2020 Specialist Mathematics Trial Exam 2 Solutions

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	A	D	E	D	E	E	E	A	C

11	12	13	14	15	16	17	18	19	20
C	A	A	D	E	B	D	B	B	C

Q1

Since $0 < x \le 1$ and $|\alpha| \ge 0$ for $\alpha \in C$,

$\left|\sqrt{-x} + \frac{1}{\sqrt{-x}}\right| = \left|i\sqrt{x} - \frac{i}{\sqrt{x}}\right| = \left|\sqrt{x} - \frac{1}{\sqrt{x}}\right| = \frac{1}{\sqrt{x}} - \sqrt{x}$ B

Q2

$\text{Arg}(\sin\theta - i\cos\theta) = \text{Arg}((\text{-}i)(\cos\theta + i\sin\theta))$

$= \text{Arg}\left(\text{cis}\left(-\frac{\pi}{2}\right)\text{cis}\theta\right) = \text{Arg}\left(\text{cis}\left(\theta - \frac{\pi}{2}\right)\right)$ A

Q3

x-intercepts: $|ax+b| = b$, $ax + b = \pm b$, $x = 0, -\frac{2b}{a}$

The enclosed region consists of 2 congruent triangles with base $= \frac{2b}{a}$ and height $= b$.: area $= 2\times\frac{1}{2}\left(\frac{2b}{a}\right)b = \frac{2b^2}{a}$ D

Q4

$b\cos^{-1}(x-a) + 2b\sin^{-1}(x-a)$

$= b\left(\cos^{-1}(x-a) + 2\left(\frac{\pi}{2} - \cos^{-1}(x-a)\right)\right)$

$= b\left(\pi - \cos^{-1}(x-a)\right) = b\cos^{-1}(a-x)$ E

Q5

$\tilde{r}_A . \tilde{r}_B = 0$.: $\tilde{r}_A$ and $\tilde{r}_B$ make a right angle.

$|\tilde{r}_A| = 12$, $\tilde{r}_B = 5$, $\angle OBA = \theta = \tan^{-1}\left(\frac{12}{5}\right)$

Shortest distance $= 5\sin\theta = \frac{60}{13}$ D

Q6

Choices D and E are unit vectors.

Only Choice E is perpendicular to both $\tilde{a}$ and $\tilde{b}$. E

Q7

$\tilde{v}_A = \tilde{i} + 2t\,\tilde{j}$, $\tilde{v}_B = 2t\,\tilde{i} - \tilde{j}$, $|\tilde{v}_A| = \sqrt{1+4t^2}$, $|\tilde{v}_B| = \sqrt{4t^2+1}$

$\tilde{a}_A = 2\tilde{j}$, $\tilde{a}_B = 2\tilde{i}$

.: same speed but different acceleration at $t > 0$. E

Q8

$f(i) = (i)^5 + (i)^4 + a(i)^3 + 5(i)^2 + b(i) + c = 0$.: $a - b = 1 + \frac{c-4}{i}$

$f(-i) = (-i)^5 + (-i)^4 + a(-i)^3 + 5(-i)^2 + b(-i) + c = 0$

.: $a - b = 1 - \frac{c-4}{i}$.: $c - 4 = 0$.: $c = 4$ E

Q9

For the rays to be defined, θ must satisfy

$-\pi < \theta + \frac{\pi}{4} \le \pi$ AND $-\pi < \theta + \frac{7\pi}{12} \le \pi$ AND $-\pi < \theta + \frac{11\pi}{12} \le \pi$

.: $-\frac{5\pi}{4} < \theta \le \frac{3\pi}{4}$ AND $-\frac{19\pi}{12} < \theta \le \frac{5\pi}{12}$ AND $-\frac{23\pi}{12} < \theta \le \frac{\pi}{12}$

.: $-\frac{5\pi}{4} < \theta \le \frac{\pi}{12}$

Ray $\text{Arg}\left(z - \sqrt{2} - \sqrt{2}\,i\right) = \theta + \frac{11\pi}{12}$ starts from $z = \sqrt{2} + \sqrt{2}\,i$.

