Victorian Certificate of Education – Free Trial Examinations

SPECIALIST MATHEMATICS

Free Trial Written Examination 1

SUGGESTED SOLUTIONS AND MARKING GUIDE BOOK

Abbreviations and Acronyms

*n*DP – correct to *n* decimal places

Marking instructions

- The relevant line(s) of working marked give an indication of what statement should be made in order to obtain that mark, but this is subject to the marker's discretion.
- Any mark related to the method can only be awarded if the student presents a convincing (and rigorous) argument.
- The final answer mark (if any) can only be awarded if the student provides the correct answer in either simplest form or the form required.
- Consequential marks can only be obtained for marks related to the method, not for any final answer.
- If elementary mathematical steps and/or logic are broken within a solution, it must be properly justified in order to obtain full marks.

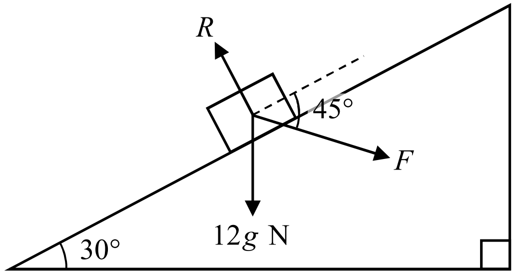
Miscellaneous notes

Some questions may have multiple methods/solutions, including some that are beyond the scope of the course. The solutions provided are the ones that were intended by the examination authors.

For any questions, concerns and/or possible errors, please contact us via the contact form on our website.
Version 2 (July 2019)

Question 1a (1 mark)

Mark	Criteria
1	Provides correct answer



Question 1b (1 mark)

Mark	Criteria
1	Writes down equation relating F and other forces
2	Provides correct answer

Resolving forces parallel to the plane, we have

$F\cos(45°) - 12g\sin(30°) = 0$ **Mark 1**

$\Rightarrow \dfrac{F}{\sqrt{2}} = 6g$

$\Rightarrow F = 6\sqrt{2}g$ N **Mark 2**

Question 2 (3 marks)

Mark	Criteria
1	Utilises double-angle formulae to write down one equation to solve for cos(θ) or sin(θ)
2	Finds both of cos(θ) and sin(θ) (except their signs)
3	Provides correct answer

Using double-angle formulae, we have

$\frac{3}{8} = 2\cos^2(\theta) - 1$ and $\frac{3}{8} = 1 - 2\sin^2(\theta)$ **Mark 1**

$\Rightarrow \cos^2(\theta) = \frac{11}{16}$ and $\sin^2(\theta) = \frac{5}{16}$.

Since $\frac{3\pi}{4} < \theta < \pi$, we must have $\cos(\theta) < 0$ and $\sin(\theta) > 0$, so

$\cos(\theta) = \frac{-\sqrt{11}}{4}$ and $\sin(\theta) = \frac{\sqrt{5}}{4}$. **Mark 2**

Therefore, $\text{cis}(\theta) = \frac{-\sqrt{11}}{4} + \frac{\sqrt{5}}{4}i$. **Mark 3**

Question 3 (4 marks)

Mark	Criteria
1	Utilises the chain rule and the product rule in implicit differentiation
2	Finds expression for dy/dx, or equivalent
3	Finds the gradient of the line perpendicular to the curve
4	Provides correct answer

$\left(y + x\dfrac{dy}{dx}\right)e^{xy} = 2y\dfrac{dy}{dx}$ **Mark 1**

$\Rightarrow\ ye^{xy} + xe^{xy}\dfrac{dy}{dx} - 2y\dfrac{dy}{dx} = 0$

$\Rightarrow\ \left(2y - xe^{xy}\right)\dfrac{dy}{dx} = ye^{xy}$

$\Rightarrow\ \dfrac{dy}{dx} = \dfrac{ye^{xy}}{2y - xe^{xy}}$ **Mark 2**

Hence, $\left.\dfrac{dy}{dx}\right|_{(0,\,2)} = \dfrac{2}{4-0} = \dfrac{1}{2}$, and so the line perpedicular to the curve at $(0, 2)$ has gradient -2 **Mark 3**

with the equation $y - 2 = -2x\ \Rightarrow\ y = 2 - 2x$. **Mark 4**

Question 4 (3 marks)

Mark	Criteria
1	Finds the standard deviation of distribution of the sample mean
2	Finds an expression for the required result in terms of the standard normal variable Z
3	Provides correct answer

$P \sim \mathrm{N}\left(200, 15^2\right)$

$\mathrm{sd}\left(\bar{P}\right) = \dfrac{15}{\sqrt{25}} = 3\ \mathrm{g}\ \Rightarrow\ \bar{P} \sim \mathrm{N}\left(200,\ 3^2\right)$ **Mark 1**

$\Pr\left(\bar{P} > 206\right) = \Pr\left(Z > \dfrac{206-200}{3}\right)$

$= \Pr(Z > 2)$ **Mark 2**

$= \dfrac{1-0.955}{2}$

$= \dfrac{0.045}{2}$

$= 0.023\quad (3\mathrm{DP})$ **Mark 3**

Question 5 (4 marks)

