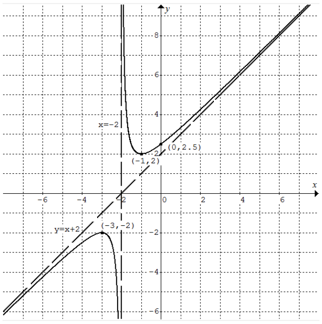
2020 Specialist Mathematics Trial Exam 1 Solutions

Q1a Asymptotes: $x=-2$, $y=x+2$, y-intercept: $\left(0,\frac{5}{2}\right)$

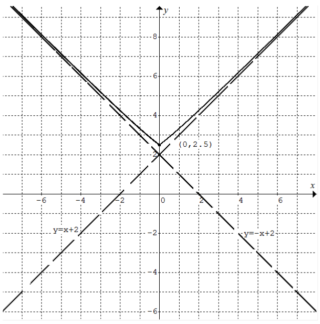
Stationary points:

$f'(x)=-\dfrac{1}{(x+2)^2}+1=0$, $x=-3,-1$, $(-3,-2)$, $(-1,2)$

Q1b



Q1c



Q2

$$\int_0^1 \frac{(x^2+x+1)}{(2x-3)(x^2+x+1)}dx=\int_0^1\frac{1}{2x-3}dx=\left[\frac{\log_e|2x-3|}{2}\right]_0^1=-\frac{1}{2}\log_e 3$$

Q3a $\tilde{c}=k\left(\dfrac{\tilde{a}}{a}+\dfrac{\tilde{b}}{b}\right)$,

$c^2=\tilde{c}.\tilde{c}=k^2\left(\dfrac{\tilde{a}}{a}.\dfrac{\tilde{a}}{a}+\dfrac{2\tilde{a}.\tilde{b}}{ab}+\dfrac{\tilde{b}}{b}.\dfrac{\tilde{b}}{b}\right)=2k^2\left(\dfrac{ab+\tilde{a}.\tilde{b}}{ab}\right)$,

$\left(\sqrt{2ab}\right)^2=2k^2\left(\dfrac{ab+\tilde{a}.\tilde{b}}{ab}\right)$, .: $k=\dfrac{ab}{\sqrt{ab+\tilde{a}.\tilde{b}}}$

.: $\tilde{c}=k\left(\dfrac{b\tilde{a}+a\tilde{b}}{ab}\right)=\dfrac{b\tilde{a}+a\tilde{b}}{\sqrt{ab+\tilde{a}.\tilde{b}}}$

Q3b $a=5$, $b=5$, $\tilde{c}=\dfrac{5\tilde{a}+5\tilde{b}}{\sqrt{25+\tilde{a}.\tilde{b}}}=5\left(\tilde{i}+\tilde{k}\right)$

Q4 $y^2=xy(x^2-1)+1$, when $y=1$, $x=0$, ±1

Implicit differentiation: $2y\dfrac{dy}{dx}=\left(y+x\dfrac{dy}{dx}\right)(x^2-1)+2x^2y$

At $(0,1)$, $\dfrac{dy}{dx}=-\dfrac{1}{2}$; at $(0,\pm1)$, $\dfrac{dy}{dx}=1$

Q5a $\text{Arg}\left(\dfrac{z+1}{z+3}\right)=\text{Arg}(z+1)-\text{Arg}(z+3)=\tan^{-1}\dfrac{y}{x+1}-\tan^{-1}\dfrac{y}{x+3}$

$\tan\left(\text{Arg}\left(\dfrac{z+1}{z+3}\right)\right)=\dfrac{\frac{y}{x+1}-\frac{y}{x+3}}{1+\frac{y^2}{(x+1)(x+3)}}=\dfrac{2y}{(x+1)(x+3)+y^2}$

.: $\text{Arg}\left(\dfrac{z+1}{z+3}\right)=\tan^{-1}\dfrac{2y}{(x+1)(x+3)+y^2}$

.: $\alpha=2y$ and $\beta=(x+1)(x+3)+y^2$

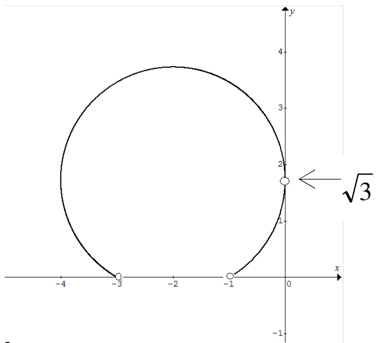
Q5b $\text{Arg}\left(\dfrac{z+1}{z+3}\right)=\tan^{-1}\dfrac{2y}{(x+1)(x+3)+y^2}=\dfrac{\pi}{6}$

$\dfrac{2y}{(x+1)(x+3)+y^2}=\tan\dfrac{\pi}{6}=\dfrac{1}{\sqrt{3}}$

$(x+1)(x+3)+y^2=2\sqrt{3}\,y$, $x^2+4x+y^2-2\sqrt{3}\,y+3=0$

Q5c $x^2+4x+y^2-2\sqrt{3}\,y+3=0$ where $x\in R^-$ and $y\in R^+$

$x^2+4x+4+y^2-2\sqrt{3}\,y+3=4$, .: $(x+2)^2+\left(y-\sqrt{3}\right)^2=4$



$\left\{z:\left|z+2-i\sqrt{3}\right|=2\right\}$ is a circle of radius 2 units centred at $\left(-2,\sqrt{3}\right)$.