$\text{Arg}\left(\sqrt{2} + \sqrt{2}\,i\right) = \frac{\pi}{4}$ and $\left|\sqrt{2} + \sqrt{2}\,i\right| = 2$

For the enclosed region to be defined, Ray $\text{Arg}(z) = \theta + \frac{\pi}{4}$ needs to pass through $z = \sqrt{2} + \sqrt{2}\,i$ or left of it .: $\theta \ge 0$, hence $0 \le \theta \le \frac{\pi}{12}$.

When $\theta = 0$, the three rays make acute angle $\frac{\pi}{3}$ with each other.

.: the enclosed region is an equilateral triangle of side length $\left|\sqrt{2} + \sqrt{2}\,i\right| = 2$ and it has an area of $\sqrt{3}$ which is a maximum.

.: $A < \sqrt{3}$ if $0 < \theta \le \frac{\pi}{12}$ A

Q10

$\sqrt{a + b\,i} = x + y\,i$, $\left(\sqrt{a^2+b^2}\,\text{cis}\theta\right)^{\frac{1}{2}} = x + y\,i$,

$\left(\sqrt{a^2+b^2}\right)^{\frac{1}{2}}\text{cis}\frac{\theta}{2} = x + y\,i$, $\left|\left(\sqrt{a^2+b^2}\right)^{\frac{1}{2}}\text{cis}\frac{\theta}{2}\right|^2 = |x + y\,i|^2$

.: $x^2 + y^2 = \sqrt{a^2+b^2}$ C

Q11

$\frac{|\vec{P}|}{\sin\left(\frac{\pi}{2}+\alpha\right)} = \frac{|\vec{Q}|}{\sin\left(\frac{\pi}{2}+\beta\right)} = \frac{|\vec{R}|}{\sin\gamma}$, $\frac{|\vec{P}|}{\cos\alpha} = \frac{|\vec{Q}|}{\cos\beta} = \frac{|\vec{R}|}{\sin\gamma}$ C

Q12

$a = -3.0$, force (friction) on the particle $= 2(-3) = -6$

.: horizontal force (friction) exerted by the particle on the floor $= 6$

Normal force exerted by the particle on the floor $= mg = 2(9.8) = 19.6$

Net force exerted by the particle on the floor $= \sqrt{6^2 + 19.6^2} \approx 20.5$ A

Q13

Let $\hat{u} = a\tilde{i} + a\tilde{j} + a\tilde{k}$ be the unit vector. $\sqrt{a^2+a^2+a^2} = a\sqrt{3} = 1$

.: $a = \frac{1}{\sqrt{3}}$, $\hat{u}.\tilde{i} = a = \frac{1}{\sqrt{3}}$, $\cos\theta = \frac{1}{\sqrt{3}}$, $\theta \approx 54.7$ A

Q14

$t = 2$, $s = u(2) + \frac{1}{2}(-9.8)2^2 = 2u - 19.6$

$t = 3$, $s = u(3) + \frac{1}{2}(-9.8)3^2 = 3u - 44.1$

Distance in the third second $= (3u - 44.1) - (2u - 19.6) = u - 24.5$

$t = 6$, $s = u(6) + \frac{1}{2}(-9.8)6^2 = 6u - 176.4$

$t = 7$, $s = u(7) + \frac{1}{2}(-9.8)7^2 = 7u - 240.1$

Distance in the seventh second $= (6u - 176.4) - (7u - 240.1) = -u + 63.7$

$u - 24.5 = -u + 63.7$, $u = 44.1$ D

Q15

$y = b\sin^{-1}\left(\frac{x}{a}\right)$, $\frac{dy}{dx} = \frac{b}{\sqrt{a^2 - x^2}}$; $y = b\cos^{-1}\left(\frac{x}{a}\right)$, $\frac{dy}{dx} = \frac{-b}{\sqrt{a^2 - x^2}}$

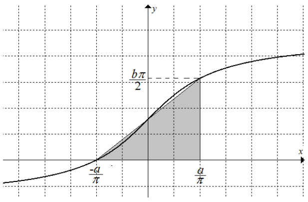
The tangents **are perpendicular** when $\frac{b}{\sqrt{a^2 - x^2}} \times \frac{-b}{\sqrt{a^2 - x^2}} = -1$

i.e. when $x^2 = a^2 - b^2 \geq 0$, $\frac{a^2}{b^2} \geq 1$, $\frac{a}{b} \geq 1$, $a \geq b$ E

Q16

The region has the same area as the triangle as shown below.

