Neap:

Trial Examination 2019

VCE Specialist Mathematics Units 3&4

Written Examination 2

Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

Question	Answer	Question	Answer
1	B	11	E
2	E	12	D
3	B	13	A
4	C	14	A
5	B	15	C
6	A	16	D
7	A	17	E
8	C	18	C
9	E	19	D
10	D	20	B

Question 1 **B**

$$-\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}$$

$$-\frac{b\pi}{2} < b\tan^{-1}(x) < \frac{b\pi}{2}$$

$$a - \frac{b\pi}{2} < a + b\tan^{-1}(x) < a + \frac{b\pi}{2}$$

$$a - \frac{b\pi}{2} < a + b\tan^{-1}(cx + d) < a + \frac{b\pi}{2}$$

So the graph of $y = a + b\tan^{-1}(cx + d)$ has asymptotes given by $y = a \pm \frac{b\pi}{2}$.

Question 2 **E**

$$\frac{dy}{dx} = \frac{m\sin(m(x-n))}{\cos^2(m(x-n))}$$

$$\frac{dy}{dx} = 0 \Rightarrow m\sin(m(x-n)) = 0$$

$$m(x-n) = 0, \pi, 2\pi, \ldots$$

$$x - n = 0, \frac{\pi}{m}, \frac{2\pi}{m}, \ldots (m > 0)$$

$$x = n, \frac{\pi}{m} + n, \frac{2\pi}{m} + n, \ldots$$

Question 3 **B**

$$\sin^2(x) + \frac{9}{16} = 1$$

$$\sin(x) = \pm\frac{\sqrt{7}}{4}$$

$$\text{cosec}(x) = \pm\frac{4}{\sqrt{7}}$$

As $\sin(x)$ is negative in the third quadrant, so is $\text{cosec}(x)$.

Hence, $\text{cosec}(x) = -\frac{4}{\sqrt{7}}$.

Question 4 **C**

The use of a proper fraction command of a CAS gives $\frac{4x^2 - 5x - 6}{(4x+3)^3(x^2-4)} = \frac{1}{(4x+3)^2(x+2)}$.

This fraction contains the repeated factor $(4x+3)^2$ and so in partial fraction form we have

$$\frac{A}{4x+3} + \frac{B}{(4x+3)^2} + \frac{C}{x+2}.$$

Question 5 B

The centre of the circle is at the midpoint of *AB*.

$$\frac{1-7i+3-i}{2} = 2-4i$$

$$\overline{AB} = \sqrt{(3-1)^2 + (-1+7)^2}$$
$$= 2\sqrt{10}$$

So the radius of *C* is $\sqrt{10}$.

The equation of *C* is $|z-(2-4i)| = \sqrt{10}$; that is, $|z-2+4i| = \sqrt{10}$.

Question 6 A

$$1+z = (1+\cos(\theta)) + i\sin(\theta)$$
$$= 2\cos^2\left(\frac{\theta}{2}\right) + 2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$
$$= 2\cos\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)$$
$$= 2\cos\left(\frac{\theta}{2}\right)\text{cis}\left(\frac{\theta}{2}\right)$$

Question 7 A

Let $u = 2x^3 + 1$.

$$\frac{du}{dx} = 6x^2 \Rightarrow \frac{1}{6}\frac{du}{dx} = x^2$$

$$\int x^2\sqrt{2x^3+1}\,dx = \frac{1}{6}\int \sqrt{u}\frac{du}{dx}dx$$
$$= \frac{1}{6}\int \sqrt{u}\,du$$

Question 8 **C**

The initial point is (0, 1); that is, $a = 0$ and $b = 1$.

Euler's method using a step size of 0.1 gives:

$a = 0$ $\quad f(a) = f(0) = 1$

$x_1 = 0.1$ $\quad f(x_1) = f(0.1) = e^{0.01}$

Using $y_{n+1} = y_n + hf(x_n)$:

$$\begin{aligned} y_1 &= b + hf(a) \\ &= 1 + 0.1e^0 \\ &= 1.1 \\ y_2 &= y_1 + hf(x_1) \\ &= 1.1 + 0.1e^{0.01} \end{aligned}$$

Question 9 **E**

Looking at the direction field, $\frac{dy}{dx} = 0$ for $y = 2x$.

