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YEAR 12 *Trial Exam Paper*

2019

SPECIALIST MATHEMATICS

Written examination 1

Worked solutions

This book presents:

- fully worked solutions
- mark allocations
- tips.

Question 1a.

Worked solution

$$\left|\overrightarrow{OA}\right| = 2\left|\overrightarrow{OC}\right|$$

$$\Rightarrow \sqrt{m^2 + \left(\sqrt{39}\right)^2} = 2\left|\overrightarrow{OC}\right| = 20$$

$$m^2 + 39 = 400$$

$$m^2 = 361$$

$$m = \pm 19$$

Mark allocation: 2 marks

- 1 mark for showing that $\sqrt{m^2 + \left(\sqrt{39}\right)^2} = 20$
- 1 mark for the correct answer

Tip

- *It is important to check the conditions given in the question to ensure that you give the correct number of solutions when more than one solution is possible.*

Question 1b.

Worked solution

$$\frac{\overrightarrow{OA}.\overrightarrow{OC}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{OC}\right|}=\frac{1}{5}$$

$$\Rightarrow \frac{-6\times m+0\times 8+0\times\sqrt{39}}{10\sqrt{m^2+\left(\sqrt{39}\right)^2}}=\frac{1}{5}$$

$$\Rightarrow \frac{-3m}{5\sqrt{m^2+39}}=\frac{1}{5}$$

$$-3m=\sqrt{m^2+39}$$

$$9m^2=m^2+39$$

$$8m^2=39$$

$$m=\pm\sqrt{\frac{39}{8}}=\pm\frac{\sqrt{78}}{4}$$

Mark allocation: 2 marks

- 1 mark for showing that $\dfrac{-3m}{5\sqrt{m^2+39}}=\dfrac{1}{5}$
- 1 mark for the correct answer

Question 2

Worked solution

Using implicit differentiation,

$$2y^2+4xy\frac{dy}{dx}-\frac{1}{\sqrt{16-x^2}}-\frac{\pi y}{3}-\frac{\pi x}{3}\frac{dy}{dx}=0$$

Substitute the coordinates of the point $\left(2,\frac{1}{2}\right)$ into the differentiated equation to find the gradient.

$$\frac{1}{2}+4\frac{dy}{dx}-\frac{1}{2\sqrt{3}}-\frac{\pi}{6}-\frac{2\pi}{3}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left(4-\frac{2\pi}{3}\right)=\frac{1}{2\sqrt{3}}+\frac{\pi}{6}-\frac{1}{2}$$

$$\frac{dy}{dx}\left(\frac{12-2\pi}{3}\right)=\frac{\sqrt{3}+\pi-3}{6}$$

$$\frac{dy}{dx}=\frac{\pi-3+\sqrt{3}}{2(12-2\pi)}=\frac{\pi-3+\sqrt{3}}{24-4\pi}$$

Mark allocation: 3 marks

- 1 mark for correctly differentiating the equation on both sides using implicit differentiation
- 1 mark for substituting the coordinates of the point into the equation
- 1 mark for giving the gradient in the correct form

Question 3

Worked solution

Since $P(1-\sqrt{3}i) = 0 \Rightarrow z = 1-\sqrt{3}i$ is a solution.

As all the coefficients are real, $z=1+\sqrt{3}i$ is also a solution (by the complex conjugate root theorem).

Therefore

$$\begin{aligned} P(z) &= \left(z-1+\sqrt{3}i\right)\left(z-1-\sqrt{3}i\right)(z-c) \\ &= \left((z-1)^2+3\right)(z-c) \\ &= \left(z^2-2z+4\right)(z-c) \end{aligned}$$

where $c \in R$.

Since $\left(z^2-2z+4\right)(z-c) = z^3-(a-1)z^2+\left(b^2+4\right)z-8$, it follows that $c = 2$, so $z = 2$ is a solution.

Therefore, the solutions of $P(z)$ are $z=1-\sqrt{3}i, 1+\sqrt{3}i, 2$.

Hence $(z^2-2z+4)(z-2) = z^3-4z^2+8z-8 = z^3-(a-1)z^2+(b^2+4)z-8$.