$\text{Area} = \frac{1}{2}\left(\frac{2a}{\pi}\right)\left(\frac{b\pi}{2}\right) = \frac{ab}{2}$ B



Q17

$y = \text{cosec}(x) = \frac{1}{\sin(x)}$, $\frac{dy}{dx} = -\frac{\cos(x)}{\sin^2(x)} = -\frac{\cos(p)}{\sin^2(p)}$

$m = \frac{\text{cosec}(p)}{p} = \frac{1}{p\sin(p)}$

$\frac{1}{p\sin(p)} = -\frac{\cos(p)}{\sin^2(p)}$.: $p = -\tan(p)$, $p \approx 2.03$ D

Q18

$\text{Var}(L_A) = \text{Var}(5L) = 5^2\,\text{Var}(L) = 5^2 \times 0.01^2 = 0.0025$

Similarly, $\text{Var}(L_B) = 0.0025$

$\text{Var}(L_{total}) = \text{Var}(L_A + L_B) = \text{Var}(L_A) + \text{Var}(L_B) = 0.0050$

$\text{sd}(L_{total}) = \sqrt{0.0050} \approx 0.071$ B

Q19

$\text{sd}(\bar{X}) = \frac{1.518 - 1.482}{4} = 0.009$

$\Pr(\bar{X} > 1.475 \mid 1.482) \approx 0.78$, $\Pr(\bar{X} > 1.475 \mid 1.518) \approx 1.00$ B

Q20

$R = x_{max} - x_{min} = 2 \times 1.96\frac{\sigma}{\sqrt{50}} = 3.92\frac{\sigma}{\sqrt{50}}$

$25\% \times R = 3.92\frac{\sigma}{\sqrt{n}}$, $\frac{1}{4} \times 3.92\frac{\sigma}{\sqrt{50}} = 3.92\frac{\sigma}{\sqrt{n}}$

.: $\sqrt{n} = 4\sqrt{50}$, $n = 800$.: extra 750 C

SECTION B

Q1a $|z+i| - |z-i| = 1$, $|z+i|^2 = (1 + |z-i|)^2$

$x^2 + (y+1)^2 = 1 + 2|z-i| + x^2 + (y-1)^2$

$4y - 1 = 2|z-i|$, $(4y-1)^2 = 4|z-i|^2$

$(4y-1)^2 = 4x^2 + 4y^2 - 8y + 4$, .: $4x^2 - 12y^2 + 3 = 0$

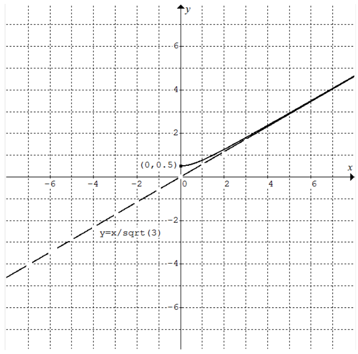
Q1b Given $\frac{dx}{dt} = \sqrt{3}$ and at time $t = 0$ the particle is at $x = 0$

.: $x = \sqrt{3}t$

Given $y > 0$ and from part a $4x^2 - 12y^2 + 3 = 0$

.: $y = \sqrt{\frac{x^2}{3} + \frac{1}{4}} = \sqrt{t^2 + \frac{1}{4}}$.: $\tilde{r}(t) = \sqrt{3}t\tilde{i} + \sqrt{t^2 + \frac{1}{4}}\tilde{j}$

