The Mathematical Association of Victoria

Trial Examination 2019

SPECIALIST MATHEMATICS

Trial Written Examination 1 - SOLUTIONS

Question 1

$z = -i$ is a given solution to the equation therefore $z + i$ is a factor of $z^3 + 2iz^2 + bz - i$.

> **Note 1:** The conjugate root theorem is NOT valid because not all of the coefficients of $z^3 + 2iz^2 + bz - i$ are real. Therefore $z = i$ is NOT necessarily a solution therefore $z - i$ is NOT necessarily a factor therefore $(z+i)(z-i) = z^2 + 1$ is NOT necessarily a factor.

$z^3 + 2iz^2 + bz - i = (z+i)\left(z^2 + az + c\right)$. .... (1)

The constant term on the left hand side of (1) is equal to $-i$ therefore $c = -1$ by inspection.

Therefore

$z^3 + 2iz^2 + bz - i = (z+i)\left(z^2 + az - 1\right)$. .... (2) **[M1]**

Equate the coefficients of z^2 on each side of (2):

$2i = i + a \qquad \Rightarrow a = i$.

Therefore

$z^3 + 2iz^2 + bz - i = (z+i)\left(z^2 + iz - 1\right)$.

Therefore

$(z+i)\left(z^2 + iz - 1\right) = 0$. **[M1]**

Apply the Null Factor Law:

$z^2 + iz - 1 = 0$. .... (3)

Solve (3) using the quadratic equation to get the other solutions:

$$z = \frac{-i \pm \sqrt{(i)^2 - 4(1)(-1)}}{2(1)} = \frac{-i \pm \sqrt{3}}{2}.$$

Answer: $z = \dfrac{-i \pm \sqrt{3}}{2}$. **[A1]**

Note 2: The value of b is not needed in order to answer this question.

Note 3: The value of b can be found by either:

1. Expanding the right hand side of $z^3 + 2iz^2 + bz - i = (z+i)\left(z^2 + iz - 1\right)$ and equating the coefficient of z.

2. Substituting $z = -i$ into $z^3 + 2iz^2 + bz - i = 0$ and solving for b.

Question 2

Substitute $y=0$ into $x\sin(2y)-y\cos(x)+x=\frac{\pi}{2}$: $x=\frac{\pi}{2}$.

Use implicit differentiation to differentiate $x\sin(2y)-y\cos(x)+x=\frac{\pi}{2}$ with respect to *x*:

$$\underbrace{\sin(2y)+2x\cos(2y)\frac{dy}{dx}}_{\text{From the product rule (use the chain rule to differentiate } \sin(2y) \text{ with respect to } x)} - \underbrace{\frac{dy}{dx}\cos(x)+y\sin(x)}_{\text{From the product rule}} +1=0\,.$$

$$\sin(2y)+2x\cos(2y)\frac{dy}{dx}-\frac{dy}{dx}\cos(x)+y\sin(x)+1=0\,.$$

[M1] **[M1]**

Substitute $y=0$ and $x=\frac{\pi}{2}$:

$$\sin(0)+2\left(\frac{\pi}{2}\right)\cos(0)\frac{dy}{dx}-\frac{dy}{dx}\cos\left(\frac{\pi}{2}\right)+(0)\sin\left(\frac{\pi}{2}\right)+1=0$$

$$\Rightarrow \pi\frac{dy}{dx}+1=0 \qquad \Rightarrow \frac{dy}{dx}=-\frac{1}{\pi}$$

therefore the gradient of the normal is $m=\pi$.

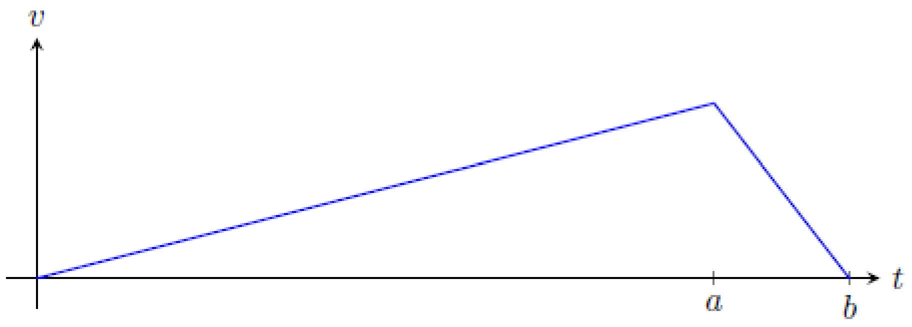
Note: A less efficient approach is to first solve for $\frac{dy}{dx}$ in terms of *y* and *x* using algebra and then to substitute $y=0$ and $x=\frac{\pi}{2}$.

