quad A1$$

b. Find the variance of the volume of a piece of chocolate in cm^6. 1 mark

$$Var(V) = Var(\pi \cdot r^2 \cdot h)$$

$$= \pi^2 r^4 \, Var(h)$$

$$= \pi^2 \times \left(\frac{1}{2}\right)^4 \times \left(\frac{1}{10}\right)^2$$

$$= \frac{\pi^2}{1600} \ (cm^6) \quad A1$$

c. Find the expected surface area of a piece of chocolate in cm^2. 1 mark

$$TSA = 2 \times \pi r^2 + 2\pi r \cdot h$$

$$= 2 \times \pi \times \left(\frac{1}{2}\right)^2 + 2\pi\left(\frac{1}{2}\right) \times 3$$

$$= \frac{7\pi}{2} \ (cm^2) \quad A1$$

Question 4 (3 marks)

The position vectors of two particles A and B at time t seconds after they have started moving are given by $\underset{\sim}{r}_A(t)=(t^2-1)\underset{\sim}{i}+\left(a+\frac{t}{3}\right)\underset{\sim}{j}$ and $\underset{\sim}{r}_B(t)=(t^3-t)\underset{\sim}{i}+\left(\arccos\left(\frac{t}{2}\right)\right)\underset{\sim}{j}$ respectively, where a is a real constant and $0 \le t \le 2$.

Find the value of a if the particles collide after they have started moving.

$x: \ t^2-1 = t^3-t, \quad (t^3-t)-(t^2-1)=0$

$(t^2-1)(t)-(t^2-1)=0$

$(t^2-1)(t-1)=0, \ (t-1)(t+1)(t-1)=0$

$2 \ge t \ge 0, \ \therefore t=1$ A1

$y: \ a+\frac{t}{3} = \cos^{-1}\left(\frac{t}{2}\right), \ t=1$

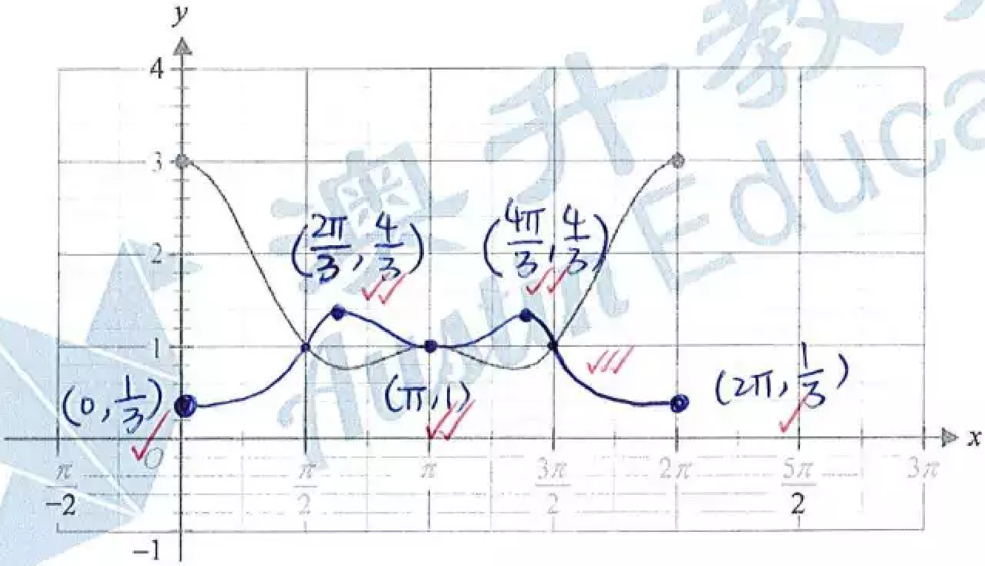
$a+\frac{1}{3} = \cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$ M1 correct substitution & eqn

therefore $a=\frac{\pi}{3}-\frac{1}{3}=\frac{(\pi-1)}{3}$

equivalent A1

Question 5 (6 marks)

The graph of $f(x) = \cos^2(x) + \cos(x) + 1$ over the domain $0 \le x \le 2\pi$ is shown below.



