YEAR 12 *Trial Exam Paper*

2019

SPECIALIST MATHEMATICS

Written examination 2

Worked solutions

This book presents:

- worked solutions
- mark allocations
- tips.

SECTION A – Multiple-choice questions

Question 1

Answer: E

Worked solution

Domain:

The maximal domain of the standard inverse cosine function is $-1 \le x \le 1$.

So, the maximal domain requires that $-1 \le bx + \pi \le 1$.

Giving $\dfrac{-1-\pi}{b} \le x \le \dfrac{1-\pi}{b}$

Range:

The range of the standard inverse cosine function is $\left[0,\ \pi\right]$.

So $0 \le \cos^{-1}\left(bx+\pi\right) \le \pi$

Then

$$0 \ge -a\cos^{-1}\left(bx+\pi\right) \ge -a\pi$$
$$-a\pi + c \le -a\cos^{-1}\left(bx+\pi\right) + c \le c$$

Therefore, the range is $[-a\pi + c,\ c]$.

Question 2

Answer: A

Worked solution

The graph displayed could either be a transformation of $y = |\sin(x)|$ or $y = |\cos(x)|$.

To achieve the graph, $y = |\sin(x)|$ would undergo the following transformations:

- a dilation by a factor of 2 in the y-direction
- a translation of 1 unit in the positive y-direction
- a translation of $\frac{\pi}{2}$ units in the positive x-direction.

This matches the transformations of option A.

For $y = |\cos(x)|$ to be transformed into the graph in the question, it would undergo the following transformations:

- a dilation by a factor of 2 in the y-direction
- a translation of 1 unit in the positive y-direction.

None of the options match these transformations, therefore option A is the correct rule.

Alternatively, by graphing each option on the CAS and matching the results to the graph given, it can be seen that option A is the correct option.

Question 3

Answer: D

Worked solution

If $\cos(\theta) = 0.8$, then by using a rearranged double angle formula, $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$.

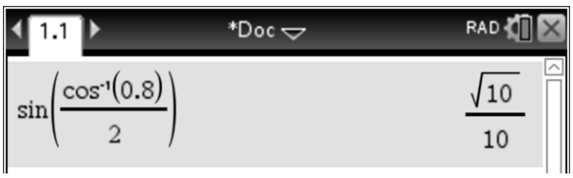
As $\theta \in \left[\frac{3\pi}{2}, 2\pi\right]$, then $\frac{\theta}{2} \in \left[\frac{3\pi}{4}, \pi\right]$, so you reject the negative branch of

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}.$$

Then

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-0.8}{2}} = \frac{\sqrt{10}}{10}$$

Alternatively, using the CAS can also give the correct answer.



Question 4

Answer: C

Worked solution

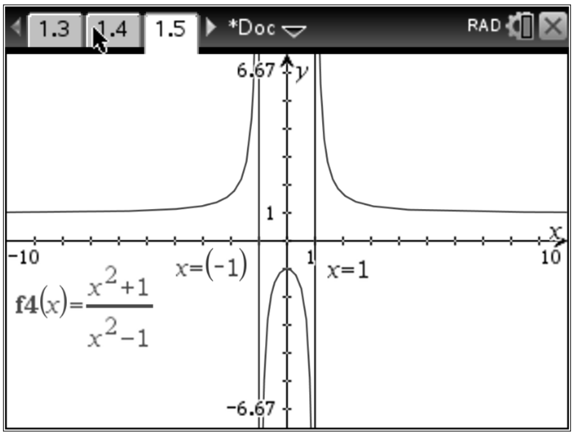
Option A is false as $\frac{d^2y}{dx^2} \neq 0$.

Option B is false as the graph has only vertical and horizontal asymptotes.

Option C is true as $\left.\frac{d^2y}{dx^2}\right|_{x=0} = -4 < 0$ and $\left.\frac{dy}{dx}\right|_{x=0} = 0$, making it concave down. Also, the point is a local maximum.

Option D is false as there are three asymptotes: $x = -1,\ x = 1,\ y = 1$.

This can be seen using the CAS.



Option E is false as the curve does not meet the conditions to be concave up at $x = 0$.

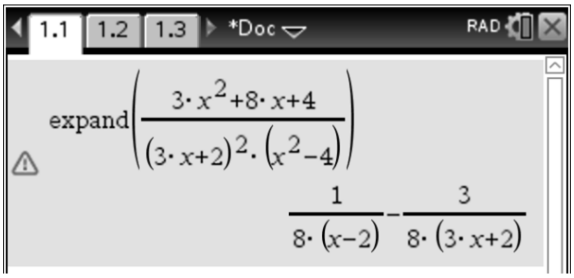
Therefore, option C is the only correct option.

Question 5

Answer: A

Worked solution

Use the expand command on the CAS.



Therefore, a partial fraction form of the expression is

$\frac{A}{x-2} + \frac{B}{3x+2}$, where $A = \frac{1}{8}$ and $B = -\frac{3}{8}$.

Question 6

Answer: A

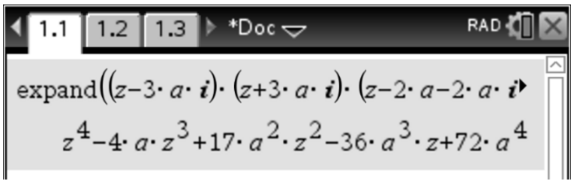
Worked solution

From the conjugate root theorem, it is known that the other solutions are $z=-3ai$ and $z=2a+2ai$.

So, the quartic polynomial can be expressed as

$$\begin{aligned}&(z-3ai)(z+3ai)(z-2a-2ai)(z-2a+2ai)\\&=\left(z^2-(3ai)^2\right)\left((z-2a)^2-(2ai)^2\right)\\&=(z^2+9a^2)(z^2-4az+8a^2)\\&=z^4-4az^3+17a^2z^2-36a^3z+72a^4\end{aligned}$$

Alternatively, use the expand function on the CAS to get the correct answer.



Tips

- *Where all coefficients are real, make use of the conjugate root theorem to find additional solutions to complex polynomials.*

Question 7

Answer: B

Worked solution

The cartesian equation of the line is $y = x - 1$.

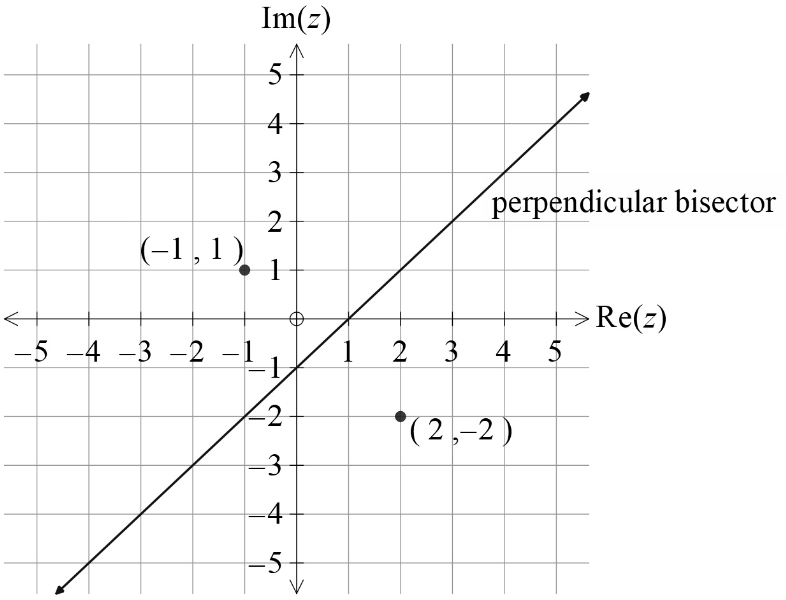
Options A and E are incorrect, as they represent rays.

Option D is incorrect, as it represents the equation of a circle centred at (–2, –3) and a radius of 4.

Options B and C represent equations of perpendicular bisectors.

Option C, however, is incorrect. Substituting $z = 2 + i$, a point on the line, into $|z - 2 - i| = |z + 1 + 2i|$ gives $|0| \neq |3 + 3i|$. Therefore, the line shown is not the perpendicular bisector described.

Option B is correct, as the perpendicular bisector of the points (2, –2) and (–1, 1) results in the line $x = -1,\ x = 1,\ y = 1$ and matches the line shown in the graph.



Alternatively, if you define z as $x + yi$ on the CAS, you can generate the cartesian relation of the line, as shown below.

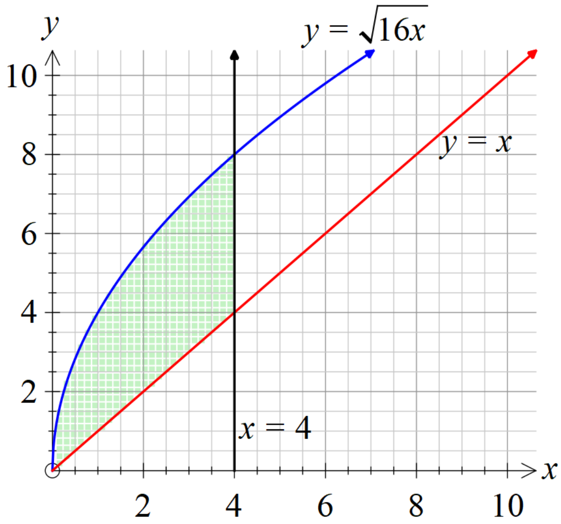
$x+y\cdot \boldsymbol{i} \to z$	$x+y\cdot \boldsymbol{i}$
$\lvert z-2+2\cdot \boldsymbol{i}\rvert = \lvert z+1-\boldsymbol{i}\rvert$	$\sqrt{x^2-4\cdot x+y^2+4\cdot y+8}=\sqrt{x^2+2\cdot x+y^2-2\cdot y+2}$
$\text{solve}\left(\sqrt{x^2-4\cdot x+y^2+4\cdot y+8}=\sqrt{x^2+2\cdot x+y^2-2\cdot y+2}, y\right)$	$y=x-1$

Question 8

Answer: D

Worked solution

A graph of the volume generated is shown below.



The volume is then given by

$$
\begin{aligned}
V &= \pi\int_0^4 \left(\sqrt{16x}\right)^2 - (x)^2\, dx \\
&= \pi\int_0^4 16x - x^2\, dx \\
&= \pi\left[8x^2 - \frac{x^3}{3}\right]_0^4 \\
&= \frac{320\pi}{3} \text{ cubic units}
\end{aligned}
$$

Question 9

Answer: D

Worked solution

For $-\sin^2(2x)\cos^2(x)+\sin^2(2x)\sin^2(x)$, take out $-\sin^2(2x)$ as a common factor:

$\Rightarrow -\sin^2(2x)(\cos^2(x)-\sin^2(x))$

Using a double angle formula gives

$-\sin^2(2x)(\cos^2(x)-\sin^2(x)) = -\sin^2(2x)\cos(2x)$

So

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} -\sin^2(2x)\cos^2(x)+\sin^2(2x)\sin^2(x)\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} -\sin^2(2x)\cos(2x)\,dx$$

Then, let $u=\sin(2x)$.