Answer: $m=\pi$. **[A1]**

Question 3

For minimum travel time the train should accelerate at $0.3\ \text{ms}^{-2}$ for a seconds and then slow down at $-2.4\ \text{ms}^{-2}$ until coming to a stop at the second station at $t = b$.

Drawing a simple velocity-time graph is useful to represent this situation:



Therefore the minimum time is the value of b.

When the train is accelerating at $0.3\ \text{ms}^{-2}$: $v = 0.3t$.

When the train is slowing down: $v = -2.4(t - b)$.

At $t = a$:

$0.3a = -2.4(a - b) \quad \Rightarrow 2.7a = 2.4b \quad \Rightarrow 27a = 24b$

$\Rightarrow 9a = 8b$. .... (1) **[M1]**

The area under the graph is 3000 and is the sum of the area of a triangle with base b and height $v = 0.3a$:

$3000 = \frac{1}{2}b(0.3a) \quad \Rightarrow 6000 = 0.3ab$

$\Rightarrow 20,000 = ab$. .... (2) **[M1]**

Solve (1) and (2) simultaneously for b.

Substitute (1) into (2):

$20,000 = \frac{8}{9}b^2$ **[M1]**

$\Rightarrow 2500 = \frac{1}{9}b^2 \quad \Rightarrow b^2 = 9 \times 2500 \quad \Rightarrow b = 3 \times 50 = 150$.

Minimum travel time is 150 seconds.

Answer: 150. **[A1]**

Question 4

• Let X be the random variable "*Mass (grams) of an apricot*".

• $X \sim \text{Normal}\left(\mu_X = 10,\ \ \sigma_X = 2\right)$.

• Let the number of apricots in a paper bag be n.

• Let W be the random variable "*Sum of mass (grams) of n apricots*".

$W = X_1 + X_2 + \cdots + X_n$

where $X_1,\ X_2,\ \ldots\ X_n$ are independent copies of X.

Note: Using the random variable nX is incorrect: $X_1 + X_2 + \cdots + X_n \neq nX$.

• W follows a normal distribution since $X_1,\ X_2,\ \ldots\ X_n$ are independent normal random variables:

• $\text{E}(W) = \mu_W = \mu_{X_1} + \mu_{X_2} + \cdots \mu_{X_n} = n\mu_X = 10n$.

• $\text{Var}(W) = \text{Var}(X_1) + \text{Var}(X_2) + \cdots \text{Var}(X_n) \qquad = n\text{Var}(X) \qquad = n(2)^2$

$\Rightarrow \text{sd}(W) = \sigma_W = 2\sqrt{n}$.

Therefore:

$W \sim \text{Normal}\left(\mu_W = 10n,\ \ \sigma_W = 2\sqrt{n}\right)$. **[M1]**

• The largest value of n such that $\Pr(W < 120) > 0.84$ is required.

• $Z = \dfrac{W - \mu_W}{\sigma_W}$ and it is given that $\Pr(Z < 1) = 0.84$.

Therefore:

$1 = \dfrac{120 - 10n}{2\sqrt{n}}$ **[M1]**

$= \dfrac{60 - 5n}{\sqrt{n}}$

$\Rightarrow \sqrt{n} = 60 - 5n$

$\Rightarrow n = (60 - 5n)^2 = 3600 - 600n + 25n^2$

$\Rightarrow 25n^2 - 601n + 3600 = 0$.

Compare with $25n^2 + an + b = 0$.

