The Mathematical Association of Victoria

SPECIALIST MATHEMATICS 2020

Trial Written Examination 2 - SOLUTIONS

SECTION A – Multiple-choice questions

ANSWERS

1	2	3	4	5	6	7	8	9	10
D	B	E	C	A	D	E	A	D	D

11	12	13	14	15	16	17	18	19	20
D	E	A	B	C	A	B	E	B	A

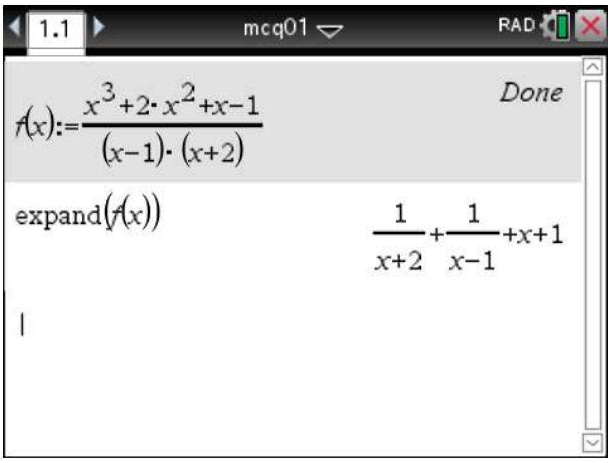
SOLUTIONS

Question 1 **Answer is D**

Use CAS to see that

$$\frac{x^3+2x^2+x-1}{(x-1)(x+2)} = x+1+\frac{1}{x+2}+\frac{1}{x-1}$$

So the asymptotes are $y = 1+x$, $x=1$ and $x=-2$.



Question 2 **Answer is B**

The point of inflection is at $(-1,1)$ and so the function is of the form $f(x)=a\arctan(x+1)+1$ where a is to be determined.

The curve passes through $\left(0,\frac{3}{2}\right)$. So $f(0)=a\arctan(1)+1=\frac{3}{2}$ giving

$$a\cdot\frac{\pi}{4}+1=\frac{3}{2}$$
$$a\cdot\frac{\pi}{4}=\frac{1}{2}$$
$$a=\frac{2}{\pi}$$

Question 3 **Answer is E**

The function is defined when $\frac{x^2}{2}\in[-1,1]$. Since $x^2\geq 0$,

$$0\leq\frac{x^2}{2}\leq 1$$
$$0\leq x^2\leq 2$$

So $x\in\left[-\sqrt{2},\sqrt{2}\right]$.

Question 4 **Answer is C**

Let $z=r\operatorname{cis}\left(\frac{\pi}{3}\right)$ where $r<1$. So

$$z^2=r^2\operatorname{cis}\left(\frac{2\pi}{3}\right)$$
$$iz^2=r^2\operatorname{cis}\left(\frac{2\pi}{3}+\frac{\pi}{2}\right)$$
$$=r^2\operatorname{cis}\left(\frac{7\pi}{6}\right),\ r^2<r$$

Question 5 **Answer is A**

Since $1+i=\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$ and $1-\sqrt{3}i=2\operatorname{cis}\left(-\frac{\pi}{3}\right)$ we have

$$z=\frac{\left(\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^2}{\left(2\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^3}$$

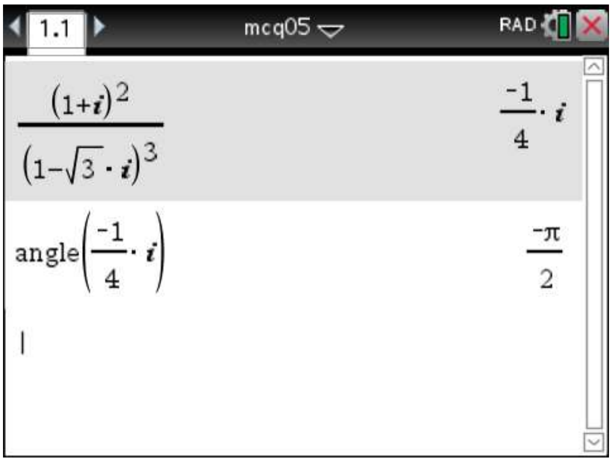
$$=\frac{2\operatorname{cis}\left(\frac{\pi}{2}\right)}{8\operatorname{cis}(-\pi)}$$