Then

$$\frac{du}{dx} = 2\cos(2x) \Rightarrow dx = \frac{du}{2\cos(2x)}$$

And when

$$x=\frac{\pi}{4} \Rightarrow u=1$$

$$x=\frac{\pi}{3} \Rightarrow u=\frac{\sqrt{3}}{2}$$

So

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} -\sin^2(2x)\cos(2x)dx$$

$$= \int_1^{\frac{\sqrt{3}}{2}} -u^2\cos(2x)\frac{du}{2\cos(2x)}$$

$$= \int_1^{\frac{\sqrt{3}}{2}} \frac{-u^2}{2}du$$

$$= \frac{1}{2}\int_{\frac{\sqrt{3}}{2}}^{1} u^2 du$$

Tips

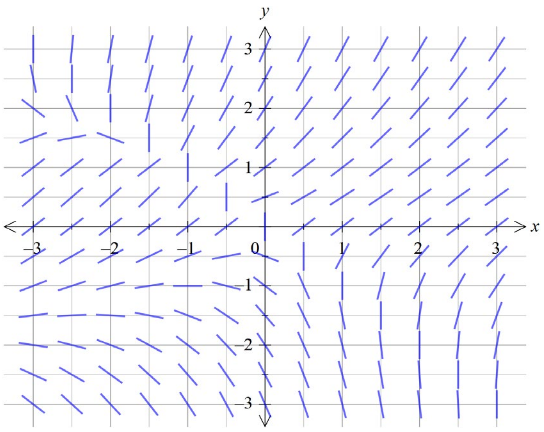
- *When a substitution is not immediately apparent, try taking out a common factor to see if the factorised form gives a more obvious substitution or reveals the use of a double angle formula that gives a suitable substitution.*
- *Don't forget to change the values of the terminals to match the substitution used.*
- *Remember to use properties of integral terminals such as* $-\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$ *to get the correct final answer.*

Question 10

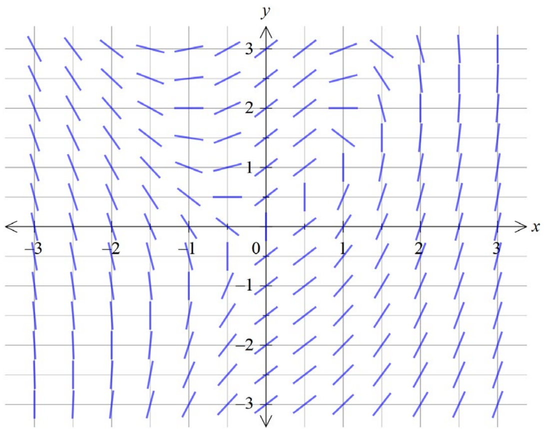
Answer: B

Worked solution

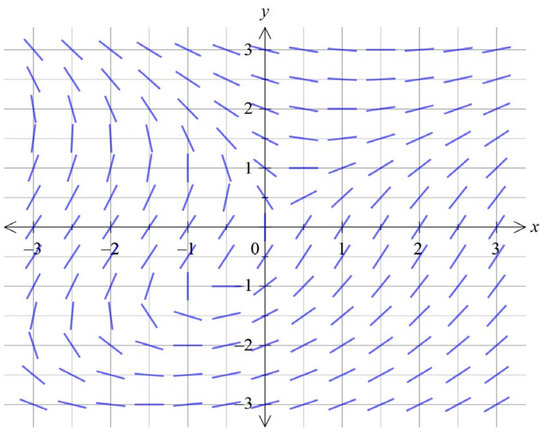
Option A has the direction field:



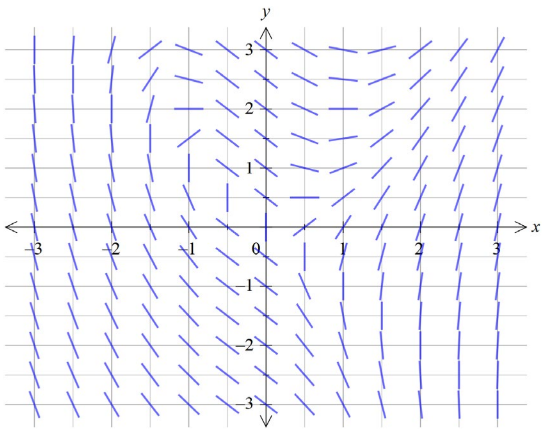
Option B has the direction field:



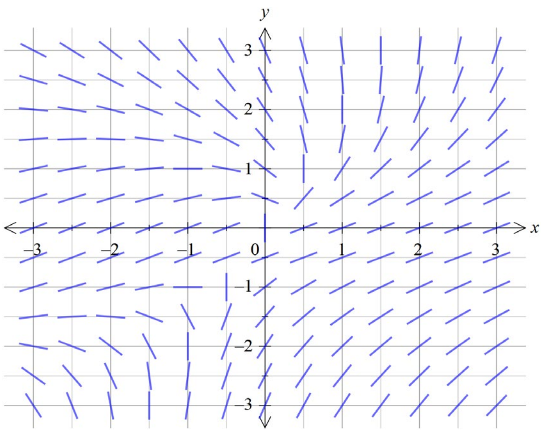
Option C has the direction field:



Option D has the direction field:



Option E has the direction field:



Therefore, option B is the correct answer.

Alternatively, looking at the behaviour of $\frac{dy}{dx}$ for $x = 0$ and $y > 0$, the direction field given is $\frac{dy}{dx} > 0$. This eliminates options C, D and E as $\frac{dy}{dx} < 0$ for these values of x and y.

Looking at the behaviour of $\frac{dy}{dx}$ for $x = 0$ and $y < 0$, the direction field given is $\frac{dy}{dx} > 0$. This then eliminates option A as $\frac{dy}{dx} < 0$ for these values of x and y.

Tips

- *When graphing the direction field on the calculator, set the scale and size of the axis to the same as that used in the question. This will make it easier to see and compare the direction field to the one given in the question.*
- *Look for points on the slope field with vertical gradients. This would correspond to points where the relation describing the gradient is undefined. For instance, the points (3, 3) and (–3, –3) have vertical gradients and relation B is the only one undefined at those points.*

Question 11

Answer: B

Worked solution

As $y = e^{2mx}$, then $\frac{dy}{dx} = 2me^{2mx}$ and $\frac{d^2y}{dx^2} = 4m^2e^{2mx}$.

Substituting into the equation gives

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 4m^2e^{2mx} - 16me^{2mx} + 12e^{2mx} = 0$$

$$\Rightarrow 4e^{2mx}(m^2 - 4m + 3) = 0$$

As $4e^{2mx} \neq 0$, then $(m^2 - 4m + 3) = 0$.

$$\Rightarrow (m-1)(m-3) = 0$$

$$\Rightarrow m = 1 \text{ or } m = 3$$

Question 12

Answer: A

Worked solution

The scalar resolute of $\underset{\sim}{b}$ in the direction of $\underset{\sim}{a}$ is given by $\underset{\sim}{b} \cdot \underset{\sim}{\hat{a}}$, where $\underset{\sim}{\hat{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \frac{1}{5}(-4\underset{\sim}{i} + 3\underset{\sim}{k})$.

So

$$\begin{aligned}\underset{\sim}{b} \cdot \underset{\sim}{\hat{a}} &= \frac{1}{5}(2\underset{\sim}{i} - 3\underset{\sim}{j} + c\underset{\sim}{k}) \cdot (-4\underset{\sim}{i} + 3\underset{\sim}{k}) \\ &= \frac{1}{5}(-8 + 3c) \\ &= 8\end{aligned}$$

$$\Rightarrow c = 16$$

Question 13

Answer: D

Worked solution

If the vectors are linearly dependent, then $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$.

So, equating the components of the vectors:

$\underset{\sim}{i}$ component: $1 = 4m - \frac{3}{2}n$

$\underset{\sim}{j}$ component: $1 = -6m + 6n$

$\underset{\sim}{k}$ component: $\gamma = 2m - 12n$

Rearranging $\underset{\sim}{j}$ component gives

$$n = \frac{1+6m}{6}$$

Substituting n into $\underset{\sim}{i}$ component and solving for m and n gives:

$$m = \frac{1}{2},\ n = \frac{2}{3}$$

Substituting m and n into $\underset{\sim}{k}$ component gives:

$$\gamma = -7$$

This value of γ is then the only one that matches any of the options. Therefore, option D is the correct option.

Question 14

Answer: A

Worked solution

The angle between the vector $\underset{\sim}{a}$ and the y-axis is given by $\theta = \cos^{-1}\left(\frac{\underset{\sim}{a}_j}{|\underset{\sim}{a}|}\right)$, where $\underset{\sim}{a}_j$ is the vector made of the components of $\underset{\sim}{a}$ in the direction of the y-axis.

So

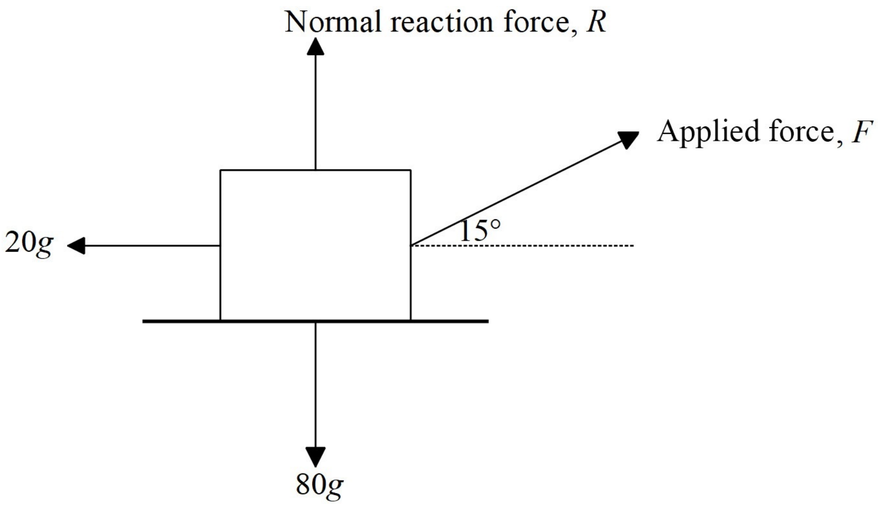
$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{-1}{|2\underset{\sim}{i} - 1\underset{\sim}{j} + 2\underset{\sim}{k}|}\right)\\ &= \cos^{-1}\left(\frac{-1}{3}\right)\\ &= 109.471\\ &\approx 109^\circ\end{aligned}$$

Question 15

Answer: D

Worked solution

A force diagram of the moving block is shown below.