Answer: $a = -601,\ \ b = 3600$. **[A1]**

Question 5

Let the area, base and altitude of the triangle at time t be $A(t)$, $b(t)$ and $h(t)$ respectively:

$$A(t)=\frac{1}{2}b(t)h(t) \qquad \text{.... (1)}$$

$$\Rightarrow \frac{dA}{dt}=\frac{1}{2}\underbrace{\left(b\frac{dh}{dt}+h\frac{db}{dt}\right)}_{\text{From the product rule}}. \qquad \text{.... (2)}$$

$$\frac{dA}{dt}=\frac{1}{2}\left(b\frac{dh}{dt}+h\frac{db}{dt}\right).$$ **[M1]**

Substitute $\frac{dh}{dt}=-2$ and $\frac{dA}{dt}=3$ into (2):

$$3=\frac{1}{2}\left(-2b+h\frac{db}{dt}\right). \qquad \text{.... (3)}$$

Substitute $h=10$ and $A=100$ into (1):

$$100=\frac{1}{2}(10)b \qquad \Rightarrow b=20.$$

Substitute $h=10$ and $b=20$ into (3):

$$3=\frac{1}{2}\left(-2(20)+10\frac{db}{dt}\right)$$ **[M1]**

$$\Rightarrow 3=-20+5\frac{db}{dt}$$

$$\Rightarrow \frac{db}{dt}=\frac{23}{5}\text{ cm/min.}$$

Answer: $\frac{23}{5}$ cm/min. **[A1]**

Unit is not required.

Question 6

a.

From the components of $\underset{\sim}{r}(t)=\left(t^2-2t\right)\underset{\sim}{i}+(t-1)\underset{\sim}{j}$:

$x=t^2-2t$. .... (1)

$y=t-1 \quad \Rightarrow t=y+1$. .... (2)

Substitute (1) into (2):

$x=(y+1)^2-2(y+1)$ **[M1]**

$=y^2+2y+1-2y-2 \quad =y^2-1$.

$x=y^2-1$ is a sideways parabola.

Given domain is $0\le t\le 2$.

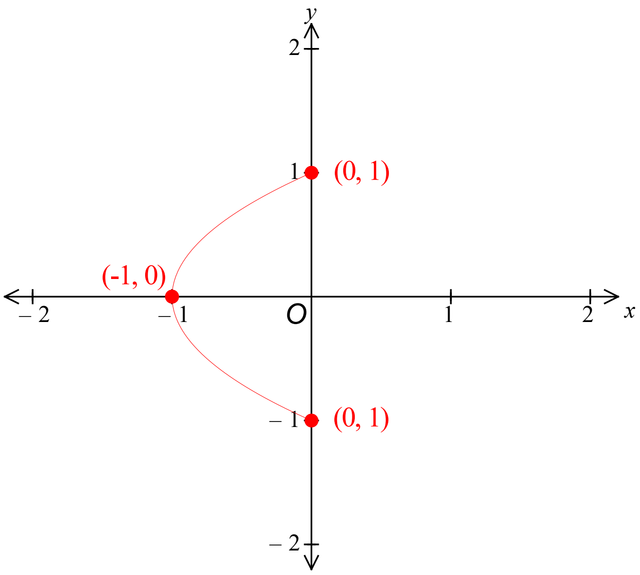
Therefore $-1\le y\le 1$. **[M1]**

Key features:

Vertex: $(-1,\ 0)$.

y-intercepts: $(0,\ -1)$ and $(0,\ 1)$.

Answer:



[A1]

Vertex and y-intercepts must be labelled.

b.

Velocity: $\frac{d\underset{\sim}{r}}{dt} = (2t-2)\underset{\sim}{i} + \underset{\sim}{j}$.

Acceleration: $\frac{d^2\underset{\sim}{r}}{dt^2} = 2\underset{\sim}{i}$.

Require $\frac{d\underset{\sim}{r}}{dt} \cdot \frac{d^2\underset{\sim}{r}}{dt^2} = 0$:

$\left((2t-2)\underset{\sim}{i} + \underset{\sim}{j}\right) \cdot 2\underset{\sim}{i} = 0$ **[M1]**

$\Rightarrow 2(2t-2) = 0$

$\Rightarrow t = 1$.