Equating the coefficients and solving for a and b gives

$a-1=4 \Rightarrow a=5$

$b^2+4=8 \Rightarrow b=\pm 2$

Mark allocation: 4 marks

- 1 mark for using the complex conjugate root theorem to find the solution $z=1+\sqrt{3}i$
- 1 mark for finding the third solution ($z = 2$)
- 1 mark for correctly calculating a
- 1 mark for correctly calculating b

Tips

- *When finding solutions to a complex polynomial in which all coefficients are real, make use of the complex conjugate root theorem to find additional solutions.*
- *When expanding* $\left(z-1-\sqrt{3}i\right)\left(z-1+\sqrt{3}i\right)$*, use the difference-of-two-squares formula:* $(z-1)^2-\left(\sqrt{(3)}i\right)^2 = (z-1)^2+3$.

Question 4a.

Worked solution

Let W be the normal random variable $X - Y$.

To find

$$\Pr(X - Y > 0) = \Pr(W > 0),$$

the mean and standard deviation of W is needed.

$$E(W) = E(X) - E(Y) = 175 - 165 = 10$$

$$\text{var}(W) = \text{var}(X) + (-1)^2 \text{var}(Y) = 3^2 + 4^2 = 25$$
$$\Rightarrow \text{SD}(W) = 5$$

So

$$\Pr(W > 0) = \Pr\left(Z > \frac{0-10}{5}\right) = \Pr(Z > -2) \approx 0.975$$

Mark allocation: 2 marks

- 1 mark for finding the mean and standard deviation of the normal random variable W
- 1 mark for calculating that the $\Pr(W > 0) \approx 0.975$

Tip

- *It is important to remember the approximate probabilities associated with z values of* $\pm 1, \pm 2$ *and* ± 3 *in a standard normal distribution.*

Question 4b.

Worked solution

Let $\overline{W}$ be the normal random distribution of the mean difference in mass, so $\overline{W} = \overline{X} - \overline{Y}$.

Then $E(\overline{W}) = E(\overline{X}) - E(\overline{Y}) = 175 - 165 = 10$.

For a 95% confidence interval, the value of z is 1.96, so the integer value to use in the calculation is 2.

The confidence interval is given by

$$\left(\overline{w} - z\frac{s}{\sqrt{n}}, \overline{w} + z\frac{s}{\sqrt{n}}\right) = \left(10 - 2\frac{5}{\sqrt{100}}, 10 + 2\frac{5}{\sqrt{100}}\right)$$

$$= (10 - 2 \times 0.5, 10 + 2 \times 0.5)$$

$$= (9, 11)$$

Mark allocation: 2 marks

- 1 mark for defining $\overline{W}$ and finding that $\mathrm{E}(\overline{W}) = 10$
- 1 mark for determining the correct interval

Question 5

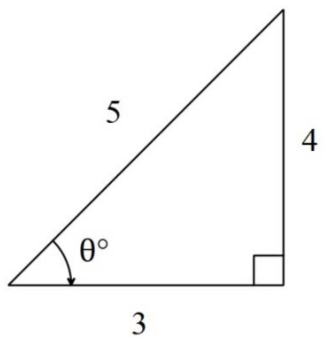
Worked solution

$$\tan(t) = \tan\left(\arcsin\left(\frac{4}{5}\right) - \arccos\left(\frac{5}{13}\right)\right)$$

A compound angle formula can be used, which leads to

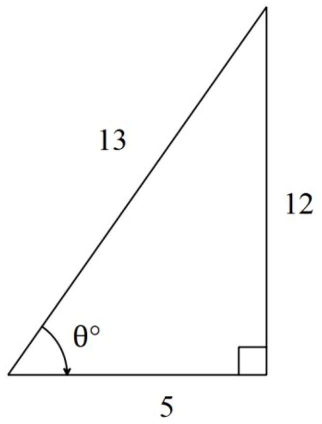
$$\tan\left(\arcsin\left(\frac{4}{5}\right) - \arccos\left(\frac{5}{13}\right)\right) = \frac{\tan\left(\arcsin\left(\frac{4}{5}\right)\right) - \tan\left(\arccos\left(\frac{5}{13}\right)\right)}{1 + \tan\left(\arcsin\left(\frac{4}{5}\right)\right)\tan\left(\arccos\left(\frac{5}{13}\right)\right)}$$