Q1c



Q1d $y = \sqrt{t^2 + \frac{1}{4}}$, $\frac{dy}{dt} = \frac{t}{\sqrt{t^2 + \frac{1}{4}}}$

$\text{Distance} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^2 \sqrt{3 + \frac{t^2}{t^2 + \frac{1}{4}}}\,dt \approx 3.83\,\text{m}$

Q1e Let $t = a$ seconds be the time. $\int_0^a \sqrt{3 + \frac{t^2}{t^2 + \frac{1}{4}}}\,dt = 5$, $a \approx 2.59$

Q2a $y = \cos^{-1}(1-x)$, $x = 1 - \cos y$

$V = \int_0^\pi \pi x^2\,dy = \int_0^\pi \pi(1 - \cos y)^2\,dy = \pi\int_0^\pi \left(\frac{3}{2} - 2\cos y + \frac{1}{2}\cos 2y\right)dy$

$= \pi\left[\frac{3}{2}y - 2\sin y + \frac{1}{4}\sin 2y\right]_0^\pi = \frac{3\pi^2}{2}$

Q2b

$V(h) = \pi\left[\frac{3}{2}y - 2\sin y + \frac{1}{4}\sin 2y\right]_0^h = \pi\left(\frac{3h}{2} - 2\sin h + \frac{1}{4}\sin 2h\right)$

Q2c $\frac{dV}{dh} \times \frac{dh}{dt} = \frac{dV}{dt}$, $\frac{dh}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dh}} = \frac{-0.010h}{\pi(1 - \cos h)^2}$

When $h = \frac{\pi}{2}$, $\frac{dh}{dt} = -0.005$ m s^{-1}

Q2d $\dfrac{dh}{dt}=\dfrac{-0.010h}{\pi(1-\cos h)^2}$, $\dfrac{dt}{dh}=\dfrac{-\pi(1-\cos h)^2}{0.010h}$

$$t=\int_{\pi}^{\frac{\pi}{2}}\frac{-\pi(1-\cos h)^2}{0.010h}dh=\int_{\frac{\pi}{2}}^{\pi}\frac{\pi(1-\cos h)^2}{0.010h}dh\approx 561.7\text{ s}$$

Q2e Time to empty the vessel $=\displaystyle\int_0^{\pi}\frac{\pi(1-\cos h)^2}{0.010h}dh\approx 652.7$ s

Average rate of outflow $\approx\dfrac{\frac{3\pi^2}{2}}{652.7}\approx 0.0227$ m^3 s^{-1}

Q2f Let $\dfrac{dV}{dt}=0.005\pi-0.010h=0$, minimum $h=\dfrac{\pi}{2}$ m

Q2g $\dfrac{dh}{dt}=\dfrac{\frac{dV}{dt}}{\frac{dV}{dh}}=\dfrac{0.005\pi-0.010h}{\pi(1-\cos h)^2}$, $\dfrac{dt}{dh}=\dfrac{\pi(1-\cos h)^2}{0.005\pi-0.010h}$

$$t=\int_{\pi}^{0.75\pi}\frac{\pi(1-\cos h)^2}{0.005\pi-0.010h}dh=\int_{0.75\pi}^{\pi}\frac{\pi(1-\cos h)^2}{0.010h-0.005\pi}dh\approx 773.6\text{ s}$$

Q3a $\overrightarrow{CB}=\tilde{a}+\tilde{b}$, $\overrightarrow{BA}=\tilde{a}-\tilde{b}$

Q3b $\overrightarrow{CB}.\overrightarrow{BA}=(\tilde{a}+\tilde{b}).(\tilde{a}-\tilde{b})=\tilde{a}.\tilde{a}-\tilde{b}.\tilde{b}=|\tilde{a}|^2-|\tilde{b}|^2=0$ since $|\tilde{a}|=|\tilde{b}|$.: $\overrightarrow{CB}$ is perpendicular to $\overrightarrow{BA}$.