The differential equation in **E** is $\frac{dy}{dx} = 2(y - 2x)$ and $(2(y - 2x) = 0) \Rightarrow y = 2x$.

Question 10 **D**

Let α be angle HOP.

$$\tan(\alpha) = \frac{h}{40}$$

$$\sec^2(\alpha)\frac{d\alpha}{dt} = \frac{1}{40}\frac{dh}{dt}$$

$\frac{dh}{dt} = 10$ and so $\frac{d\alpha}{dt} = \frac{1}{4\sec^2(\alpha)}$.

As $\sec(\alpha) = \frac{5}{4}$, $\frac{d\alpha}{dt} = \frac{4}{25}$.

Question 11 **E**

$\overrightarrow{QR} = 6\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$

$\overrightarrow{QP} = -2\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k}$

$$\begin{aligned} \angle PQR &= \arccos\left(\frac{(-2\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k}) \cdot (6\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k})}{\sqrt{(-2)^2 + (-6)^2 + 4^2}\sqrt{6^2 + 2^2 + (-4)^2}}\right) \\ &= \arccos\left(-\frac{5}{7}\right) \end{aligned}$$

Question 12 **D**

As $\underset{\sim}{a} = 4\underset{\sim}{b} - \underset{\sim}{c}$, each of these vectors can be expressed as a linear combination of the other two vectors. This is the definition for linearly dependent vectors.

Question 13 **A**

$$\underset{\sim}{r}'(t) = -k\sin(t)\underset{\sim}{i} + k(1 + \cos(t))\underset{\sim}{j}$$

$$\begin{aligned}|\underset{\sim}{r}'(t)| &= \sqrt{(-k\sin(t))^2 + k^2(1 + \cos(t))^2} \\ &= \sqrt{k^2\sin^2(t) + k^2 + 2k^2\cos(t) + k^2\cos(t)} \\ &= k\sqrt{2 + 2\cos(t)} \\ &= k\sqrt{4\cos^2\left(\frac{t}{2}\right)} \\ &= 2k\cos\left(\frac{t}{2}\right)\end{aligned}$$

Question 14 **A**

The parametric equations are:

$x = 2\operatorname{cosec}(t)$ (1)

$y = 2\cot(t)$ (2)

Using $1 + \cot^2(t) = \operatorname{cosec}^2(t)$, we obtain $1 + \frac{y^2}{4} = \frac{x^2}{4}$.

Hence $\frac{x^2}{4} - \frac{y^2}{4} = 1$ for $x \geq 2$ and $y \geq 0$.

Question 15 **C**

Let the normal reaction force be R.

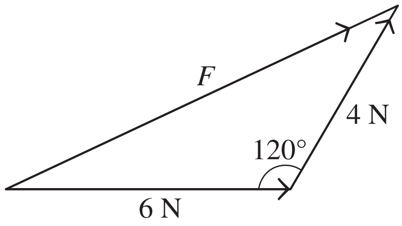
$$\sum F = ma$$

Taking the downwards direction as positive:

$$40g - R = 40 \times \frac{g}{5}$$

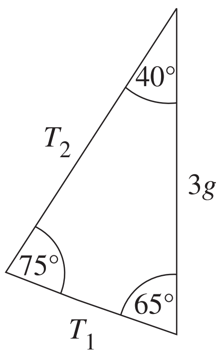
Solving for R we obtain $R = 32g$.

Question 16 D



$$F^2 = 4^2 + 6^2 - 2(4)(6)\cos(120°)$$
$$= 76$$
$$F = \sqrt{76}$$
$$a = \frac{\sqrt{76}}{2}$$
$$= \sqrt{19} \text{ (m s}^{-2}\text{)}$$

Question 17 E



Applying the sine rule to the triangle, we obtain:

$$\frac{T_1}{\sin(40°)} = \frac{T_2}{\sin(65°)} = \frac{3g}{\sin(75°)}$$

$$T_1 = \frac{3g\sin(40°)}{\sin(75°)} \text{ and } T_2 = \frac{3g\sin(65°)}{\sin(75°)}$$

Question 18 C

The p-value is the probability of observing a value of the sample statistic as extreme or more extreme than the one observed, assuming that the null hypothesis is true.