.: $\left\{z:\text{Arg}\left(\dfrac{z+1}{z+3}\right)=\dfrac{\pi}{6}\right\}\subset\left\{z:\left|z+2-i\sqrt{3}\right|=2\right\}$

Q6a $\theta=\sin^{-1}t$, $\sin\theta=t$, $\cos\theta=\sqrt{1-t^2}$

$\sin^{-1}t=\cos^{-1}\dfrac{t}{2}$, $\cos\left(\sin^{-1}t\right)=\dfrac{t}{2}$, $\sqrt{1-t^2}=\dfrac{t}{2}$, $1-t^2=\dfrac{t^2}{4}$,

$t^2=\dfrac{4}{5}$, $t=\dfrac{2}{\sqrt{5}}$

Q6b $v=\dfrac{dx}{dt}=\dfrac{-1}{\sqrt{4-t^2}}$, as $t\to\dfrac{2}{\sqrt{5}}$, $v\to\dfrac{\sqrt{5}}{-4}$, .: $a=5$ and $b=-4$

Q7a Acceleration is maximum when friction between the mass and the rough floor is zero. This occurs when the normal reaction force of the floor is zero when the weight force + vert. comp. pulling force is zero. .: $F\sin 60°=9.8$ where F newtons is the pulling force.

.: $F=\dfrac{19.6}{\sqrt{3}}$ N and the horizontal component of F is $F\cos 60°=\dfrac{9.8}{\sqrt{3}}$ N

.: maximum acceleration is $\dfrac{9.8}{\sqrt{3}}$ ms^{-2}.

Q7b Distance $=s=\dfrac{1}{2}at^2$, average speed $=\dfrac{s}{t}=\dfrac{1}{2}\times\dfrac{9.8}{\sqrt{3}}\times\sqrt{3}=4.9$ ms^{-1}

Q8 Let X kg be mass of an orange from the orchard.
Given $\mu = 0.18$, $\sigma = 0.01$, $\text{Var}(X) = \sigma^2 = 0.01^2$

A bag of 18 oranges: Let $Y = X_1 + X_2 + ... + X_{18}$ kg be the mass of a bag of 18 oranges.

$\text{E}(Y) = \text{E}(X_1 + X_2 + ... + X_{18})$

$= \text{E}(X_1) + \text{E}(X_2) + ... + \text{E}(X_{18}) = 18 \times 0.18$

$\text{Var}(Y) = \text{Var}(X_1 + X_2 + ... + X_{18}) = \text{Var}(X_1) + \text{Var}(X_2) + ... + \text{Var}(X_{18})$

$= 18\text{Var}(X) = 18 \times 0.01^2$, $\text{sd}(Y) = 3\sqrt{2} \times 0.01$

A bag of 6 oranges: Let $Z = X_1 + X_2 + ... + X_6$ kg be the mass of a bag of 6 oranges.

$\text{E}(Z) = 6\text{E}(X) = 6 \times 0.18$, $\text{Var}(Z) = 6\text{Var}(X) = 6 \times 0.01^2$

3 bags of 6 oranges: Let $W = Z_1 + Z_2 + Z_3$ kg be the total mass.

$\text{E}(W) = \text{E}(Z_1) + \text{E}(Z_2) + \text{E}(Z_3) = 3 \times \text{E}(Z) = 18 \times 0.18$

$\text{Var}(W) = \text{Var}(Z_1) + \text{Var}(Z_2) + \text{Var}(Z_3) = 3 \times \text{Var}(Z) = 18 \times 0.01^2$

$\text{sd}(W) = 3\sqrt{2} \times 0.01$

The mass of a bag of 18 oranges and the mass of 3 bags of 6 oranges have the same mean and standard deviation.

Q9a Height of the cone $= \sqrt{3}\cos 30° = \frac{3}{2}$

Length of $PQ = \sqrt{3}$, radius of the cone $= \frac{\sqrt{3}}{2}$

Volume of the cone $= \frac{1}{3}\pi\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{3}{2}\right) = \frac{3\pi}{8}$

Q9b Let V be the vertex of the cone.

Length of $OV = \frac{\sqrt{3}}{\cos 30°} = 2$, required distance $= 2 - \frac{3}{2} = \frac{1}{2}$

Q9c Radius of the sphere (circle) $= \sqrt{3}\tan 30° = 1$

Let O be $(0, 0)$. Equation of the circle $= x^2 + y^2 = 1$

Volume of revolution $= \frac{3\pi}{8} + \int_{-\frac{1}{2}}^{1} \pi x^2\, dy = \frac{3\pi}{8} + \int_{-\frac{1}{2}}^{1} \pi(1 - y^2)dy$

$= \frac{3\pi}{8} + \pi\left[y - \frac{y^3}{3}\right]_{-\frac{1}{2}}^{1} = \frac{3\pi}{8} + \pi\left[1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{24}\right] = \frac{3\pi}{2}$

Q10a Let h cm be the mean height of the remaining 20 adults in the sample.

$\frac{80 \times 175.5 + 20h}{100} = 174.4$, $h = 170.0$

Q10b $\text{E}(\overline{X}) = \mu = 175.0$, $\text{sd}(\overline{X}) = \frac{\sigma}{\sqrt{100}} = \frac{\sigma}{10}$

$\Pr(\overline{X} > 178.0) \approx 0.025$, $\Pr\left(Z > \frac{178.0 - 175.0}{\frac{\sigma}{10}}\right) \approx 0.025$

$\therefore \frac{178.0 - 175.0}{\frac{\sigma}{10}} \approx 2$, $\sigma \approx 15$

Please inform mathline@itute.com re conceptual and/or mathematical errors