Resolving the forces in the horizontal direction means that the applied force in the horizontal direction is $20g$.

Therefore, resolving the forces in the vertical direction gives

$$\sum F = R + 20g\tan(15°) - 80g = 0$$

$$\Rightarrow R = 80g - 20g\tan(15°)$$

$$= 731.482$$

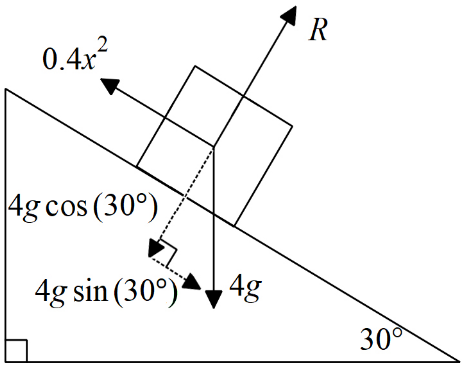
$$= 731 \text{ newtons}$$

Question 16

Answer: A

Worked solution

A force diagram of the sliding block on the ramp is shown below.



Assuming that the motion down the plane is positive, then resolving the forces parallel to the plane gives

$$\sum F = 2g - 0.4x^2 = 4a$$

$$\Rightarrow a = \frac{g}{2} - 0.1x^2$$

Using the variable acceleration formula $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$:

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{g}{2} - 0.1x^2$$

$$\Rightarrow \frac{1}{2}v^2 = \int \frac{g}{2} - 0.1x^2\,dx$$

So, the magnitude of the velocity at 5 m can be given by

$$v = \sqrt{2\int_0^5 \frac{g}{2} - 0.1x^2\,dx}$$

As the question asks for the magnitude of the velocity, the positive branch of the square root is evaluated.

Therefore, $v = \sqrt{2\int_0^5 \frac{g}{2} - 0.1x^2\,dx} = 6.38\ \text{ms}^{-1}$

Question 17

Answer: A

Worked solution

If $\underset{\sim}{a} = -t\underset{\sim}{i} + 2\underset{\sim}{j}\ \text{ms}^{-2}$, then $\underset{\sim}{v} = \int -t\underset{\sim}{i} + 2\underset{\sim}{j}\ \text{ms}^{-2}dt = -\dfrac{t^2}{2}\underset{\sim}{i} + 2t\underset{\sim}{j} + \underset{\sim}{c}$.

Applying the initial conditions gives

$$\underset{\sim}{v} = \left(-\frac{t^2}{2} + 2\right)\underset{\sim}{i} + \left(2t - 3\right)\underset{\sim}{j}$$

Momentum is given by $\underset{\sim}{p} = m\underset{\sim}{v}$.

So, the momentum at two seconds is

$$\begin{aligned}\underset{\sim}{p} &= 0.25 \times \left(-\frac{2^2}{2} + 2\right)\underset{\sim}{i} + 0.25 \times \left(2 \times 2 - 3\right)\underset{\sim}{j} \\ &= 0\underset{\sim}{i} + 0.25\underset{\sim}{j}\ \text{kg ms}^{-1} \\ &= 0.25\underset{\sim}{j}\ \text{kg ms}^{-1}\end{aligned}$$

Tip

- *Momentum is a vector quantity, not scalar. Therefore, the answer must indicate this vector quality. If the magnitude of the momentum is asked for, then the magnitude of the vector will need to be calculated.*

Question 18

Answer: C

Worked solution

The sample mean is the average of a confidence interval:

$$\bar{x} = \frac{65.18 + 71.32}{2} = 68.25$$

For the lower limit of a confidence interval, you have $65.18 = 68.25 - z\dfrac{s}{\sqrt{60}}$.

For a 95% confidence interval, use $z = 1.96$.

$$\Rightarrow s = \frac{\sqrt{60}\left(68.25 - 65.18\right)}{1.96} = 12.13$$

This is closest to option C: 68.25 and 12.15.

Question 19

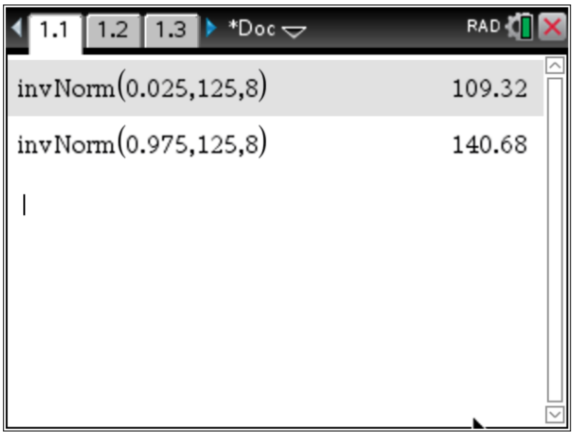
Answer: C

Worked solution

Since $\Pr(m < X < n) = 0.95$ and $\frac{m+n}{2} = 125$,

then $\Pr(X < m) = 0.025$ and $\Pr(X > n) = 0.025$.

Using the inverse normal function, you get $m = 109.3$ and $n = 140.7$, correct to one decimal place.



Note: $\frac{m+n}{2} = 125$ indicates that 125 is the midpoint between m and n. This means m and n are the same distance away from 125, making them symmetrical about the mean.

Question 20

Answer: D

Worked solution

For this question you are trying to find $\Pr(-10 < Y - X < 10)$.

So

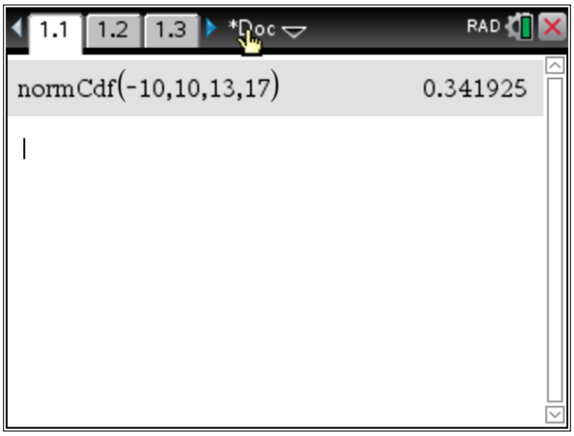
$$E(Y-X) = E(Y) - E(X) = 138 - 125 = 13$$

$$\text{var}(Y-X) = \text{var}(Y) + (-1)^2\,\text{var}(X) = 15^2 + 8^2 = 289$$

$$\Rightarrow s(Y-X) = \sqrt{289} = 17$$

Therefore, using the normal CDF function on the CAS gives

$\Pr(-10 < Y - X < 10) = 0.3419$



Note: The wording 'within 10 minutes' means that the recharge time of brand Y could be 10 minutes greater or 10 minutes less than that of brand X, leading to $\Pr(-10 < Y - X < 10)$.

SECTION B

Question 1a.

Worked solution

The denominator of f factorises to $(x-2)(x+1)$.

Hence, two vertical asymptotes exist.

Therefore, the maximal domain is $x \in R \setminus \{-1,\ 2\}$.

Mark allocation: 1 mark

- 1 mark for stating $x \in R \setminus \{-1,\ 2\}$

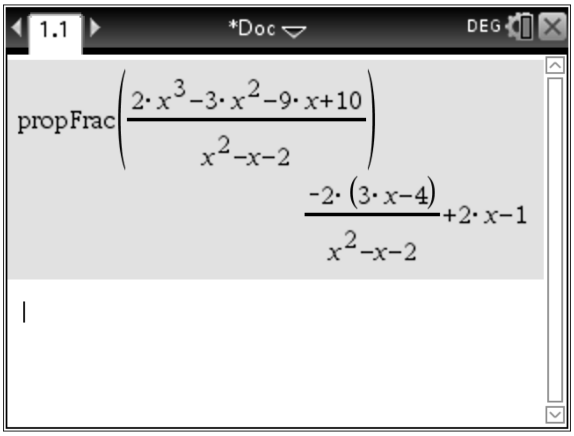
Tips

- *For questions that ask for the domain of the function, solve for the denominators equal to zero for rational functions, to determine for which values of x the function is undefined.*
- *When a curve is going to be used repeatedly throughout a question, it is useful to define the equation of the curve in the CAS at the start to avoid having to retype it multiple times.*

Question 1b.

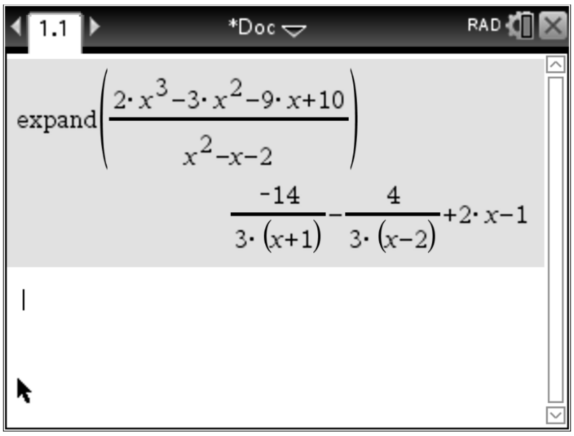
Worked solution

The function f can be expressed as $f(x) = \dfrac{-2(3x-4)}{x^2-x-2} + 2x - 1$.



Therefore, using the result from **part a.**, there are two vertical asymptotes: $x=-1, x=2$, and one oblique asymptote, $y=2x-1$.

Alternatively, using the expand command on the CAS will yield an expression where all three asymptotes, $x=-1, x=2$ and $y=2x-1$, can be seen.



Mark allocation: 2 marks

- 1 mark for stating $x=-1, x=2$
- 1 mark for stating $y=2x-1$

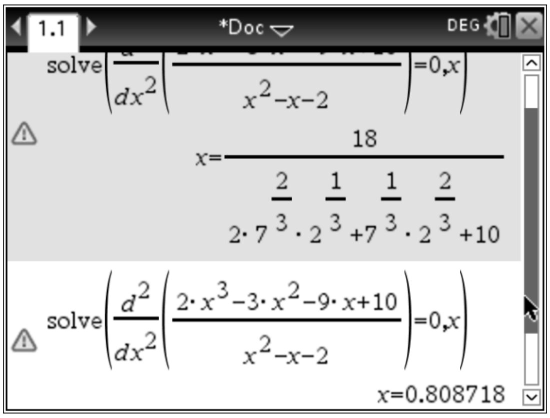
Tip

- *Rational functions, whereby the degree of the numerator – the degree of the denominator = 1, will have an oblique asymptote of the form* $y = mx + c$.