Substitute $t = 1$ into $x = t^2 - 2t$ and $y = t - 1$:

$x = -1$, $y = 0$.

Answer: $(-1, 0)$. **[A1]**

Question 7

a.

$$\text{Left Hand Side} = \frac{\sec^2\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}{\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}$$

$$= \frac{1}{\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}$$

$$= \frac{2}{2\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}$$ **[M1]**

Use the double angle formula $\underbrace{\sin(2x) = 2\sin(x)\cos(x)}_{\text{See VCAA Formula Sheet}}$ with $x = \frac{\theta}{2}+\frac{\pi}{4}$:

$$= \frac{2}{\sin\left(2\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right)}$$

$$= \frac{2}{\sin\left(\theta+\frac{\pi}{2}\right)}$$ **[M1]**

$$= \frac{2}{\cos(\theta)}$$

$= 2\sec(\theta) = \text{Right Hand Side.}$

Deduct 1 mark for poor setting out. Must have LHS = ... = RHS

b.

From the VCAA formula sheet:

Arc length $L = \int_0^{\frac{\pi}{6}} \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx$.

$y = \log_e\left(\cos(x)\right)$

therefore $\frac{dy}{dx} = -\frac{\sin(x)}{\cos(x)} = -\tan(x)$.

Therefore

$$L = \int_0^{\frac{\pi}{6}} \sqrt{\left(-\tan(x)\right)^2 + 1}\, dx$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{\tan^2(x) + 1}\, dx$$

$$= \int_0^{\frac{\pi}{6}} \sqrt{\sec^2(x)}\, dx \qquad = \int_0^{\frac{\pi}{6}} |\sec(x)|\, dx$$

$$= \int_0^{\frac{\pi}{6}} \sec(x)\, dx$$ **[M1]**

since $\sec(x) > 0$ for $x \in \left[0, \frac{\pi}{6}\right]$.

Substitute $2\sec(\theta) = \dfrac{\sec^2\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)}{\tan\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)}$, $\theta \in \left[0, \dfrac{\pi}{2}\right)$, from **part a.**:

$$L = \int_0^{\frac{\pi}{6}} \sec(x)\, dx = \int_0^{\frac{\pi}{6}} \frac{\sec^2\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)}{2\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)}\, dx .$$

Substitute $u = \tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$:

$$\frac{du}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}+\frac{\pi}{4}\right) \quad \Rightarrow dx = \frac{2}{\sec^2\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)}du .$$

$$x = 0 \Rightarrow u = \tan\left(\frac{\pi}{4}\right) = 1 .$$

$$x = \frac{\pi}{6} \Rightarrow u = \tan\left(\frac{\pi}{12}+\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3} .$$

Therefore

$$L = \int_1^{\sqrt{3}} \frac{\sec^2\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)}{2u} \times \frac{2}{\sec^2\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)} du$$

$$= \int_1^{\sqrt{3}} \frac{1}{u}\, du$$ **[M1]**

$$= \left[\log_e|u|\right]_1^{\sqrt{3}} \qquad = \log_e\left(\sqrt{3}\right) - \log_e(1) \qquad = \log_e\left(\sqrt{3}\right) \qquad = \log_e\left(3^{1/2}\right)$$

$$= \frac{1}{2}\log_e(3) .$$

Answer: $\dfrac{1}{2}\log_e(3)$. **[A1]**

Accept $\log_e\left(\sqrt{3}\right)$.

Question 8

a.

Let $y = \sin^{-1}\left(\frac{2}{x-1}\right)$.

Use the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Let $u = \frac{2}{x-1}$:

$$\frac{du}{dx} = -\frac{2}{(x-1)^2}.$$

$y = \sin^{-1}(u)$.