Q3c $\overrightarrow{CB}+\overrightarrow{BA}=\overrightarrow{CA}$, $(\overrightarrow{CB}+\overrightarrow{BA}).(\overrightarrow{CB}+\overrightarrow{BA})=\overrightarrow{CA}.\overrightarrow{CA}$

.: $\overrightarrow{CB}.\overrightarrow{CB}+2\overrightarrow{CB}.\overrightarrow{BA}+\overrightarrow{BA}.\overrightarrow{BA}=\overrightarrow{CA}.\overrightarrow{CA}$

.: $\overrightarrow{CB}.\overrightarrow{CB}+\overrightarrow{BA}.\overrightarrow{BA}=\overrightarrow{CA}.\overrightarrow{CA}$ since $\overrightarrow{CB}.\overrightarrow{BA}=0$

.: $|\overrightarrow{CB}|^2+|\overrightarrow{BA}|^2=|\overrightarrow{CA}|^2$

For a right-angle triangle the sum of the squares of the two shorter sides equals the square of the longest side.

Q3d Let $\angle ACB=\theta$.

Scalar resolute of $\overrightarrow{CA}$ in the direction of $\overrightarrow{CB}$ $=CB=2|\tilde{a}|\cos\theta$

Scalar resolute of $\overrightarrow{CO}$ in the direction of $\overrightarrow{CB}$ $=CP=|\tilde{a}|\cos\theta$

.: P is the mid point of line segment CB .

Q4a Resultant force $\tilde{R}$

$=1\tilde{j}+(2\sin 30°\tilde{i}+2\cos 30°\tilde{j})+(4\sin 60°\tilde{i}+4\cos 60°\tilde{j})$

$\quad+(8\sin 120°\tilde{i}+8\cos 120°\tilde{j})$

$=(6\sqrt{3}+1)\tilde{i}+(\sqrt{3}-1)\tilde{j}$ N

$\tilde{a}=\dfrac{\tilde{R}}{m}=\left(3\sqrt{3}+\dfrac{1}{2}\right)\tilde{i}+\left(\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\right)\tilde{j}$ m s^{-2}

Q4b $\Delta\tilde{v}=\displaystyle\int_0^2\left(3\sqrt{3}+\frac{1}{2}\right)\tilde{i}+\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\tilde{j}\,dt=(6\sqrt{3}+1)\tilde{i}+(\sqrt{3}-1)\tilde{j}$

$\tilde{v}=2\tilde{j}+(6\sqrt{3}+1)\tilde{i}+(\sqrt{3}-1)\tilde{j}=(6\sqrt{3}+1)\tilde{i}+(\sqrt{3}+1)\tilde{j}$ m s^{-1}

Q4c

Change in momentum $=m\Delta\tilde{v}=2(6\sqrt{3}+1)\tilde{i}+2(\sqrt{3}-1)\tilde{j}$ kg m s^{-1}

Q4d Initial speed $=|2\tilde{j}|=2$

Final speed ($t=2$) $=\sqrt{(6\sqrt{3}+1)^2+(\sqrt{3}+1)^2}\approx 11.7$

Change in speed $\approx 11.7-2=9.7$ m s^{-1}

Q4e No. The direction of motion is not constant.

The particle has an initial velocity of $2\tilde{j}$ and a velocity of $(6\sqrt{3}+1)\tilde{i}+(\sqrt{3}+1)\tilde{j}$ after 2 seconds.

Q5a Given the pulling force is constant.

.: Tension in the cord is constant mg newtons.

Resultant force in the direction of motion is

$R=mg\cos\theta-0.75mg$ where $\cos\theta=\dfrac{5-x}{\sqrt{1+(5-x)^2}}$

.: $R=mg\left(\dfrac{5-x}{\sqrt{1+(5-x)^2}}-0.75\right)$, $a=\dfrac{R}{m}=\left(\dfrac{5-x}{\sqrt{1+(5-x)^2}}-0.75\right)g$

Q5b Let $\left(\dfrac{5-x}{\sqrt{1+(5-x)^2}}-0.75\right)g=0$, $x\approx 3.87$ (3.8661)

Q5c $a=\dfrac{d\left(\frac{1}{2}v^2\right)}{dx}=\left(\dfrac{5-x}{\sqrt{1+(5-x)^2}}-0.75\right)g$