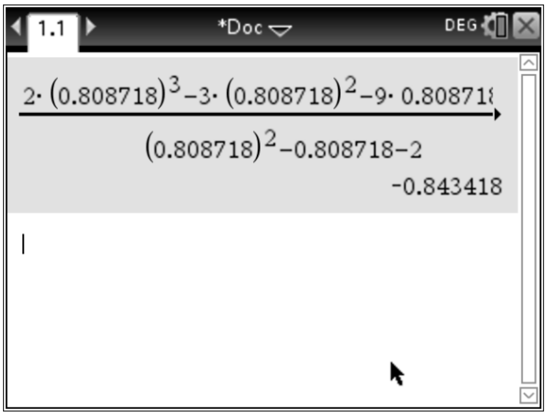
Question 1c.

Worked solution

Solving for the second derivative equal to zero gives $x = 0.808718$

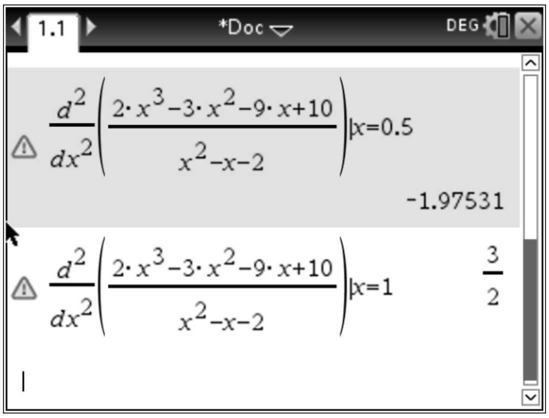


Substituting $x = 0.808718$ into f gives $f(0.808718) = -0.84$, to two decimal places.



Checking the second derivative either side of $x = 0.808718$:

$f''(0.5) = -1.97531$ and $f''(1) = 1.5$



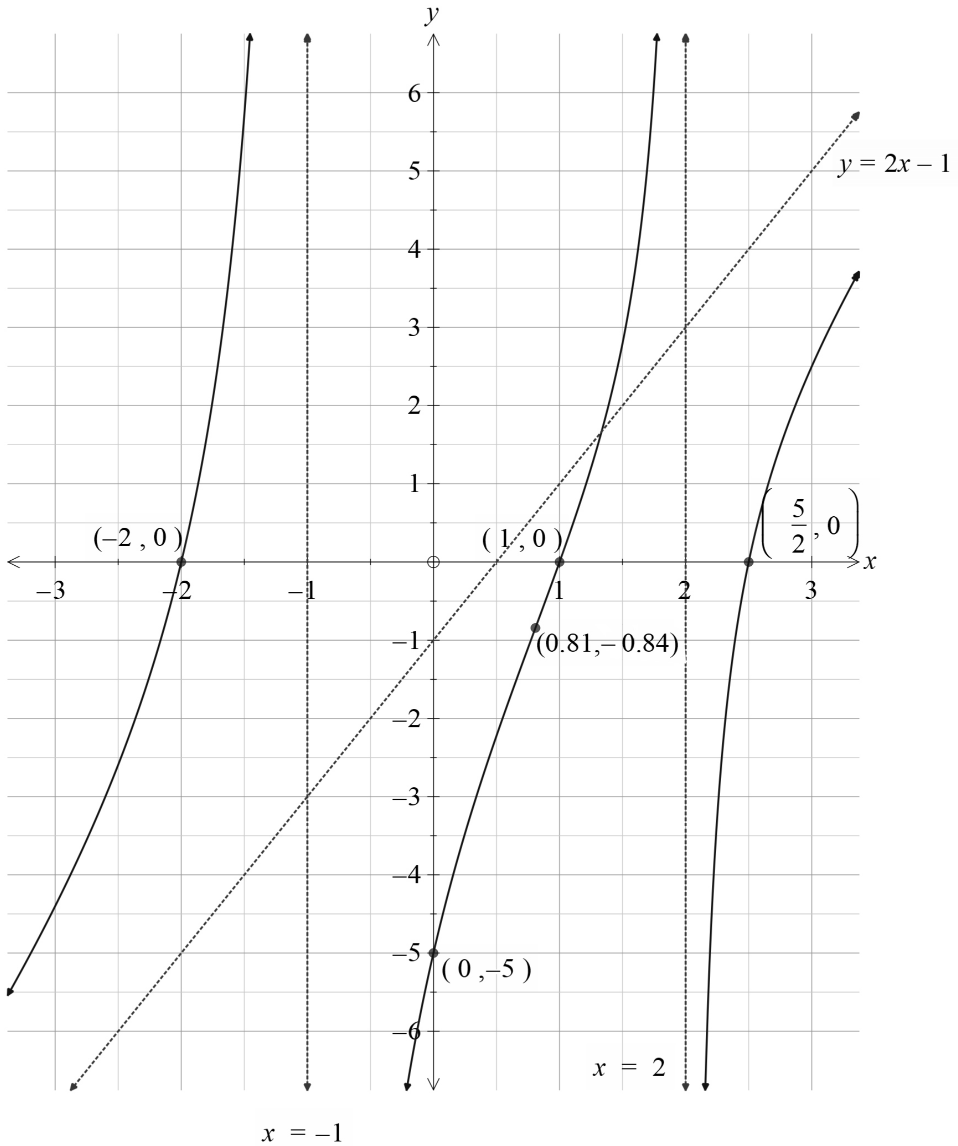
As there is a change of sign either side of $x = 0.808718$, the point $(0.81, \ -0.84)$, to two decimal places, is a point of inflection.

Mark allocation: 2 marks

- 1 mark for finding $(0.81, \ -0.84)$
- 1 mark for showing that the second derivative changes sign either side of $x = 0.808718$

Question 1d.

Worked solution



Mark allocation: 3 marks

- 1 mark for the accurate shape of the graph, showing behaviours towards the asymptotes
- 1 mark for accurately labelling intercepts and point of inflection (mark is awarded for plotting the point of inflection within the vicinity of its location)
- 1 mark for showing asymptotes with equations

Tip

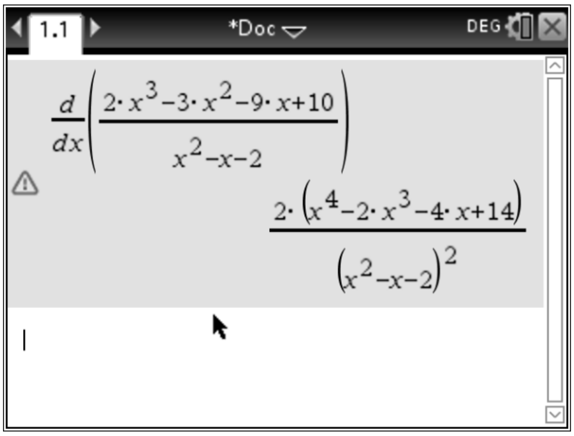
- *When using the CAS to aid your sketching, match the window settings on the CAS to the same domain, range and intervals as the axes provided in the exam to increase the accuracy of sketching.*

Question 1e.

Worked solution

The length of the curve is given by $L=\int_{x_1}^{x_2}\sqrt{1+(f'(x))^2}\,dx$.

Use the CAS to find the derivative.



Then the integral is

$$L=\int_{0.5}^{1.5}\sqrt{1+\left(\frac{2x^4-2x^3-8x+28}{\left(x^2-x-2\right)^2}\right)^2}\,dx$$

Evaluating the integral gives

$$L=\int_{0.5}^{1.5}\sqrt{1+\left(\frac{2x^4-2x^3-8x+28}{\left(x^2-x-2\right)^2}\right)^2}\,dx=5.12327\ldots$$

The length, correct to two decimal places, is 5.12.

Mark allocation: 2 marks

- 1 mark for showing $L=\int_{0.5}^{1.5}\sqrt{1+\left(\frac{2x^4-2x^3-8x+28}{\left(x^2-x-2\right)^2}\right)^2}\,dx$
- 1 mark for stating a length of 5.12

Question 2a.

Worked solution

Let $z = x+iy$, then $|z+1+i| = \sqrt{2}\,|z-1-i$.

$\Rightarrow |x+iy+1+i| = \sqrt{2}\,|x+iy-1-i|$

$\Rightarrow \sqrt{(x+1)^2+(y+1)^2} = \sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$

$$(x+1)^2+(y+1)^2 = 2(x-1)^2+2(y-1)^2$$

$$x^2+2x+1+y^2+2y+1 = 2x^2-4x+2+2y^2-4y+2$$

$$0 = x^2-6x+1+y^2-6y+1$$

Completing the square:

$0 = (x-3)^2-9+1+(y-3)^2-9+1$

$\Rightarrow (x-3)^2+(y-3)^2 = 16$

Alternatively, squaring both sides gives

$|z+1+i|^2 = 2|z-1-i|^2$

Using the fact that $z\overline{z} = |z|^2$ gives

$$(z+(1+i))(\overline{z}+(1-i)) = 2(z-(1+i))(\overline{z}-(1-i))$$

$$z\overline{z}+(1-i)z+(1+i)\overline{z}+(1+i)(1-i) = 2\left[z\overline{z}-(1-i)z-(1+i)\overline{z}+(1+i)(1-i)\right]$$

$$(z-3(1+i))(\overline{z}-3(1-i))-18+2 = 0$$

$$(z-3(1+i))(\overline{z}-3(1-i)) = 16$$

Once again using the fact that $z\overline{z} = |z|^2$, the equation can be rewritten as

$|z-3(1+i)|^2 = 16$

$|z-(3+3i)| = 4$

From here, the cartesian equation of the circle can be read as $(x-3)^2+(y-3)^2 = 16$.

Alternatively, if you define z as $x+yi$ on the CAS, you can generate the cartesian relation of the curve, $(x-3)^2+(y-3)^2=16$, as shown below.