From the VCAA formula sheet:

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}.$$

Therefore

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times \left(-\frac{2}{(x-1)^2}\right).$$

Substitute $u = \frac{2}{x-1}$:

$$\frac{dy}{dx}=\frac{-2}{(x-1)^2\sqrt{1-\left(\frac{2}{x-1}\right)^2}}$$ **[M1]**

$$=\frac{-2}{(x-1)^2\sqrt{\frac{(x-1)^2-4}{(x-1)^2}}}$$

$$=\frac{-2\,|x-1|}{(x-1)^2\sqrt{(x-1)^2-4}}$$ **[M1]**

since $\sqrt{(x-1)^2}=|x-1|$

$$=\frac{-2\,|x-1|}{|x-1|^2\sqrt{(x-1-2)(x-1+2)}}$$

since $(x-1)^2=|x-1|^2$

$$=\frac{-2}{|x-1|\sqrt{(x-3)(x+1)}}$$

where $k(x)=|x-1|$, $a=-2$, $b=-3$, $c=1$.

Answer: $\dfrac{-2}{|x-1|\sqrt{(x-3)(x+1)}}$. **[A1]**

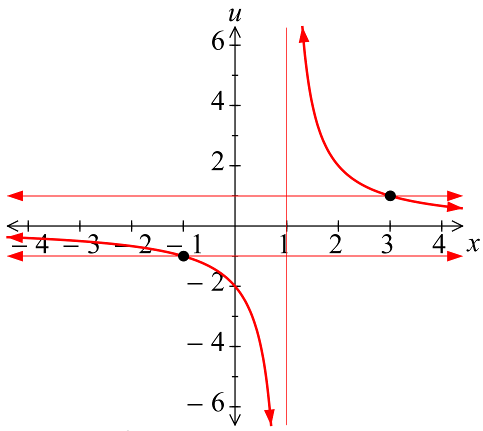
b.

Let $u = \dfrac{2}{x-1}$.

Then $y = \sin^{-1}(u)$.

Maximal domain *D*: Solve $-1 \le u \le 1$.

Use a simple graph of $u = \dfrac{2}{x-1}$:



Solve $u = 1$: $\dfrac{2}{x-1} = 1 \quad \Rightarrow x = 3$.

Solve $u = -1$: $\dfrac{2}{x-1} = -1 \quad \Rightarrow x = -1$.

[M1]

From the graph of $u = \dfrac{2}{x-1}$ it can be seen that the values of x that solve $-1 \le u \le 1$ are

$x \in (-\infty, \ -1] \cup [3, \ +\infty)$.

Answer: $x \in (-\infty, \ -1] \cup [3, \ +\infty)$. **[A1]**

Range:

• From the graph of $u = \dfrac{2}{x-1}$ it can be seen that only the subset $[-1, \ 0) \cup (0, \ 1]$ of the possible values $-1 \le u \le 1$ are used.

Therefore the range is $\left[-\dfrac{\pi}{2}, \ 0\right) \cup \left(0, \ \dfrac{\pi}{2}\right]$.

Answer: $\left[-\dfrac{\pi}{2}, \ 0\right) \cup \left(0, \ \dfrac{\pi}{2}\right]$ **[A1]**

c.

- Shape: **[A1]**

  From **part a.** $\frac{dy}{dx} = \frac{-2}{|x-1|\sqrt{(x-3)(x+1)}} < 0$ over D

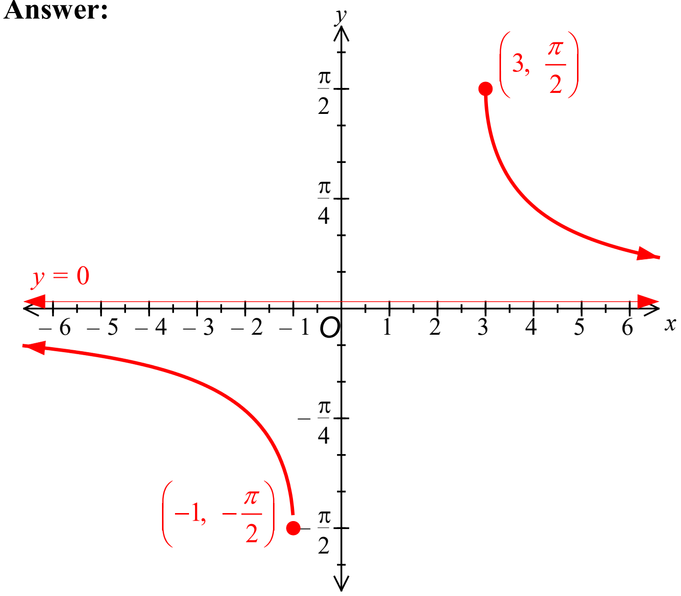
  therefore $y = \sin^{-1}\left(\frac{2}{x-1}\right)$ is a decreasing function over D.