Mark	Criteria
1	Converts $2\text{cis}(\pi/3)$ to cartesian form
2	Finds a quadratic factor of g using the conjugate root theorem
3	Find remaining quadratic factor by comparison or long division, or equivalent
4	Provides correct answer

$2\text{cis}\left(\frac{\pi}{3}\right)=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=1+\sqrt{3}i$ **Mark 1**

By the conjugate root theorem, $z-1-\sqrt{3}i$ and $z-1+\sqrt{3}i$ are factors and so a quadratic factor of g is

$$\left(z-1-\sqrt{3}i\right)\left(z-1+\sqrt{3}i\right)=z^2-z-z+1+3$$
$$=z^2-2z+4.$$ **Mark 2**

Therefore, $g(z)=\left(z^2-2z+4\right)\left(az^2+bz+c\right)$ for some $a,b,c\in\mathbb{R}$

$$\Rightarrow\begin{cases}a=1 & \left(\text{comparing } z^4 \text{ coefficient}\right)\\ b-2a=-2 & \left(\text{comparing } z^3 \text{ coefficient}\right)\\ 4c=-16 & (\text{comparing constant term})\end{cases}$$

$\Rightarrow\ a=1,\ b=0$ and $c=-4$ **Mark 3**

Therefore, we have

$$g(z)=\left(z^2-2z+4\right)\left(z^2-4\right)$$
$$=\left(z-1-\sqrt{3}i\right)\left(z-1+\sqrt{3}i\right)(z+2)(z-2),$$

and so the solutions of $g(z)=0$ are

$z=1+\sqrt{3}i,\ 1-\sqrt{3}i,\ -2,\ 2$ **Mark 4**

Question 6 (4 marks)

Mark	Criteria
1	Writes integrand in partial fraction form
2	Finds partial fraction decomposition constants
3	Finds an antiderivative of the integrand
4	Provides correct answer

$I=\int_1^2\left(\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2}\right)dx,$ where **Mark 1**

$Ax(x+2)+B(x+2)+Cx^2=4$ for all $x\in\mathbb{R}\setminus\{-2,0\}.$

$x\to0\ \Rightarrow\ 0+2B+4=4\ \Rightarrow\ B=2$

$x\to-2\ \Rightarrow\ 0+0+4C=4\ \Rightarrow\ C=1$ **Mark 2**

$x=1\ \Rightarrow\ 3A+6+1=4\ \Rightarrow\ A=-1$

Therefore, we have

$$I=\int_1^2\left(\frac{-1}{x}+\frac{2}{x^2}+\frac{1}{x+2}\right)dx$$

$$=\left[-\log_e|x|-\frac{2}{x}+\log_e|x+2|\right]_1^2$$ **Mark 3**

$$=-\log_e(2)-1+\log_e(4)-(0-2+\log_e(3))$$

$$=1+\log_e\left(\frac{2}{3}\right)$$ **Mark 4**

Question 7 (4 marks)

Mark	Criteria
1	Forms equation involving m
2	Finds m
3	Finds sine of angle AOB
4	Provides correct answer

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$, where $\theta = \angle AOB$.

$\Rightarrow 1 - 1 + m = \sqrt{3}\sqrt{2 + m^2} \times \frac{-1}{\sqrt{6}}$ **Mark 1**

$\Rightarrow \sqrt{2}m = -\sqrt{2 + m^2}$

$\Rightarrow m^2 = 2$

$\Rightarrow m = -\sqrt{2}$, $\left[\text{since } \underset{\sim}{a} \cdot \underset{\sim}{b}\big|_{m=\sqrt{2}} = \sqrt{2} > 0 \text{ and } \cos(\theta) < 0\right]$ **Mark 2**

$\sin(\theta) = \sqrt{1 - \left(\frac{-1}{\sqrt{6}}\right)^2} = \sqrt{\frac{5}{6}}$ $[\sin(\theta) > 0 \text{ since } 0 < \theta < \pi]$ **Mark 3**

Therefore, the area of the triangle AOB is

$A = \frac{1}{2}|\underset{\sim}{a}||\underset{\sim}{b}|\sin(\theta)$

$= \frac{1}{2} \times \sqrt{3} \times 2\sqrt{\frac{5}{6}}$

$= \frac{\sqrt{10}}{2}$ units^2. **Mark 4**

Question 8a (1 mark)

Mark	Criteria
1	Provides a correct method

$\frac{d}{dx}\left[\log_e\left(x + \sqrt{x^2 + k}\right)\right] = \frac{1}{x + \sqrt{x^2 + k}}\left(1 + \frac{x}{\sqrt{x^2 + k}}\right)$

$= \frac{1}{x + \sqrt{x^2 + k}} \times \frac{\sqrt{x^2 + k} + x}{\sqrt{x^2 + k}}$ **Mark 1**

$= \frac{1}{\sqrt{x^2 + k}}$, as required.