a. **i.** Find $f'(x)$. 1 mark

$$f'(x) = -2\cos(x)\cdot\sin(x) - \sin(x) \quad \text{A1}$$

ii. Hence, find the coordinates of the turning points of the graph in the interval $(0, 2\pi)$. 2 marks

$$f'(x) = 0, \quad -2\sin(x)\cos(x) - \sin(x) = 0$$

$$\sin(x)(-2\cos(x) - 1) = 0$$

$$\sin(x) = 0 \Rightarrow x = 0 \text{ (reject)}, \ x = \pi, \ x = 2\pi \text{ (reject)}$$

$$\cos(x) = -\tfrac{1}{2}, \quad x = \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ only}$$

T.p: $(\pi, 1)$ A1

$(\frac{2\pi}{3}, \frac{3}{4})$, $(\frac{4\pi}{3}, \frac{3}{4})$ A1

b. Sketch the graph of $y = \dfrac{1}{f(x)}$ on the set of axes above. Clearly label the turning points and endpoints of this graph with their coordinates. 3 marks

A1 endpoints × 2

A1 T.p × 3

A1 accuracy

Question 6 (3 marks)

Find the value of d for which the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j} - 8\underset{\sim}{k}$ and $\underset{\sim}{c} = -6\underset{\sim}{i} + 2\underset{\sim}{j} + d\underset{\sim}{k}$ are **linearly dependent**.

Method 1:

Let $A = \begin{bmatrix} 2 & -3 & 4 \\ -2 & 4 & -8 \\ -6 & 2 & d \end{bmatrix}$ M1

$\det(A) = 2\begin{vmatrix} 4 & -8 \\ 2 & d \end{vmatrix} + (-3)\begin{vmatrix} -8 & -2 \\ -6 & d \end{vmatrix} + 4\begin{vmatrix} -2 & 4 \\ -6 & 2 \end{vmatrix}$

$= 2(d-16)$ A1

For linear dependence $\det(A) = 0$

$\therefore d - 16 = 0,\ d = 16$ A1

Method 2:

Let $m\underset{\sim}{a} + n\underset{\sim}{b} = \underset{\sim}{c}$

$\begin{cases} 2m - 2n = -6 \\ -3m + 4n = 2 \\ 4m - 8n = d \end{cases}$ M1 $\Rightarrow$ $\begin{matrix} 4m - 4n = -12 \\ -3m + 4n = 2 \end{matrix}$ $\Rightarrow$ $\begin{matrix} m = -10 \\ n = -7 \end{matrix}$ A1

$\therefore d = -40 + 56 = 16$ A1

Question 7 (5 marks)

a. Show that $3-\sqrt{3}i = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$. 1 mark

$r = \sqrt{3^2+(-\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$ M1

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{3}\right) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ $\therefore 3-\sqrt{3}i = 2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)$

b. Find $\left(3-\sqrt{3}i\right)^3$, expressing your answer in the form $x+iy$, where $x, y \in R$. 2 marks

$\left(2\sqrt{3}\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^3 = -24\sqrt{3}i$ A1

$(2\sqrt{3})^3 = 8\times 3\sqrt{3} = 24\sqrt{3}$

$\text{cis}\left(-\frac{\pi}{2}\right) = -i$ M1

use de Moivre's Theorem

c. Find the integer values of n for which $\left(3-\sqrt{3}i\right)^n$ is real. 1 mark

$(2\sqrt{3})^n\,\text{cis}\left(-\frac{n\pi}{6}\right) \in \mathbb{R}$, $-\frac{n\pi}{6} = 0, \pm\pi, \pm 2\pi, \ldots$

$\therefore \underbrace{n = 6k}, k \in \mathbb{Z}$ A1

d. Find the integer values of n for which $\left(3-\sqrt{3}i\right)^n = ai$, where a is a real number. 1 mark

$-\frac{n\pi}{6} = \frac{\pi}{2} + 2k\pi = \frac{\pi(1+4k)}{2}$

$\underbrace{n = 6k+3, k \in \mathbb{Z}}$

A1 or $6k-3$ or equivalent

can be other letters, i.e. n

but must have $k \in \mathbb{Z}$

Question 8 (4 marks)