Since the angle $\arcsin\left(\frac{4}{5}\right)$ is from a 3–4–5 triangle, the corresponding arctan expression is $\arctan\left(\frac{4}{3}\right)$.



$$\theta = \arcsin\left(\frac{4}{5}\right) = \arctan\left(\frac{4}{3}\right)$$

Since the angle $\arccos\left(\frac{5}{13}\right)$ is from a 5–12–13 triangle, the corresponding arctan expression is $\arctan\left(\frac{12}{5}\right)$.



$$\theta = \arccos\left(\frac{5}{13}\right) = \arctan\left(\frac{12}{5}\right)$$

Substituting these new expressions into the compound angle formula gives

$$\frac{\tan\left(\arctan\left(\frac{4}{3}\right)\right)-\tan\left(\arctan\left(\frac{12}{5}\right)\right)}{1+\tan\left(\arctan\left(\frac{4}{3}\right)\right)\tan\left(\arctan\left(\frac{12}{5}\right)\right)}$$

$$=\frac{\frac{4}{3}-\frac{12}{5}}{1+\frac{4}{3}\times\frac{12}{5}}=\frac{-\frac{16}{15}}{\frac{63}{15}}$$

$$=-\frac{16}{63}$$

Therefore

$$\tan(t)=-\frac{16}{63}$$

Mark allocation: 3 marks

- 1 mark for using the appropriate compound angle formula
- 1 mark for finding $\arctan\left(\frac{4}{3}\right)$ and $\arctan\left(\frac{12}{5}\right)$
- 1 mark for the correct answer

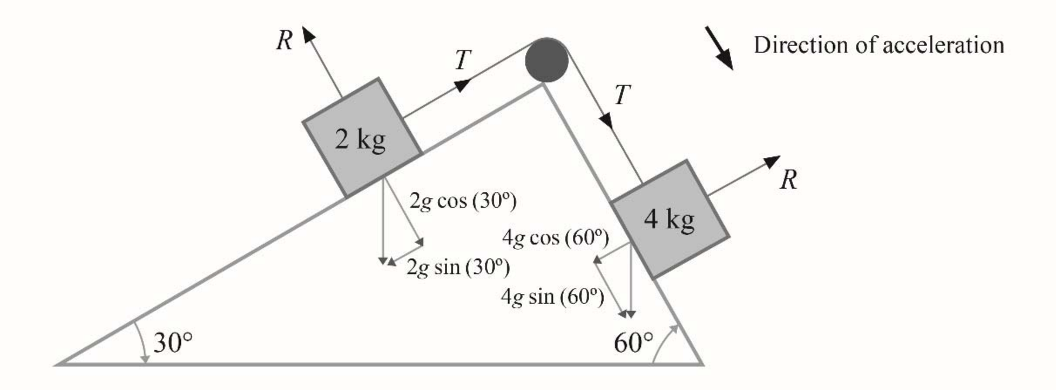
Tips

- *Become familiar with recognising when to apply compound angle formulas.*
- *It is important to be familiar with Pythagorean triples that are commonly used in exams, such as 3–4–5, 6–8–10 and 5–12–13.*

Question 6a.

Worked solution

A free-body diagram of the system is shown below.



Take the direction of acceleration as positive.

From Newton's second law, $\sum F = ma$, and considering only forces parallel to a plane:

$$4g\sin(60°) - T + T - 2g\sin(30°) = 6a$$

$$2g\sqrt{3} - g = 6a$$

$$\Rightarrow a = \frac{g(2\sqrt{3}-1)}{6}\ \text{ms}^{-2}$$

Mark allocation: 2 marks

- 1 mark for using Newton's second law
- 1 mark for correctly calculating the acceleration

Tips

- *Drawing a free-body diagram is a good way to visualise what forces are acting on an object and in which direction. This can help you see what forces to use in Newton's second law and other calculations.*
- *Always take the direction of motion as positive when setting up a vector equation representing motion.*

Question 6b.

Worked solution

Solve this by considering the forces on either object.