Question 8b (4 marks)

Mark	Criteria
1	Writes down a definite integral expression for the volume
2	Finds an antiderivative of one function
3	Finds an antiderivative of other function, or otherwise
4	Provides correct answer

$V = \pi\int_{-\sqrt{2}}^{\sqrt{2}} \frac{x + 2}{\sqrt{x^2 + 6}}\,dx$ **Mark 1**

$= \pi \underbrace{\int_{-\sqrt{2}}^{\sqrt{2}} \frac{x}{\sqrt{x^2 + 6}}\,dx}_{\text{integral of an odd function over a symmetric interval}} + \pi\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2}{\sqrt{x^2 + 6}}\,dx$

$= 0 + 2\pi\left[\log_e\left(x + \sqrt{x^2 + 6}\right)\right]_{-\sqrt{2}}^{\sqrt{2}}$ [using **part a**] **Marks 2 and 3**

$= 2\pi\log_e\left(3\sqrt{2}\right) - 2\pi\log_e\left(\sqrt{2}\right)$

$= 2\pi\log_e(3)$

$= \pi\log_e(9)$ units^3 **Mark 4**

Question 9a.i (1 mark)

Mark	Criteria
1	Provides correct answer

$$\dot{\underset{\sim}{r}}(t) = 2\times\frac{1}{2}[1-\cos(2t)]\underset{\sim}{i} + 4\times\frac{1}{2}[1+\cos(2t)]\underset{\sim}{j}$$

$$= [1-\cos(2t)]\underset{\sim}{i} + [2+2\cos(2t)]\underset{\sim}{j}$$ **Mark 1**

Question 9a.ii (2 marks)

Mark	Criteria
1	Finds an antiderivative of the velocity vector
2	Provides correct answer

$$\underset{\sim}{r}(t) = \int_0^t \dot{\underset{\sim}{r}}(\tau)\,d\tau + \underset{\sim}{i} + 2\underset{\sim}{j}$$

$$= \left[\left(\tau - \frac{1}{2}\sin(2\tau)\right)\mathrm{i} + [2\tau + \sin(2\tau)]\right]_0^t + \underset{\sim}{i} + 2\underset{\sim}{j}$$ **Mark 1**

$$= \left(1 + t - \frac{1}{2}\sin(2t)\right)\underset{\sim}{i} + [2 + 2t + \sin(2t)]\underset{\sim}{j}$$ **Mark 2**

Question 9b (3 marks)

Mark	Criteria
1	Finds an expression for the force vector
2	Removes square root from equation
3	Provides correct answer

$$\ddot{\underset{\sim}{r}}(t) = 2\sin(2t)\underset{\sim}{i} - 4\sin(2t)\underset{\sim}{j}$$

$$\Rightarrow \underset{\sim}{F}(t) = 10\left[\sin(2t)\underset{\sim}{i} - 2\sin(2t)\underset{\sim}{j}\right]$$ **Mark 1**

$$\Rightarrow |\underset{\sim}{F}(t)| = 10\sqrt{\sin^2(2t) + 4\sin^2(2t)}$$

$$= 10\sqrt{5}|\sin(2t)|$$

Then, we have

$$|\underset{\sim}{F}(t)| = 5\sqrt{5} \Rightarrow |\sin(2t)| = \frac{1}{2}$$ **Mark 2**

$$\Rightarrow \sin(2t) = \begin{cases} 1/2 & \text{for } 0 \le 2t < \pi \\ -1/2 & \text{for } \pi \le 2t \le 2\pi \end{cases}$$

$$\Rightarrow 2t = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\Rightarrow t = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12} \text{ seconds}$$ **Mark 3**

Question 10 (4 marks)

Mark	Criteria
1	Writes down a definite integral for the arc length
2	Expands expression inside square root
3	Finds an antiderivative of the integrand
4	Provides correct answer

We have $\frac{dy}{dx}=\frac{e^x-e^{-x}}{2}$, and so the arc length is

$$L=\int_{-1}^{1}\sqrt{1+\left(\frac{e^x-e^{-x}}{2}\right)^2}\,dx$$ **Mark 1**

$$=\int_{-1}^{1}\sqrt{1+\frac{1}{4}\left(e^{2x}-2+e^{-2x}\right)}\,dx$$

$$=\frac{1}{2}\int_{-1}^{1}\sqrt{e^{2x}+2+e^{-2x}}\,dx$$ **Mark 2**

$$=\frac{1}{2}\int_{-1}^{1}\sqrt{\left(e^x+e^{-x}\right)^2}\,dx$$

$$=\frac{1}{2}\int_{-1}^{1}\left(e^x+e^{-x}\right)dx \quad \left[e^x+e^{-x}>0 \text{ for all } x\in\mathbb{R}\right]$$

$$=\frac{1}{2}\left[e^x-e^{-x}\right]_{-1}^{1}$$ **Mark 3**

$$=\frac{1}{2}\left(e-\frac{1}{e}\right)-\frac{1}{2}\left(\frac{1}{e}-e\right)$$

$$=e-\frac{1}{e}\text{ units}$$ **Mark 4**

END OF SUGGESTED SOLUTIONS AND MARKING GUIDE BOOK