$\dfrac{1}{2}v^2=\displaystyle\int\left(\frac{5-x}{\sqrt{1+(5-x)^2}}-0.75\right)g\,dx$

$v^2=\left(-2\sqrt{1+(5-x)^2}-1.5x+c\right)g$

When $x=0$, $v=0.1$, .: $c\approx 10.1991$

$v^2\approx\left(-2\sqrt{1+(5-x)^2}-1.5x+10.1991\right)g$

Maximum speed occurs when $x\approx 3.8661$ (i.e. acceleration $=0$)

$v^2\approx\left(-2\sqrt{1+(5-3.8661)^2}-1.5(3.8661)+10.1991\right)g$

Max $v\approx 3.67$

Q5d $v=\dfrac{dx}{dt}\approx\sqrt{g}\sqrt{-2\sqrt{1+(5-x)^2}-1.5x+10.1991}$

$\dfrac{dt}{dx}=\dfrac{1}{\sqrt{g}\sqrt{-2\sqrt{1+(5-x)^2}-1.5x+10.1991}}$

$t\approx\dfrac{1}{\sqrt{g}}\displaystyle\int_0^5\frac{1}{\sqrt{-2\sqrt{1+(5-x)^2}-1.5x+10.1991}}dx\approx 2.19$

Q5e $R = mg\cos\theta - \alpha mg$, $a = g\cos\theta - \alpha g$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = \left(\frac{5-x}{\sqrt{1+(5-x)^2}} - \alpha\right)g$$

$$v^2 \approx \left(-2\sqrt{1+(5-x)^2} - 2\alpha x + 10.1991\right)g$$

When $x = 5$, $v = 0$, .: $0 \approx -2 - 2\alpha(5) + 10.1991$

$\alpha \approx 0.82$

Q6a $\Pr(X > 365\times 24) \approx 0.4142$

Q6b $\Pr(X > 9000 \mid X > 8000) = \dfrac{\Pr(X > 9000)}{\Pr(X > 8000)} \approx 0.5116$

Q6c Total life $X_{total} = X_1 + X_2$

$\mathrm{E}(X_{total}) = \mathrm{E}(X_1) + \mathrm{E}(X_2) = 8500 + 8500 = 17000$

$\mathrm{Var}(X_{total}) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 1200^2 + 1200^2 = 2\times 1200^2$

$\mathrm{sd}(X_{total}) = \sqrt{2\times 1200^2} = 1200\sqrt{2} \approx 1697$

Q6d $\mathrm{E}(\overline{X}) = \mathrm{E}(X) = 8500$, $\mathrm{sd}(\overline{X}) = \dfrac{\mathrm{sd}(X)}{\sqrt{n}} = \dfrac{1200}{5} = 240$

$\Pr(\overline{X} > 9000) \approx 0.0186$ (0.01861)

Q6e

$\Pr(\text{at least one sample}) = 1 - \Pr(\text{none}) = 1 - (1 - 0.01861)^3 \approx 0.0548$

Q6f $\bar{x} = 8700$, $\mathrm{sd}(\overline{X}) = 240$

$8700 - 1.96\times 240 = 8229.6$, $8700 + 1.96\times 240 = 9170.4$

$8500 \in (8229.6, 9170.4)$.: the claim is to be accepted.

Q6g $\bar{x} = 8300$, $\mathrm{sd}(\overline{X}) = \dfrac{\mathrm{sd}(X)}{\sqrt{n}} = \dfrac{1200}{\sqrt{250}} \approx 75.8947$

$8300 - 1.96\times 75.8947 \approx 8151.25$, $8300 + 1.96\times 75.8947 \approx 8448.75$

$8500 \notin (8151.25, 8448.75)$.: the claim is to be rejected.

Q6h The claim is to be rejected.
A larger sample provides a more accurate determination.

Q6i A Type II error occurs if the claim is accepted when it is to be rejected.

Please inform mathline@itute.com re conceptual and mathematical errors