  The maximal domain is $x \in (-\infty, \ -1] \cup [3, \ +\infty)$.

- Horizontal asymptote: $y = 0$. **[A1]**

- Endpoints: $\left(-1, \ -\frac{\pi}{2}\right)$ and $\left(3, \ \frac{\pi}{2}\right)$. **[A1]**

Answer:



d.

Background theory:

• The concavity of the solution curve over the interval $x \in [4, 5]$ is required:

• If the solution curve is **concave down** over the interval $x \in [4, 5]$ then Euler's Method **over**estimates the exact value of y.

• If the solution curve is **concave up** over the interval $x \in [4, 5]$ then Euler's Method **under**estimates the exact value of y.

• $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave up $\Rightarrow$ underestimate.

• $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave down $\Rightarrow$ overestimate.

The sign of $\frac{d^2y}{dx^2}$ for $x \in [4, 5]$ is therefore required.

There are two possible explanations that can be give:

Explanation 1: From the given differential equation: $\frac{dy}{dx} = \sin^{-1}\left(\frac{2}{x-1}\right)$ therefore $\frac{d^2y}{dx^2}$ gives the gradient of the graph of $y = \sin^{-1}\left(\frac{2}{x-1}\right)$.

By inspection of the graph in **part c.**: $\frac{d^2y}{dx^2} < 0$ for D. Therefore the solution curve is **concave down** over the interval $x \in [4, 5]$ therefore Euler's method gives an **over**estimate of the value of y when $x = 5$. **[A1]**

Explanation 2: From the given differential equation: $\frac{dy}{dx} = \sin^{-1}\left(\frac{2}{x-1}\right)$ therefore

$\frac{d^2y}{dx^2} = \frac{-2}{|x-1|\sqrt{(x-3)(x+1)}}$ from **part a.**

$\frac{-2}{|x-1|\sqrt{(x-3)(x+1)}} < 0$ for D therefore $\frac{d^2y}{dx^2} < 0$ for D.

Therefore the solution curve is **concave down** over the interval $x \in [4, 5]$ therefore Euler's method gives an **over**estimate of the value of y when $x = 5$. **[A1]**

Question 9

a.

$$I = \int \frac{e^{\frac{x}{2}}}{\sqrt{3e^{-x} - e^{x} + 4}}\, dx$$

Substitute $u = e^x$: $u = e^x \quad \Rightarrow \frac{du}{dx} = e^x = u \quad \Rightarrow dx = \frac{1}{u} du$.

$$I = \int \frac{u^{\frac{1}{2}}}{\sqrt{\frac{3}{u} - u + 4}}\, \frac{1}{u} du$$ **[M1]**

$$= \int \frac{u^{\frac{1}{2}}}{\sqrt{\frac{3 - u^2 + 4u}{u}}}\, \frac{1}{u} du$$

$$= \int \frac{u^{\frac{1}{2}}}{\frac{\sqrt{3 - u^2 + 4u}}{\sqrt{u}}}\, \frac{1}{u} du$$ **[M1]**

$$= \int \frac{u}{\sqrt{3 + 4u - u^2}}\, \frac{1}{u} du$$

which was to be shown.

b.

$I = \int \frac{1}{\sqrt{3+4u-u^2}}\,du$ (from **part a.**)

$= \int \frac{1}{\sqrt{7-(u-2)^2}}\,du$ **[M1]**

$= \sin^{-1}\left(\frac{u-2}{\sqrt{7}}\right) + c$.

Substitute $u = e^x$: $I = \sin^{-1}\left(\frac{e^x-2}{\sqrt{7}}\right) + c$.

Answer: $I = \sin^{-1}\left(\frac{e^x-2}{\sqrt{7}}\right) + c$. **[A1]**

The arbitrary constant is required.

END OF SOLUTIONS