Find the volume of the solid of revolution formed when the graph of $y = \sqrt{\dfrac{1+2x}{1+x^2}}$ is rotated about the x-axis over the interval $[0, 1]$.

$$V = \underbrace{\pi \int_0^1 \frac{1+2x}{1+x^2}\,dx}_{\text{M1}} = \pi \int_0^1 \frac{1}{1+x^2}\,dx + \pi \int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= \pi \left[\tan^{-1}(x)\right]_0^1 + \pi \left[\log_e |1+x^2|\right]_0^1$$

A1 A1

$$= \pi \times \left(\frac{\pi}{4} - 0\right) + \pi \left(\log_e(2) - \log_e(1)\right)$$

$$= \frac{\pi^2}{4} + \pi \cdot \log_e(2)$$

A1 equivalent answers exist.

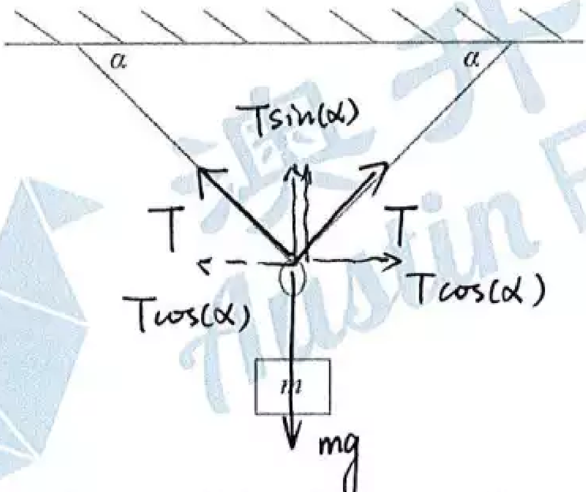
i.e.: $\pi\left(\frac{\pi}{4} + \log_e(2)\right)$

or

$\pi\left(\dfrac{\pi + 4\ln(2)}{4}\right)$ etc.

Question 9 (4 marks)

a. A light inextensible string is connected at each end to a horizontal ceiling. A mass of m kilograms hangs in equilibrium from a smooth ring on the string, as shown in the diagram below. The string makes an angle α with the ceiling.

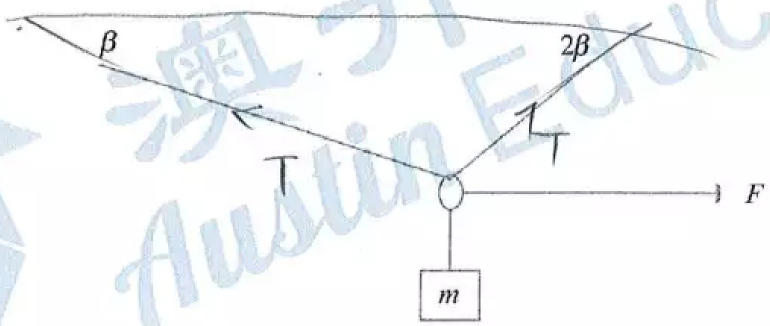


Express the tension, T newtons, in the string in terms of m, g and α. 1 mark

$$2T\sin(\alpha) = mg$$

$$T = \frac{mg}{2\sin(\alpha)}$$ A1

b. A different light inextensible string is connected at each end to a horizontal ceiling. A mass of m kilograms hangs from a smooth ring on the string. A horizontal force of F newtons is applied to the ring. The tension in the string has a constant magnitude and the system is in equilibrium. At one end the string makes an angle β with the ceiling and at the other end the string makes an angle 2β with the ceiling, as shown in the diagram below.