$x+y\cdot \boldsymbol{i} \rightarrow z$	$x+y\cdot \boldsymbol{i}$
$\lvert z+1+\boldsymbol{i}\rvert=\sqrt{2}\cdot\lvert z-1-\boldsymbol{i}\rvert$	$\sqrt{x^2+2\cdot x+y^2+2\cdot y+2}=\sqrt{2\cdot(x^2-2\cdot x+y^2-2\cdot y+2)}$
completeSquare$(x^2+2\cdot x+y^2+2\cdot y+2=2\cdot(x^2-2\cdot x+y^2-2\cdot y+2),x,y)$	$-(x-3)^2-(y-3)^2=-16$

Mark allocation: 3 marks

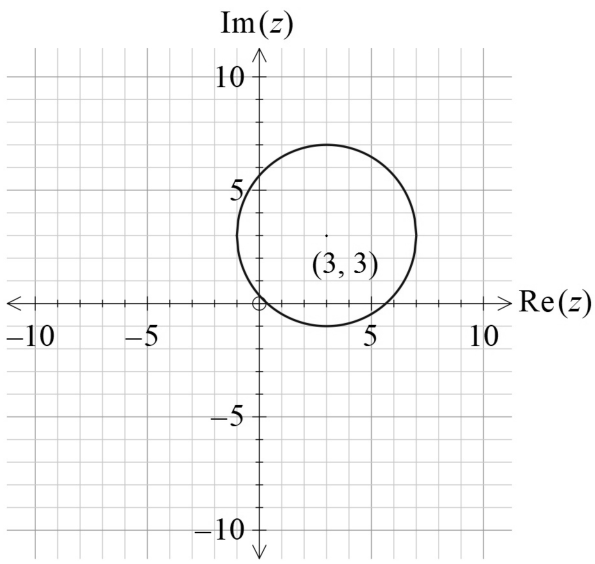
- 1 mark for showing $\sqrt{(x+1)^2+(y+1)^2}=\sqrt{2}\sqrt{(x-1)^2+(y-1)^2}$
- 1 mark for showing $0=x^2-6x+1+y^2-6y+1$ or similar
- 1 mark for $(x-3)^2+(y-3)^2=16$

OR

- 1 mark for expressing $|z+1+i|^2=2|z-1-i|^2$ as $(z+(1+i))(\overline{z}+(1-i))=2(z-(1+i))(\overline{z}-(1-i))$
- 1 mark for showing $|z-(3+3i)|=4$
- 1 mark for $(x-3)^2+(y-3)^2=16$

Question 2b.

Worked solution



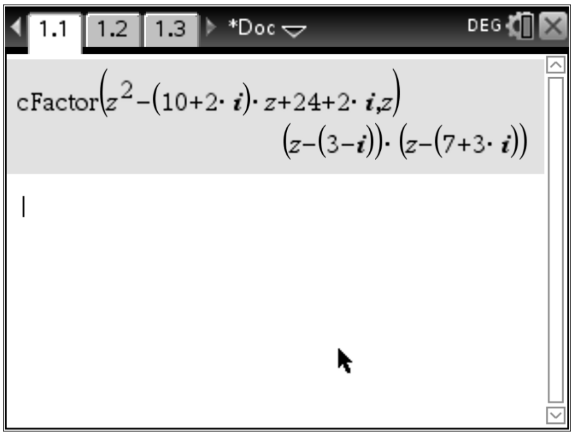
Mark allocation: 2 marks

- 1 mark for sketching a circle centred at (3, 3)
- 1 mark for sketching the correct radius of the circle

Question 2c.

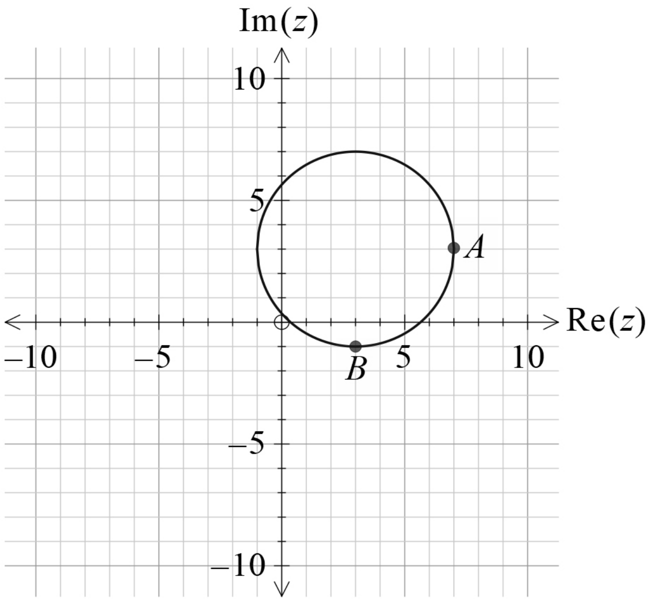
Worked solution

Factorising the equation using the CAS gives $(z-3+i)(z-7-3i)=0$.



So, the solutions are $A=7+3i$ and $B=3-i$.

The plotted points are



Mark allocation: 2 marks

- 1 mark for $A=7+3i$ and $B=3-i$
- 1 mark for correctly plotting and labelling points

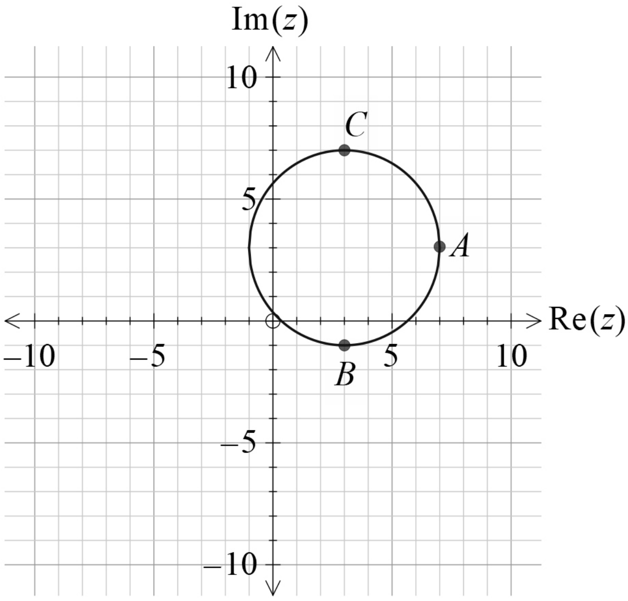
Tip

- *When labelling plotted coordinates on the complex plane, an 'i' should not be included on the vertical coordinate, as the labelling of the Im(z) axes already implies that the coordinate is imaginary.*

Question 2d.

Worked solution

$$\begin{aligned} z_2 &= \overline{z}_1 i \\ &= (7-3i)i \\ &= 3+7i \end{aligned}$$



Mark allocation: 2 marks

- 1 mark for $z_2 = 3+7i$
- 1 mark for correctly plotting and labelling the point

Question 2e.

Worked solution

As position vectors are, $A = 7\underset{\sim}{i} + 3\underset{\sim}{j}$, $B = 3\underset{\sim}{i} - \underset{\sim}{j}$, $C = 3\underset{\sim}{i} + 7\underset{\sim}{j}$

then, $\overrightarrow{AB} = -4\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\overrightarrow{AC} = -4\underset{\sim}{i} + 4\underset{\sim}{j}$.

Finding the dot product:

$$\begin{aligned} \overrightarrow{AB} \bullet \overrightarrow{AC} &= (-4\underset{\sim}{i} - 4\underset{\sim}{j}) \bullet (-4\underset{\sim}{i} + 4\underset{\sim}{j}) \\ &= 16-16 \\ &= 0 \end{aligned}$$

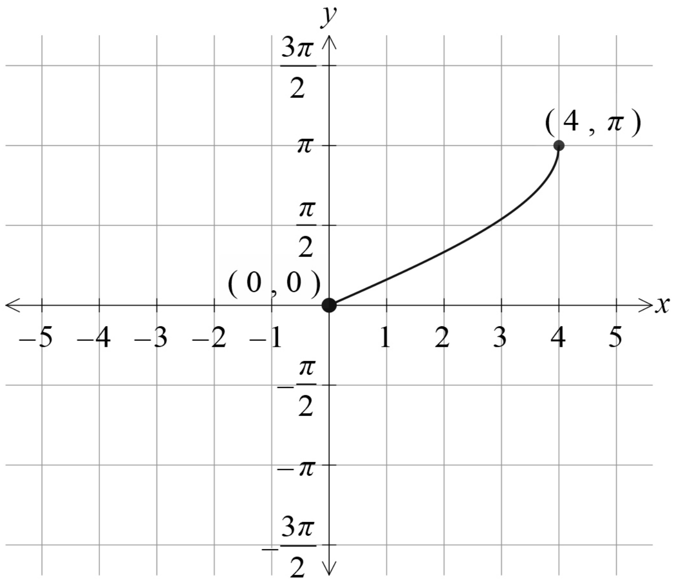
As $\overrightarrow{AB} \bullet \overrightarrow{AC} = 0$, then the vectors meet at a right angle.

Mark allocation: 2 marks

- 1 mark for $\overrightarrow{AB} = -4\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\overrightarrow{AC} = -4\underset{\sim}{i} + 4\underset{\sim}{j}$
- 1 mark for finding the dot product is zero and concluding that the vectors meet at a right angle

Question 3a.

Worked solution



Mark allocation: 2 marks

- 1 mark for accurate shape
- 1 mark for correctly labelling endpoints

Question 3b.

Worked solution

$V = \pi\int_a^b x^2 dy$

Rearranging the equation to make x the subject gives

$x = 4\sin\left(\frac{y}{2}\right)$

$$\Rightarrow V = \pi\int_0^h \left(4\sin\left(\frac{y}{2}\right)\right)^2 dy$$
$$= 16\pi\int_0^h \sin^2\left(\frac{y}{2}\right) dy$$

Using the trigonometric double angle formula $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$:

$$\Rightarrow V = 16\pi\int_0^h \frac{1}{2}\left(1-\cos(y)\right) dy$$
$$= 8\pi\left[y - \sin(y)\right]_0^h$$
$$= 8\pi\left[h - \sin(h) - 0 + \sin(0)\right]$$
$$= 8\pi[h - \sin(h)]$$

Mark allocation: 2 marks

- 1 mark for $8\pi\left[y-\sin(y)\right]_0^h$ or using an appropriate trigonometric identity to get $16\pi\int_0^h \frac{1}{2}\left(1-\cos(y)\right)dy$
- 1 mark for correctly showing the volume is $8\pi[h-\sin(h)]$

Question 3c.i.

Worked solution

$\dfrac{dV}{dt} = 16\pi^2\sqrt{h}$ and $\dfrac{dV}{dh} = 8\pi(1-\cos(h))$

Then

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV}\frac{dV}{dt}\\ &= \frac{1}{8\pi(1-\cos(h))}\times 16\pi^2\sqrt{h}\\ &= \frac{2\pi\sqrt{h}}{1-\cos(h)}\end{aligned}$$

Mark allocation: 2 marks

- 1 mark for $\dfrac{dV}{dh} = 8\pi(1-\cos(h))$
- 1 mark for showing $\dfrac{dh}{dt} = \dfrac{dh}{dV}\dfrac{dV}{dt} = \dfrac{2\pi\sqrt{h}}{1-\cos(h)}$

Question 3c.ii.

Worked solution

One-quarter of the depth is $\dfrac{\pi}{4}$ cm.