Consider the 4 kg object. Using Newton's second law and considering only those forces acting parallel to the incline:

$$\sum F = 4g\sin(60°) - T = 4a$$
$$\Rightarrow T = 4g\sin(60°) - 4a$$

Substituting the value of a, $\dfrac{g(2\sqrt{3}-1)}{6}$, derived in **part a.** gives

$$T = 2\sqrt{3}g - \frac{2g(2\sqrt{3}-1)}{3}$$
$$\Rightarrow T = \frac{(2\sqrt{3}+2)g}{3} \text{ newtons}$$

Alternatively, considering the 2 kg object:

$$\sum F = T - 2g\sin(30°) = 2a$$
$$\Rightarrow T = 2a + 2g\sin(30°)$$

Substituting a and solving for T gives

$$T = \frac{g(2\sqrt{3}-1)}{3} + g$$
$$\Rightarrow T = \frac{(2\sqrt{3}+2)g}{3} \text{ newtons}$$

Mark allocation: 1 mark

- 1 mark for the correct answer

Question 7a.

Worked solution

The inflow of salt in kgs^{-1} is $0 \times 5 = 0$.

The outflow of salt in kgs^{-1} is $\dfrac{x}{200+5t-10t} \times 10 = \dfrac{10x}{200-5t} = \dfrac{2x}{40-t}$

$$\frac{dx}{dt} = \text{inflow} - \text{outflow} = 0 - \frac{2x}{40-t}$$
$$\frac{dx}{dt} = -\frac{2x}{40-t}$$

Mark allocation: 1 mark

- 1 mark for showing appropriate working

Question 7b.

Worked solution

From the result given in **part a.**

$\Rightarrow \dfrac{dx}{dt}+\dfrac{2x}{40-t}=0$

Separate the variables:

$\dfrac{dx}{x}=\dfrac{2.dt}{t-40}$

Integrate both sides of the equation:

$\displaystyle\int\frac{dx}{x}=\int\frac{2.dt}{t-40}$

$\ln(x)+c=2\ln(t-40)$

$\ln(x)+c=\ln(t-40)^2$

where *c* is a real constant and $x>0$.

Apply the initial conditions to find *c*:

$\ln(10)+c=\ln(-40)^2$

$c=\ln(1600)-\ln(10)=\ln(160)$

Solve for *x*:

$\ln(x)+\ln(160)=\ln(t-40)^2$

$\ln(x)=\ln(t-40)^2-\ln(160)$

$\ln(x)=\ln\left(\dfrac{(t-40)^2}{160}\right)$

$x=\dfrac{(t-40)^2}{160}$

Mark allocation: 3 marks

- 1 mark for separating the variables and correctly integrating both sides of the differential equation
- 1 mark for applying the initial conditions to find *c*
- 1 mark for providing the correct answer of $x=\dfrac{(t-40)^2}{160}$

Question 8a.

Worked solution

Express the acceleration as $v\dfrac{dv}{dx}$:

$$v\frac{dv}{dx}=v^3+16v$$

Separate the variables and integrate both sides:

$$\int\frac{v\cdot dv}{v^3+16v}=\int dx$$

$$\int\frac{dv}{v^2+16}=\int dx$$

$$\Rightarrow \frac{1}{4}\arctan\left(\frac{v}{4}\right)=x+c$$

where c is a real constant.

Apply the initial conditions to find c:

$$\frac{1}{4}\arctan\left(\frac{0}{4}\right)=\frac{\pi}{16}+c$$

$$\Rightarrow c=-\frac{\pi}{16}$$

$$\Rightarrow \frac{1}{4}\arctan\left(\frac{v}{4}\right)=x-\frac{\pi}{16}$$

Rearrange to get the velocity:

$$v=4\tan\left(4x-\frac{\pi}{4}\right)$$

Mark allocation: 3 marks

- 1 mark for selecting an appropriate acceleration formula, such as $a=v\dfrac{dv}{dx}$, to equate the acceleration to, and for appropriate integration techniques
- 1 mark for using the initial conditions to find c
- 1 mark for finding the equation of the velocity in terms of x

Tip

- *When given acceleration as a function of velocity and the initial conditions are given in terms of displacement and velocity, express the acceleration in the form* $a=v\dfrac{dv}{dx}$.

Question 8b.