Question 19 D

An approximate 95% confidence interval for μ is $\left(\bar{x} - 1.96\frac{s}{\sqrt{n}}, \bar{x} + 1.96\frac{s}{\sqrt{n}}\right)$.

Using the z interval feature of a CAS with $\bar{x} = 497$, $s^2 = 13$ and $n = 100$, we obtain $(496.3, 497.7)$.

Question 20 B

Let P be the random variable that represents the perimeter of the manufactured part.

$P = 2L + 2W$

$\text{E}(P) = 2 \times 2 + 2 \times 5$

$= 14$

$\text{Var}(P) = 4 \times (0.1)^2 + 4 \times (0.2)^2$

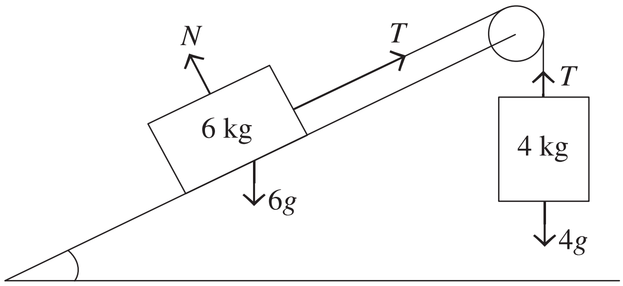
$= 0.2$

$\Pr(P > 14.5) = 0.1318$

SECTION B

Question 1 (10 marks)

a.



A1

b. $\sin(\alpha) = \frac{3}{5}$

$4g - T = 4a$ and $T - 6g\sin(\alpha) = 6a$ A1

Adding the two equations gives $4g - 6g \times \frac{3}{5} = 10a$ or equivalent. M1

So, $a = \frac{g}{25}$ (m s^{-2}).

c. $T = 4g - 4 \times \frac{g}{25}$ or $T = 6 \times \frac{g}{25} + 6g \times \frac{3}{5}$

$= \frac{96g}{25}$ (N) A1

d. **Method 1:**

$\frac{d}{ds}\left(\frac{1}{2}v^2\right) = \frac{g}{25} \Rightarrow \frac{1}{2}v^2 = \frac{gs}{25} + c$ M1

When $s = 0$, $v = 0$ and so $c = 0$.

So, $v^2 = \frac{2gs}{25} \Rightarrow v = \frac{\sqrt{2gs}}{5} (v > 0)$.

When $s = \frac{3}{2}$, $v = \frac{\sqrt{3g}}{5}$(m s^{-1}). A1

Method 2:

Use of $v^2 = u^2 + 2as$ with $u = 0$, $a = \frac{g}{25}$ and $s = \frac{3}{2}$. M1

$v = \frac{\sqrt{3g}}{5}$ (m s^{-1}) A1

e. **Method 1:**

$0 - 6g \times \frac{3}{5} = 6a \Rightarrow a = -\frac{3g}{5}$ A1

$\frac{dv}{dt} = -\frac{3g}{5} \Rightarrow v = -\frac{3gt}{5} + d$ M1

When $t = 0$, $u = \frac{\sqrt{3g}}{5}$ and so $d = \frac{\sqrt{3g}}{5}$.

Solving $0 = \frac{\sqrt{3g}}{5} - \frac{3gt}{5}$ for t gives $t = 0.1844\ldots$ M1

The required time is 0.37 (s) (correct to two decimal places). A1

Method 2:

$0 - 6g \times \frac{3}{5} = 6a \Rightarrow a = -\frac{3g}{5}$ A1

Use of $v = u + at$ with $v = 0$, $u = \frac{\sqrt{3g}}{5}$ and $a = -\frac{3g}{5}$. M1

Solving $0 = \frac{\sqrt{3g}}{5} - \frac{3g}{5}$ for t gives $t = 0.1844\ldots$ M1