So, $\dfrac{dh}{dt} = \dfrac{2\pi\sqrt{\frac{\pi}{4}}}{1-\cos\left(\frac{\pi}{4}\right)} = 19.01 \text{ cms}^{-1}$, correct to two decimal places.

Mark allocation: 1 mark

- 1 mark for $\dfrac{dh}{dt} = 19.01 \text{ cms}^{-1}$

Question 3d.i.

Worked solution

Euler's method, in the context of this question, is

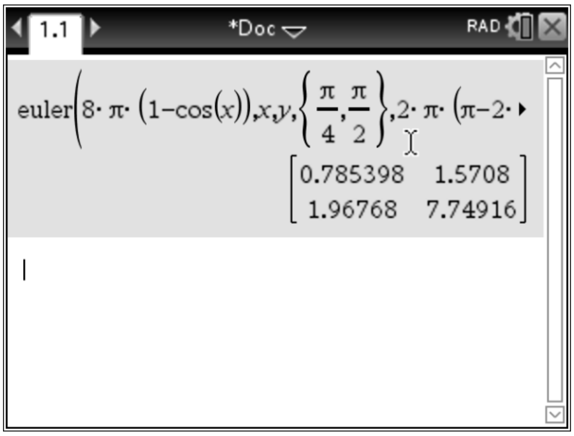
$h_{n+1} = h_n + \text{ step size}$, $V_{n+1} = V_n + \text{step size} \times \dfrac{dV}{dh_n}$, where $\text{step size} = \dfrac{\pi}{4}$, $h_0 = \dfrac{\pi}{4}$ and

$V_0 = 2\pi(\pi - 2\sqrt{2})$, using the result from **part b.**

When $\dfrac{dV}{dh} = 8\pi(1 - \cos(h))$, then, $h_1 = \dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2}$ and

$$
\begin{aligned}
V_1 &= V_0 + \frac{\pi}{4} \times \frac{dV}{dh_0} \\
&= 2\pi(\pi - 2\sqrt{2}) + \frac{\pi}{4} \times 8\pi\left(1 - \cos\left(\frac{\pi}{4}\right)\right) \\
&\approx 7.7492\ ... \\
&= 7.75 \text{ cm}^3 \text{ (correct to two decimal places)}
\end{aligned}
$$

Euler's method can also be used with the CAS.



Mark allocation: 2 marks

- 1 mark for evidence of using Euler's method to find the answer
- 1 mark for an estimate of 7.75 cm^3

Question 3d.ii.

Worked solution

From **part b.**, the volume is given by $8\pi[h - \sin(h)]$, so if $h = \frac{\pi}{2}$ cm,

then the volume is $8\pi\left[\frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)\right] \approx 14.346 \text{ cm}^3$.

Therefore, the estimate is an underestimate of the volume.

Mark allocation: 1 mark

- 1 mark for finding the correct volume and concluding that the estimate is an underestimate

Question 4a.

Worked solution

Starting with drone S:

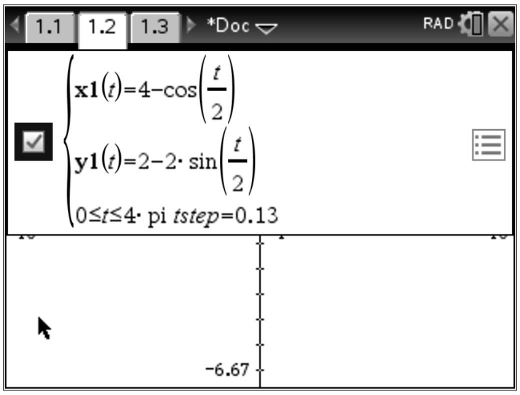
Let $x = 4 - \cos\left(\frac{t}{2}\right)$ and $y = 2 - 2\sin\left(\frac{t}{2}\right)$.

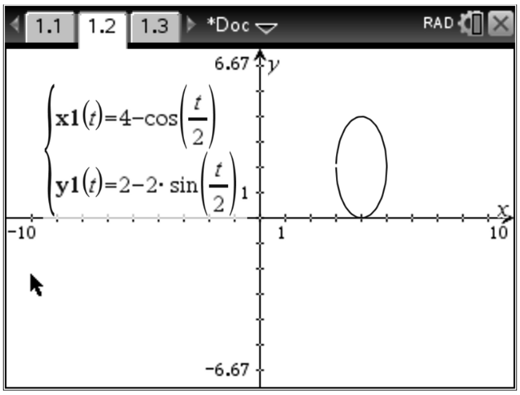
$\Rightarrow -(x-4) = \cos\left(\frac{t}{2}\right)$ and $\frac{y-2}{-2} = \sin\left(\frac{t}{2}\right)$

If $\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right) = 1$, then the cartesian equation of drone S can be modelled by the ellipse

$$(x-4)^2 + \frac{(y-2)^2}{4} = 1.$$

Alternatively, using the parametric graphing feature on the CAS, you could graph the parametric equations to get an ellipse and be able to read the cartesian equation from the graph.





Test some values of t to determine the direction of movement.

When $t = 0$, $\underset{\sim}{r}_S(0) = (4 - \cos(0))\underset{\sim}{i} + (2 - 2\sin(0))\underset{\sim}{j} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

When $t = \dfrac{\pi}{2}$:

$$\underset{\sim}{\mathrm{r}}_{\mathrm{S}}\left(\frac{\pi}{4}\right) = \left(4 - \cos\left(\frac{\pi}{4}\right)\right)\underset{\sim}{\mathrm{i}} + \left(2 - 2\sin\left(\frac{\pi}{4}\right)\right)\underset{\sim}{\mathrm{j}} = \left(4 - \frac{\sqrt{2}}{2}\right)\underset{\sim}{\mathrm{i}} + \left(2 - \frac{\sqrt{2}}{2}\right)\underset{\sim}{\mathrm{j}}$$

$$\approx 3.29\underset{\sim}{\mathrm{i}} + 1.29\underset{\sim}{\mathrm{j}}$$

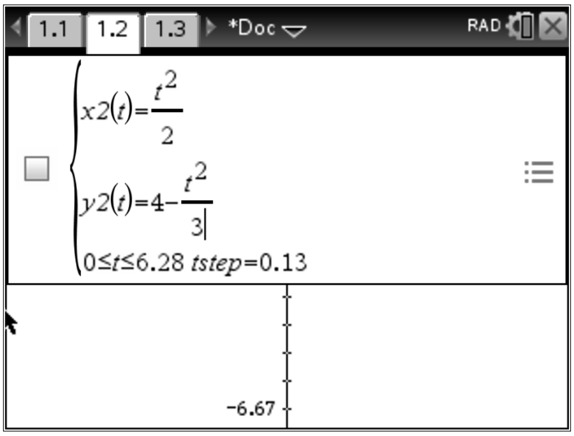
So, drone S is moving in an anticlockwise direction.

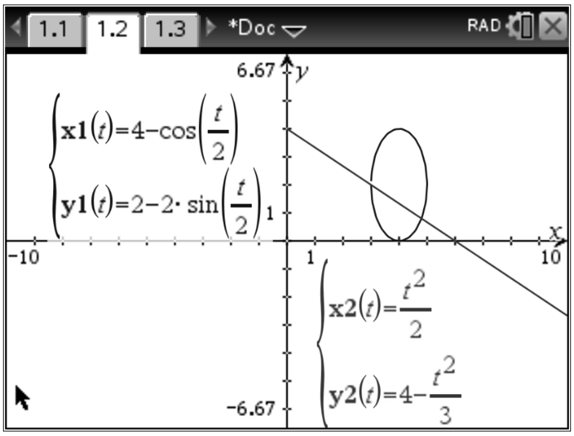
For the delivery drone:

Let $x = \dfrac{t^2}{2}$ and $y = 4 - \dfrac{t^2}{3}$, then, $2x = t^2 \;\Rightarrow\; y = 4 - \dfrac{2x}{3}$.

Therefore, the cartesian equation of drone S can be modelled by the straight line $y = 4 - \dfrac{2x}{3}$.

Alternatively, using the parametric graphing feature on the CAS, you could graph the parametric equations to get a straight line and be able to read the cartesian equation from the graph.





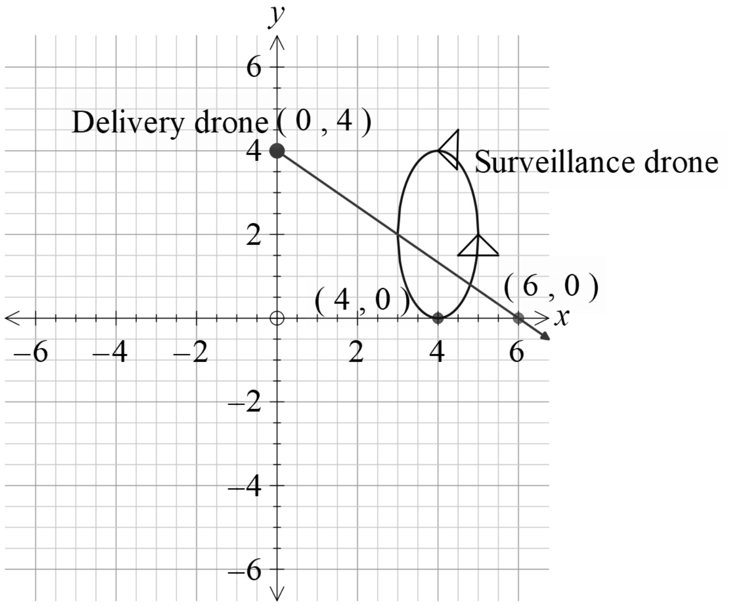
Test some values of t to determine the direction of movement.

When $t=0$: $\underset{\sim}{r}_{\text{D}}(0)=\left(\dfrac{0^2}{2}\right)\underset{\sim}{i}+\left(4-\dfrac{0^2}{3}\right)\underset{\sim}{j}=0\underset{\sim}{i}+4\underset{\sim}{j}$

When $t=\dfrac{\pi}{2}$: $\underset{\sim}{r}_{\text{S}}(1)=\left(\dfrac{1^2}{2}\right)\underset{\sim}{i}+\left(4-\dfrac{1^2}{3}\right)\underset{\sim}{j}=\dfrac{1}{2}\underset{\sim}{i}+\dfrac{11}{3}\underset{\sim}{j}$

Drone D is moving in in the positive x-direction and negative y-direction.

The graph is shown below.



Mark allocation: 3 marks

- 1 mark for showing correct curves with labels
- 1 mark for showing the direction of movement of the drones
- 1 mark for correctly labelling intercepts

Question 4b.