Worked solution

Substitute $x=\frac{\pi}{8}$ into this equation:

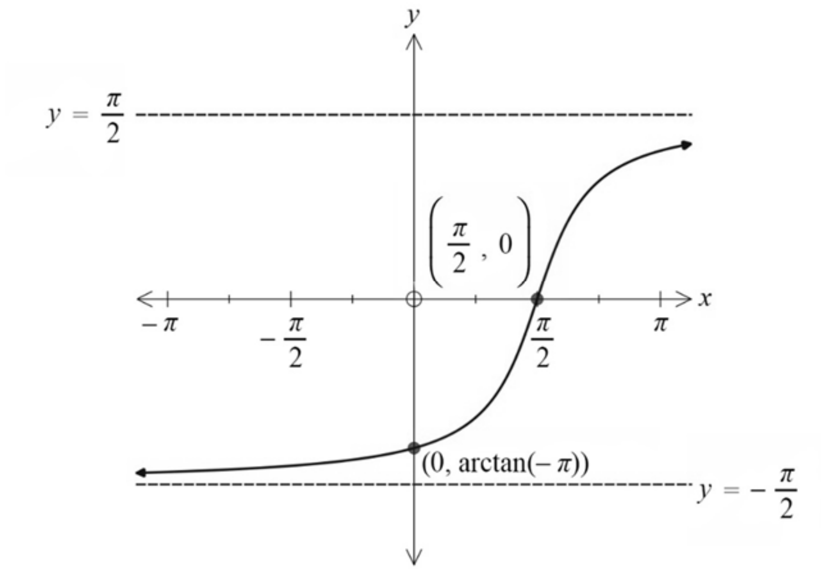
$$\begin{aligned} v &= 4\tan\left(4\times\frac{\pi}{8}-\frac{\pi}{4}\right) \\ &= 4\tan\left(\frac{\pi}{2}-\frac{\pi}{4}\right) \\ &= 4\text{ m s}^{-1} \end{aligned}$$

Mark allocation: 1 mark

- 1 mark for correctly calculating the velocity

Question 9a.

Worked solution



Mark allocation: 2 marks

- 1 mark for accurately sketching the graph and showing its asymptotic behaviour
- 1 mark for accurately labelling the intercepts and asymptotes

Question 9b.

Worked solution

The translated function is $g(x) = f\left(x+\frac{\pi}{2}\right) = \arctan\left(2\left(x+\frac{\pi}{2}\right)-\pi\right) = \arctan(2x)$.

The volume when rotated about the *y*-axis is given by

$$V = \pi\int_{y_1}^{y_2} x^2 dy$$

Rearranging the function to make *x* the subject gives $x = \frac{1}{2}\tan(y)$.

Therefore, the volume is

$$\begin{aligned}
V &= \pi\int_0^{\frac{\pi}{3}} \frac{1}{4}\tan^2(y)dy \\
&= \frac{\pi}{4}\int_0^{\frac{\pi}{3}} \tan^2(y)dy = \int_0^{\frac{\pi}{3}} \sec^2(y) - 1\ dy \\
&= \frac{\pi}{4}\left[\tan(y) - y\right]_0^{\frac{\pi}{3}} \\
&= \frac{\pi}{4}\left[\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{3} - \tan(0) + 0\right] \\
&= \frac{\pi}{4}\left(\sqrt{3} - \frac{\pi}{3}\right) \\
&= \frac{\pi}{12}\left(3\sqrt{3} - \pi\right)
\end{aligned}$$

Mark allocation: 4 marks

- 1 mark for defining $g(x)$
- 1 mark for deriving the equation $V = \pi\int_0^{\frac{\pi}{3}} \frac{1}{4}\tan^2(y)dy$
- 1 mark for correctly integrating the integrand
- 1 mark for deriving the equation $V = \frac{\pi}{12}\left(3\sqrt{3} - \pi\right)$

Tip

- *Remember to use the identity* $1 + \tan^2(\theta) = \sec^2(\theta)$ *to integrate the volume.*

Question 10a.