Show that $F = mg\left(\dfrac{1-\cos(\beta)}{\sin(\beta)}\right)$. 3 marks

$+\uparrow \quad T\sin(\beta)+T\sin(2\beta)=mg$

$T=\dfrac{mg}{\sin(\beta)+\sin(2\beta)}$

$+\rightarrow$

$F+T\cos(2\beta)=T\cos(\beta)$

$F=T(\cos(\beta)-\cos(2\beta))$

$=\dfrac{mg}{\sin(\beta)+\sin(2\beta)}\cdot(\cos(\beta)-\cos(2\beta))$

$=\dfrac{mg}{\sin(\beta)(1+2\cos(\beta))}\cdot(\cos(\beta)-(2\cos^2(\beta)-1))$

$=\dfrac{mg}{\sin(\beta)\cancel{(1+2\cos(\beta))}}\cdot(1-\cos(\beta))\cancel{(1+2\cos}$

Question 10 (5 marks)

Find $\frac{dy}{dx}$ at the point $\left(\frac{\sqrt{\pi}}{\sqrt{6}}, \frac{\sqrt{\pi}}{\sqrt{3}}\right)$ for the curve defined by the relation $\sin(x^2) + \cos(y^2) = \frac{3\sqrt{2}}{\pi}xy$.

Give your answer in the form $\frac{\pi - a\sqrt{b}}{\sqrt{a}\left(\pi + \sqrt{b}\right)}$, where $a, b \in Z^+$.

Imp diff: $\underbrace{2x \cdot \cos(x^2)}_{A1} - \underbrace{2y\frac{dy}{dx}\sin(y^2)}_{A1} = \underbrace{\frac{3\sqrt{2}}{\pi}\left(y + x\frac{dy}{dx}\right)}_{A1}$

Sub in $x = \sqrt{\frac{\pi}{6}}$, $y = \sqrt{\frac{\pi}{3}}$

$$2\sqrt{\frac{\pi}{6}} \cdot \cos\left(\frac{\pi}{6}\right) - 2\sqrt{\frac{\pi}{3}}\frac{dy}{dx}\sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{2}}{\pi}\left(\sqrt{\frac{\pi}{3}} + \sqrt{\frac{\pi}{6}} \cdot \frac{dy}{dx}\right)$$

M1 $\cancel{2}\sqrt{\frac{\pi}{6}} \times \frac{\sqrt{3}}{\cancel{2}} - \cancel{2}\sqrt{\frac{\pi}{3}} \cdot \frac{\sqrt{3}}{\cancel{2}}\frac{dy}{dx} = \frac{3\sqrt{2}}{\pi} \times \frac{\sqrt{\pi}}{\sqrt{3}} + \frac{\cancel{3\sqrt{2}}^{\sqrt{3}}}{\cancel{\pi}_{\sqrt{\pi}}} \times \frac{\cancel{\sqrt{\pi}}}{\cancel{\sqrt{6}}_{\cancel{\sqrt{3}}}}\frac{dy}{dx}$

$$\sqrt{\frac{\pi}{2}} - \sqrt{\pi}\frac{dy}{dx} = \frac{\sqrt{6}}{\sqrt{\pi}} \times \frac{\sqrt{3}}{\sqrt{\pi}}\frac{dy}{dx}$$

$$\frac{\sqrt{\pi}}{\sqrt{2}} - \sqrt{\pi}\frac{dy}{dx} = \frac{3\sqrt{2}}{\pi}\frac{dy}{dx}$$

$$\frac{\sqrt{\pi}}{\sqrt{2}} = \frac{dy}{dx}\left(\frac{3\sqrt{2}}{\pi} + \sqrt{\pi}\right) = \frac{dy}{dx}\left(\frac{\pi\sqrt{\pi} + 3\sqrt{2}}{\pi}\right)$$

$$\frac{dy}{dx} = \frac{\pi\sqrt{\pi}}{\sqrt{2}(\pi\sqrt{\pi} + 3\sqrt{2})} = \frac{\sqrt{2}\pi\sqrt{\pi}\,(\pi\sqrt{\pi} - 3\sqrt{2})}{2(\pi\sqrt{\pi} + 3\sqrt{2})(\pi\sqrt{\pi} - 3\sqrt{2})}$$

$$\therefore \frac{dy}{dx} = \frac{\pi - 2\sqrt{3}}{\sqrt{2}(\pi + \sqrt{3})}$$ A1

END OF QUESTION AND ANSWER BOOK