Worked solution

The angle between the drones is given by

$$\theta = \cos^{-1}\left(\frac{\underset{\sim}{v}_S \cdot \underset{\sim}{v}_D}{|\underset{\sim}{v}_S||\underset{\sim}{v}_D|}\right) \text{ at } t = \sqrt{6}.$$

So

$$\underset{\sim}{v}_S = \frac{\sin\left(\frac{t}{2}\right)}{2}\underset{\sim}{i} - \cos\left(\frac{t}{2}\right)\underset{\sim}{j}$$

$$\underset{\sim}{v}_D = t\underset{\sim}{i} - \frac{2t}{3}\underset{\sim}{j}$$

Therefore

$$\theta = \cos^{-1}\left(\frac{\left(\frac{1}{2}\sin\left(\frac{\sqrt{6}}{2}\right)\underset{\sim}{i} - \cos\left(\frac{\sqrt{6}}{2}\right)\underset{\sim}{j}\right) \bullet \left(\sqrt{6}\underset{\sim}{i} - \frac{2\sqrt{6}}{3}\underset{\sim}{j}\right)}{\left|\frac{1}{2}\sin\left(\frac{\sqrt{6}}{2}\right)\underset{\sim}{i} - \cos\left(\frac{\sqrt{6}}{2}\right)\underset{\sim}{j}\right|\left|\sqrt{6}\underset{\sim}{i} - \frac{2\sqrt{6}}{3}\underset{\sim}{j}\right|}\right)$$

$$= 2.106\ldots$$

$$= 2.1°$$

Mark allocation: 2 marks

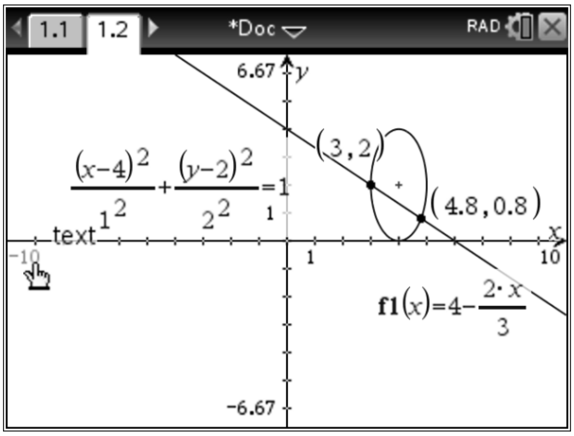
- 1 mark for finding the velocity vectors
- 1 mark for an angle of $2.1°$, correct to one decimal place

Question 4c.

Worked solution

Both drones must be shown to go through the same point but at different times.

It can be found that the curves of the drones intersect at the points (3, 2) and (4.8, 0.8).



Take the points and find the value of t for which the delivery drone is at those positions. Then substitute this value of t into the position vector of drone S to show that the drones cross paths but don't collide.

So, for point (3, 2):

$$\underset{\sim}{r}_D(t) = \frac{t^2}{2}\underset{\sim}{i} + \left(4 - \frac{t^2}{3}\right)\underset{\sim}{j} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$$

$$\Rightarrow t = \sqrt{6}$$

Then, the position of drone S at $t = \sqrt{6}$ is

$$\underset{\sim}{r}_S(\sqrt{6}) = \left(4 - \cos\left(\frac{\sqrt{6}}{2}\right)\right)\underset{\sim}{i} + \left(2 - 2\sin\left(\frac{\sqrt{6}}{2}\right)\right)\underset{\sim}{j}$$

$$\approx 3.66\underset{\sim}{i} + 0.12\underset{\sim}{j}$$

Then, for point (4.8, 0.8)

$$\underset{\sim}{r}_D(t) = \frac{t^2}{2}\underset{\sim}{i} + \left(4 - \frac{t^2}{3}\right)\underset{\sim}{j} = 4.8\underset{\sim}{i} + 0.8\underset{\sim}{j}$$

$$\Rightarrow t = \sqrt{9.6}$$

Then, the position of drone S at $t = \sqrt{9.6}$ is

$$\underset{\sim}{r}_S(\sqrt{9.6}) = \left(4 - \cos\left(\frac{\sqrt{9.6}}{2}\right)\right)\underset{\sim}{i} + \left(2 - 2\sin\left(\frac{\sqrt{9.6}}{2}\right)\right)\underset{\sim}{j}$$

$$\approx 3.98\underset{\sim}{i} + 0.0004\underset{\sim}{j}$$

Therefore, the paths of each drone intersect, as the points (3, 2) and (4.8, 0.8) are common to each path. However, the drones are not at these points at the same time and therefore do not collide.

Mark allocation: 3 marks

- 1 mark for finding the positions that the drones have in common
- 1 mark for showing that the drones are not at (3, 2) at the same time
- 1 mark for showing that the drones are not at (4.8, 0.8) at the same time

Question 4d.

Worked solution

The speed of the surveillance drone is given by

$$|\underset{\sim}{v}_S| = \left|\frac{1}{2}\sin\left(\frac{t}{2}\right)\underset{\sim}{i} - \cos\left(\frac{t}{2}\right)\underset{\sim}{j}\right|$$

$$= \sqrt{\left(\frac{1}{2}\sin\left(\frac{t}{2}\right)\right)^2 + \left(-\cos\left(\frac{t}{2}\right)\right)^2}$$

$$= \sqrt{\frac{1}{4}\sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right)} = \sqrt{\frac{1}{4} + \frac{3}{4}\cos^2\left(\frac{t}{2}\right)}$$

As the maximum value of $\cos^2\left(\frac{t}{2}\right) = 1$

$$\Rightarrow \underset{\sim}{v}_S = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 \text{ km min}^{-1}$$

Mark allocation: 2 marks

- 1 mark for showing $\sqrt{\frac{1}{4}\sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right)}$ or similar
- 1 mark for a maximum speed

Question 4e.

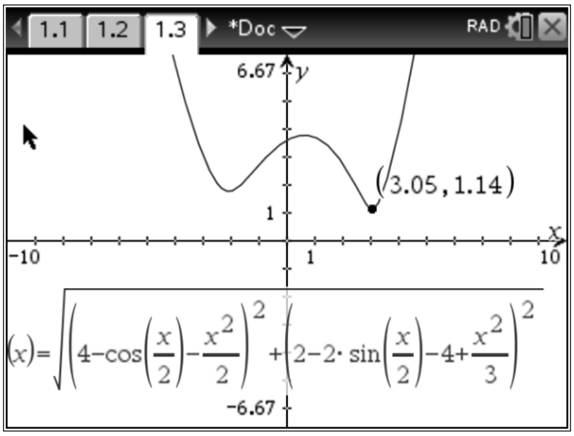
Worked solution

The distance between the two drones is

$$\left|\underset{\sim}{r}_S - \underset{\sim}{r}_D\right| = \left|\left(4-\cos\left(\frac{t}{2}\right)-\frac{t^2}{2}\right)\underset{\sim}{i} + \left(2-2\sin\left(\frac{t}{4}\right)-4+\frac{t^2}{3}\right)\underset{\sim}{j}\right|$$

$$= \sqrt{\left(4-\cos\left(\frac{t}{2}\right)-\frac{t^2}{2}\right)^2 + \left(2-2\sin\left(\frac{t}{4}\right)-4+\frac{t^2}{3}\right)^2}$$

Graphing the distance equation and finding the minimum point yields a minimum distance of 1.1 km at a time of 3.1 minutes.

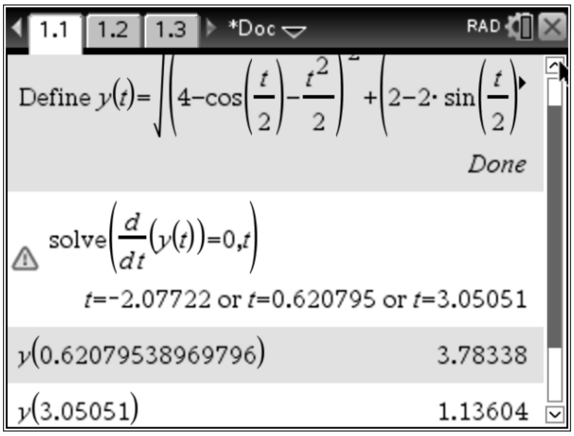


Alternatively, using calculus:

Let y be the distance between the drones, then

$$y = \sqrt{\left(4-\cos\left(\frac{t}{2}\right)-\frac{t^2}{2}\right)^2 + \left(2-2\sin\left(\frac{t}{4}\right)-4+\frac{t^2}{3}\right)^2}$$

Now, differentiate the distance and set the derivative equal to zero to find the time, and then substitute the times into the distance equation to find the minimum distance.



Therefore, the minimum distance is 1.1 km at a time of 3.1 minutes.

Mark allocation: 2 marks

- 1 mark for setting up the distance relationship

  $$\left|\underset{\sim}{\mathrm{r}}_{\mathrm{S}}-\underset{\sim}{\mathrm{r}}_{\mathrm{D}}\right|=\sqrt{\left(4-\cos\left(\frac{t}{2}\right)-\frac{t^2}{2}\right)^2+\left(2-2\sin\left(\frac{t}{4}\right)-4+\frac{t^2}{3}\right)^2}$$

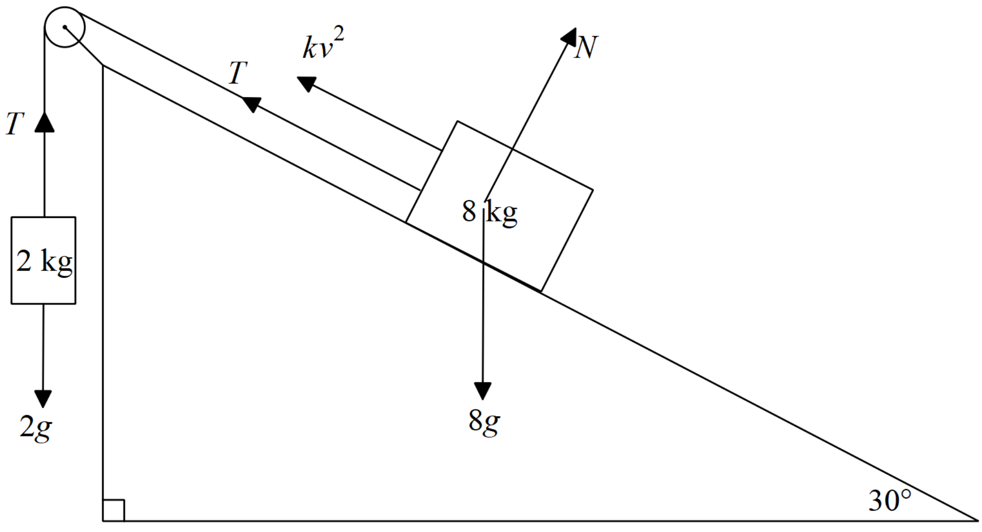
- 1 mark for a minimum distance of 1.1 km at a time of 3.1 minutes

Tip

- *When finding the maximum/minimum distance between two moving objects, graphing the distance formula is a useful way of visualising where or what is the maximum/minimum point and thus finding the required value.*

Question 5a.