Worked solution

Use the following formula for the length of a curve:

$$\int_{t_1}^{t_2} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\ dt$$

The parametric equations for the position vector are

$x(t) = 2t^6 + 3$ and $y(t) = \dfrac{6}{7}t^7 + 1$

$\Rightarrow x'(t) = 12t^5$ and $y'(t) = 6t^6$

Substituting these values into the formula for the length of a curve and simplifying gives the required result:

$$\int_0^2 \sqrt{\left(12t^5\right)^2 + \left(6t^6\right)^2}\ dt$$

$$= \int_0^2 \sqrt{144t^{10} + 36t^{12}}\ dt$$

$$= \int_0^2 6t^5\sqrt{4 + t^2}\ dt$$

$$= \int_0^2 6t^5\sqrt{t^2 + 4}\ dt$$

Mark allocation: 1 mark

- 1 mark for showing the appropriate working to achieve the required result

Question 10b.

Worked solution

Beginning with

$$\int_0^2 6t^5\sqrt{t^2+4}\ dt$$

Let $u = t^2 + 4 \Rightarrow$ when $t = 0$ then $u = 4$ and when $t = 2$ then $u = 8$

Therefore

$$\frac{du}{dt} = 2t \Rightarrow dt = \frac{du}{2t}$$

So

$$\int_0^2 6t^5\sqrt{t^2+4}\ dt = \int_4^8 6t^5\sqrt{u}\ \frac{du}{2t} = \int_4^8 3t^4\sqrt{u}\ du$$

Since $u = t^2 + 4 \Rightarrow t^2 = u - 4$

$$\int_4^8 3t^4\sqrt{u}\ du = 3\int_4^8 (u-4)^2 \times u^{\frac{1}{2}}\ du$$

$$= 3\int_4^8 (u^2 - 8u + 16) \times u^{\frac{1}{2}}\ du$$

$$= 3\int_4^8 u^{\frac{5}{2}} - 8u^{\frac{3}{2}} + 16u^{\frac{1}{2}}\ du$$

$$= 3\left[\frac{2}{7}u^{\frac{7}{2}} - \frac{16}{5}u^{\frac{5}{2}} + \frac{32}{3}u^{\frac{3}{2}}\right]_4^8$$

This is as required. It follows that

$$a = \frac{2}{7},\ b = -\frac{16}{5},\ c = \frac{32}{3}$$

Alternatively:

To evaluate $\int_0^2 6t^5\sqrt{t^2+4}\,dt$,

let $u = t^2 + 4$ so that $\frac{du}{dt} = 2t \Rightarrow \frac{du}{2} = t\,dt$. Note also that $t^2 = u - 4 \Rightarrow t^4 = (u-4)^2$.

Thus when $t = 0$, $u = 4$ and when $t = 2$, $u = 8$.

The integral becomes

$$\frac{6}{2}\int_4^8 (u-4)^2\sqrt{u}\,du = 3\int_4^8 \left(u^2-8u+16\right)u^{\frac{1}{2}}\,du$$

$$= 3\int_4^8 \left(u^{\frac{5}{2}}-8u^{\frac{3}{2}}+16u^{\frac{1}{2}}\right)du$$

$$= 3\left[\frac{2}{7}u^{\frac{7}{2}}-8\times\frac{2}{5}u^{\frac{5}{2}}+16\times\frac{2}{3}u^{\frac{3}{2}}\right]_4^8$$

$$= 3\left[\frac{2}{7}u^{\frac{7}{2}}-\frac{16}{5}u^{\frac{5}{2}}+\frac{32}{3}u^{\frac{3}{2}}\right]_4^8$$

$a=\frac{2}{7},\ b=-\frac{16}{5},\ c=\frac{32}{3}$

Mark allocation: 4 marks

- 1 mark for demonstrating the appropriate substitution $u=t^2+4$ and adjusting the terminals
- 1 mark for correctly stating $3\int_4^8 (u-4)^2\times u^{\frac{1}{2}}\ du$
- 1 mark for correctly deriving $d=3\left[au^{\frac{7}{2}}-bu^{\frac{5}{2}}+cu^{\frac{3}{2}}\right]_4^8$
- 1 mark for correctly determining the value of *a*, *b* and *c*

Tips

- *Remember to adjust the integrand terminals when performing integration by substitution.*
- *Be aware that once a variable has been substituted, a linear substitution may be possible to simplify the integral.*

END OF WORKED SOLUTIONS