The required time is 0.37 (s) (correct to two decimal places). A1

Question 2 (9 marks)

a. As $0 < n < m$, $me^x + n > 0$ for $x \in R$. A1

Hence the graph of f has no vertical asymptotes.

b. **i.** $f''(x) = \frac{(m^2 - n^2)e^x(me^x - n)}{(me^x + n)^3}$ A1

Attempting to solve $f''(x) = 0$ for x. M1

Coordinates of the point of inflection are $\left(\log_e\left(\frac{n}{m}\right), \frac{m^2 + n^2}{2mn}\right)$. A1

ii. $x < \log_e\left(\frac{n}{m}\right)$ A1

iii. As $0 < n < m$, $m^2 + n^2 > 0$ and $2mn > 0$. So $\frac{m^2 + n^2}{2mn} > 0$. A1

Hence the point of inflection on the graph of f is always above the x-axis.

c. When $m = 3$ and $n = 1, f(x) = \dfrac{e^x + 3}{3e^x + 1}$.

Solving $f(x) = \dfrac{2}{3}$ for x gives $x = \log_e\left(\dfrac{7}{3}\right)$. M1

Let the volume of the solid formed be V.

$$V = \pi\int_0^{\log_e\left(\frac{7}{3}\right)}\left(\left(\frac{e^x + 3}{3e^x + 1}\right)^2 - \left(\frac{2}{3}\right)^2\right)dx$$ M1

$= 0.625$ (correct to three decimal places) A1

Question 3 (10 marks)

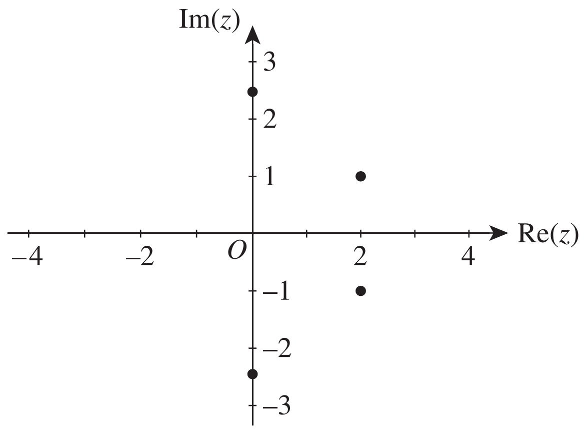
a. The coefficients of the equation are real. By the conjugate root theorem, all complex roots occur in conjugate pairs, so there must be more than one possible value for k. A1

b. $(ki)^4 - 4(ki)^3 + a(ki)^2 - 24(ki) + 30 = (k^4 - ak^2 + 30) + (4k^3 - 24k)i$ M1

Solving $4k^3 - 24k = 0$ for k gives $k = \pm\sqrt{6}$ (reject $k = 0$). A1

Substituting $k = \pm\sqrt{6}$ into $k^4 - ak^2 + 30 = 0$ gives $(\sqrt{6})^4 - a(\sqrt{6})^2 + 30 = 0 \Rightarrow a = 11$. A1

c. The roots are $\pm\sqrt{6}i,\ 2 \pm i$. A1



four roots plotted correctly A1

d. $A = \dfrac{1}{2}(2\sqrt{6} + 2)(2) = 2(\sqrt{6} + 1)$ M1 A1

e. Comparing $z^4 - 4z^3 + 11z^2 - 24z + 30 = 0$ and $1 - 4w + 11w^2 - 24w^3 + 30w^4 = 0$ we obtain $z = \dfrac{1}{w}$; that is, $w = \dfrac{1}{z}$. A1

$w = \pm\dfrac{i}{\sqrt{6}}, \dfrac{1}{5}(2 \pm i)$ A1

Question 4 (12 marks)

a. Vertical movement is denoted by $e^{0.75t}\underset{\sim}{j}$.