Worked solution



Mark allocation: 1 mark

- 1 mark for showing all forces acting on the masses.

Tip

- *When drawing and labelling the forces on a diagram, write the weight forces with the value of the mass. For instance, don't leave the force, as mg if the value of the mass is known.*

Question 5bi.

Worked solution

For the 8 kg mass, the equation of motion parallel to the direction of travel is:

$$\sum F = ma$$

$$\Rightarrow 8g\sin(30°) - T - kv^2 = 8a$$

$$4g - T - kv^2 = 8a$$

For the 2 kg mass, the equation of motion parallel to the direction of travel is:

$$\sum F = ma$$

$$\Rightarrow T - 2g = 2a$$

Mark allocation: 2 marks

- 1 mark for $4g - T - kv^2 = 8a$
- 1 mark for $T - 2g = 2a$

Question 5b.ii.

Worked solution

Adding the two equations of motion gives

$$(4g - T - kv^2) + (T - 2g) = 8a + 2a$$
$$2g - kv^2 = 10a$$
$$a = \frac{2g - kv^2}{10}$$

Mark allocation: 2 marks

- 1 mark for adding together the two equations of motion from **part b.i.**
- 1 mark for $a = \frac{2g - kv^2}{10}$

Question 5c.

Worked solution

When the velocity is constant, then the acceleration is 0 ms^{-2}.

$$0 = \frac{2g - kv^2}{10} = \frac{2g - k \times 10^2}{10}$$
$$0 = 2g - k \times 10^2$$
$$k = \frac{2g}{10^2} = \frac{g}{50} = 0.196$$

Mark allocation: 1 mark

- 1 mark for $k = \frac{g}{50}$ or $k = 0.196$

Question 5d.

Worked solution

Start by expressing the acceleration as $a = v\dfrac{dv}{dx}$.

Then

$$\frac{2g - 0.2v^2}{10} = v\frac{dv}{dx}$$
$$\Rightarrow \int dx = \int \frac{10v}{2g - 0.2v^2}\,dv$$
$$x + c = -25\ln\left(2g - 0.2v^2\right),$$

where c is a constant of integration.

Rearranging to make v^2 the subject gives

$$v^2 = 10g - 5e^{\frac{x+c}{-25}} = 10g - 5e^{\frac{c}{25}}e^{-\frac{x}{25}}$$

Letting $A = 5e^{\frac{c}{25}}$ now gives $v^2 = 10g - Ae^{-\frac{x}{25}}$.

Applying the initial conditions $x = 0,\ v = \sqrt{96}$ gives

$$\sqrt{96}^2 = 10g - Ae^{-\frac{0}{25}}$$
$$\Rightarrow A = 2$$

So

$$v^2 = 10g - 2e^{-\frac{x}{25}}$$
$$\Rightarrow v = \pm\sqrt{10g - 2e^{-\frac{x}{25}}}$$

Rejecting the negative branch (as the initial velocity was positive) gives

$$v = \sqrt{10g - 2e^{-\frac{x}{25}}}$$

Mark allocation: 3 marks

- 1 mark for setting up the separation of variables integration of $\int dx = \int \frac{10v}{2g - 0.2v^2}\, dv$
- 1 mark for applying the initial conditions to find constants of integration
- 1 mark for $v = \sqrt{10g - 2e^{-\frac{x}{25}}}$

Tip

- *The following is a useful way to determine which version of the variable acceleration formula to use in a question:*
  - *When* $a = f(x)$ *and the initial conditions are in terms of x and v, use* $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.
  - *When* $a = f(v)$ *and the initial conditions are in terms of x and v, use* $a = v\frac{dv}{dx}$.

Question 6a.

Worked solution

$H_0 : \mu = 270$ cm

$H_1 : \mu > 270$ cm

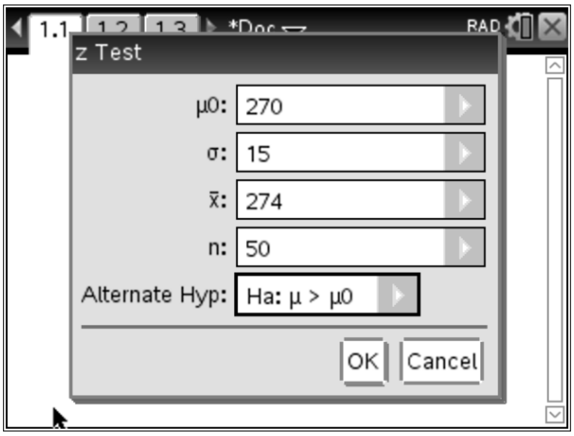
Mark allocation: 1 mark

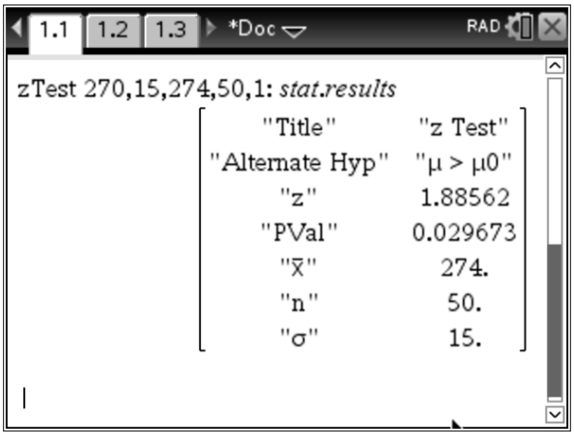
- 1 mark for stating the correct hypotheses

Question 6b.

Worked solution

p value $= \Pr(\bar{X} > 274 \mid \mu = 270) = 0.0297$, correct to four decimal places.





As $p < 0.05$, then H_0 should be rejected at the 5% significance level.

Mark allocation: 2 marks

- 1 mark for a p value of 0.0297
- 1 mark for stating with a reason that H_0 should be rejected

Question 6c.

Worked solution

For H_0 not to be rejected at the 5% significance level, p value > 0.05.

The critical value for a right-tail test at the 5% significance level is $z = 1.645$.

So

$$1.645 = \frac{274 - 270}{\left(\frac{15}{\sqrt{n}}\right)}$$

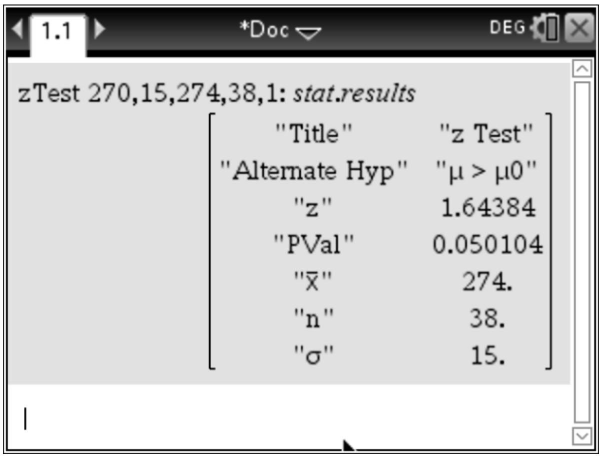
Rearranging for n gives

$$n = \left(\frac{15 \times 1.645}{274 - 270}\right)^2$$

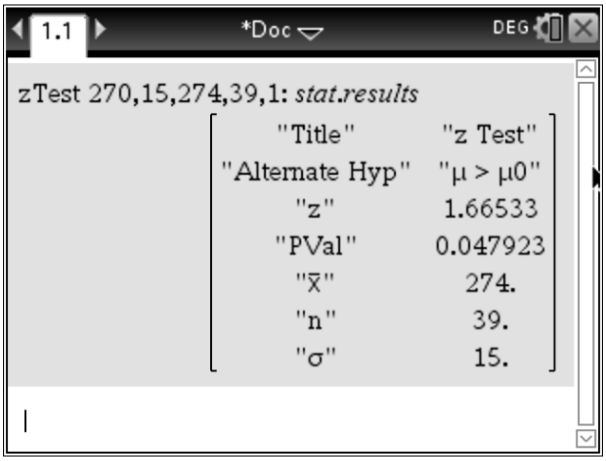
$$n = 38.0535$$

As n must be an integer, sample sizes of 38 and 39 must be checked to determine if their respective p values are greater than 0.05.

For $n = 38$, $p = 0.0501 > 0.05$.



For $n = 39$, $p = 0.0479 < 0.05$.



So, the largest sample size is 38.

Mark allocation: 2 marks

- 1 mark for $1.645 = \dfrac{274-270}{\left(\dfrac{15}{\sqrt{n}}\right)}$
- 1 mark for a sample size of 38

Tip

- *Remember that both the rounded up and rounded down answers, however unlikely one or the other may appear to be, must be checked to ensure that they meet the requirements of the question.*

Question 6d.

Worked solution

that the fertiliser increases the height of mature sunflowers when it actually has no effect

Mark allocation: 1 mark

- 1 mark for describing the correct conclusion

Question 6e.

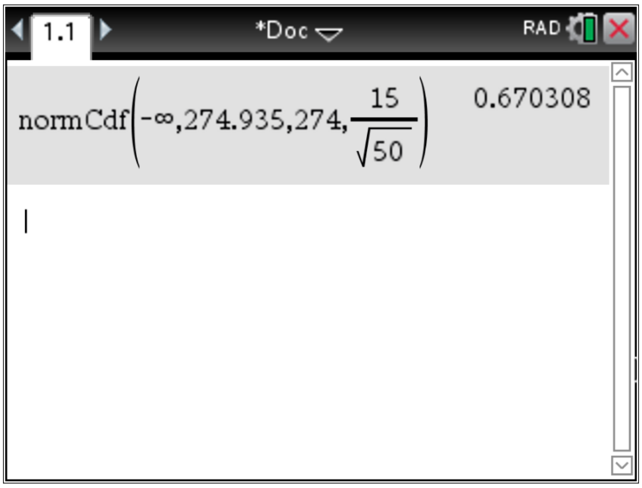
A type II error is failing to reject H_0 when it is false, which would mean selecting a sample mean less than 274.935 cm.

So

$\Pr(\text{type II error}) = \Pr(\text{accept } H_0 \text{ when false} \mid H_1 = \mu)$

$= \Pr(\bar{X} < 274.935 \mid \mu = 274)$

$= 0.6703$



Mark allocation: 2 marks

- 1 mark for $\Pr(\text{type II error}) = \Pr(\bar{X} < 274.935 \mid \mu = 274)$
- 1 mark for a probability of 0.6703

END OF WORKED SOLUTIONS