At $t = 2$, the vertical movement is denoted by $e^{1.5}\underset{\sim}{j}$. M1

Since $e^{1.5} > 0$, vertical movement is in a northwards direction. A1

b. At $t = 2$, $x = 4 + \int_1^2 \cos(t^4)\,dt$. M1

The particle's x-coordinate is 3.88 (correct to two decimal places). A1

c. $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{e^{0.75t}}{\cos(t^4)}$ A1

Attempting to solve $\dfrac{e^{0.75t}}{\cos(t^4)} = 3$ for t. M1

$t = 0.95$ (s) (correct to two decimal places) A1

d. $\left|\underset{\sim}{v}(t)\right| = \sqrt{\cos^2(t^4) + e^{1.5t}}$ A1

Attempting to solve $\sqrt{\cos^2(t^4) + e^{1.5t}} = 5$ for t. M1

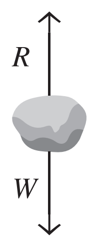
$t = 2.14$ (s) (correct to two decimal places) A1

e. $d = \int_0^1 \sqrt{\cos^2(t^4) + e^{1.5t}}\,dt$ M1

$= 1.78$ (m) (correct to two decimal places) A1

Question 5 (11 marks)

a.



A1

b. $F = ma \Rightarrow mv\dfrac{dv}{dx} = mg - \dfrac{mv^2}{1000}$ A1

Hence, $v\dfrac{dv}{dx} = g - \dfrac{v^2}{1000}$.

c. $v\frac{dv}{dx} = g - \frac{v^2}{1000}$

$\int \frac{v}{g - \frac{v^2}{1000}}\, dv = dx$ (or equivalent) M1

$\int \frac{1000v}{1000g - v^2}\, dv = \int dx$

$-500\log_e\left|1000g - v^2\right| = x + c$ M1

$1000g - v^2 = Ae^{-\frac{x}{500}}$

$v^2 = 1000g - Ae^{-\frac{x}{500}}$ A1

When $x = 0$, $v = 0$ and so $A = 1000g$.

$v = \sqrt{1000g\left(1 - e^{\frac{-x}{500}}\right)}$ and when $x = 25$, $v = 21.86$ (m s^{-1}) (correct to two decimal places). A1

d. **Method 1:**

$mv\frac{dv}{dx} = mg - 5mv \Rightarrow v\frac{dv}{dx} = g - 5v$

$\int \frac{v}{g - 5v}\, dv = \int dx$ M1

$x = \int_{21.86\ldots}^{10} \frac{v}{g - 5v}\, dv$ M1 A1

So $x = 2.7$ (m). A1

Method 2:

$mv\frac{dv}{dx} = mg - 5mv \Rightarrow v\frac{dv}{dx} = g - 5v$

$\int \frac{v}{g - 5v}\, dv = \int dx$ M1

$\frac{1}{5}\int \left(-1 + \frac{g}{g - 5v}\right) dv = \int dx$

$-\frac{v}{5} - \frac{g}{25}\log_e|g - 5v| = x + d$ A1

When $x = 25$, $v = 21.86$ and so $d = -31.17\ldots$

Solving the above equation for x when $v = 10$ gives $x = 27.72\ldots$ M1

The distance below the surface of the lake at which the speed of the object is reduced to 10 m s^{-1} is 2.7 m (correct to one decimal place). A1

e. $0 = g - 5v \Rightarrow v = \frac{g}{5}$ (m s^{-1}) A1

Question 6 (8 marks)

a. $H_0 : \mu = 125, H_1 : \mu < 125$ A1

b. $\overline{W} \sim \mathrm{N}\left(125, \left(\frac{4}{\sqrt{100}}\right)^2\right)$ A1

$p = \Pr(\overline{W} < 124 \mid \mu = 125)$ M1

$= 0.0062$ (correct to four decimal places) A1

c. $0.0062 < 0.05$ and so we reject the null hypothesis. A1

d. Let $\overline{w}$ be the minimum sample mean weight that could be observed for the null hypothesis to not be rejected.

$\Pr(\overline{W} < \overline{w} \mid \mu = 125) = 0.05$

$\overline{w} = 124.34$ (g) (correct to two decimal places) A1

e. From **part d.**, the minimum value for $\overline{w}$ is $\overline{w} = 124.34$ for the null hypothesis to be accepted at the 5% level of significance.

$\Pr(\overline{W} > 124.34 \mid \mu = 124) = 0.20$ M1 A1