$$=\frac{1}{4}\operatorname{cis}\left(\frac{3\pi}{2}\right)$$

$$=\frac{1}{4}\operatorname{cis}\left(-\frac{\pi}{2}\right)$$

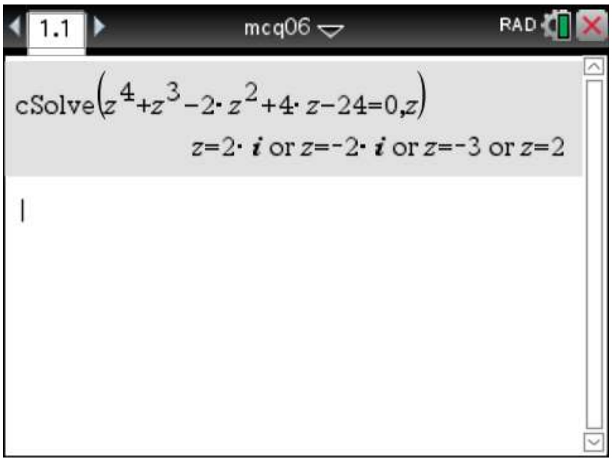
So the modulus is $\frac{1}{4}$ and the principal argument is $-\frac{\pi}{2}$.

This result can also be obtained using CAS:



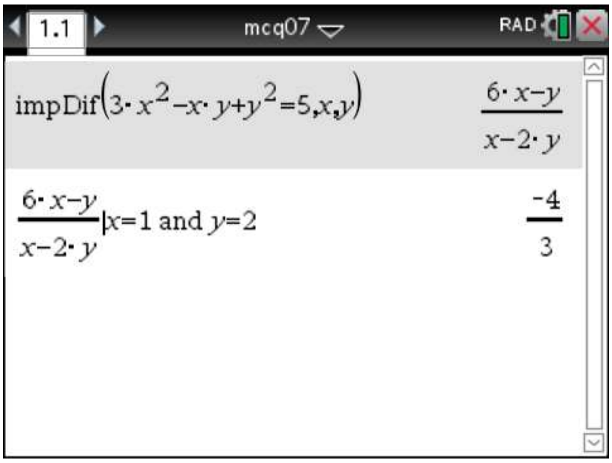
Question 6 **Answer is D**

The solutions to the equation $z^4 + z^3 - 2z^2 + 4z - 24 = 0$ are $z = \pm 2i$, $z = -3$ and $z = 2$. The sum of the solutions is therefore -1.
Solving the equation should be performed using CAS:



Question 7 **Answer is E**

Use implicit differentiation to find $\frac{dy}{dx} = \frac{6x - y}{x - 2y}$ and so when $x = 1$ and $y = 2$, $\frac{dy}{dx} = -\frac{4}{3}$.
This can be done efficiently using CAS:



The equation of the tangent to the curve is $y - 2 = -\frac{4}{3}(x - 1)$ and so $y = -\frac{4}{3}x + \frac{10}{3}$.

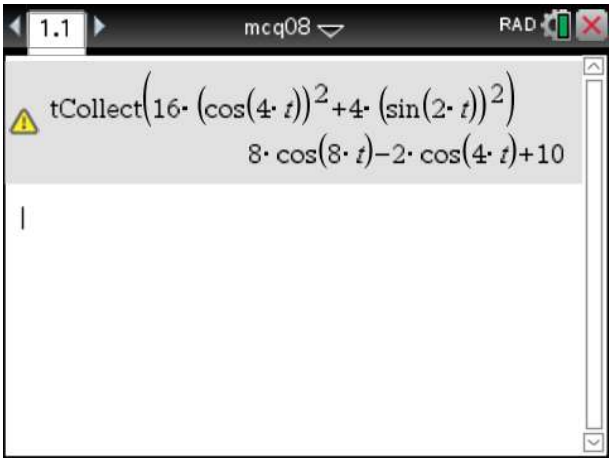
Question 8 **Answer is A**

The integrand is $\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$. Now $\frac{dx}{dt}=4\cos(4t)$ and $\frac{dy}{dt}=-2\sin(2t)$ and so

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2&=16\cos^2(4t)+4\sin^2(2t)\\&=8\big(1+\cos(8t)\big)+2\big(1-\cos(4t)\big)\\&=8+8\cos(8t)+2-2\cos(4t)\\&=10-2\cos(4t)+8\cos(8t)\end{aligned}$$

Therefore, the length of the curve is $\int_0^{\frac{\pi}{4}}\sqrt{10-2\cos(4t)+8\cos(8t)}\,dt$.

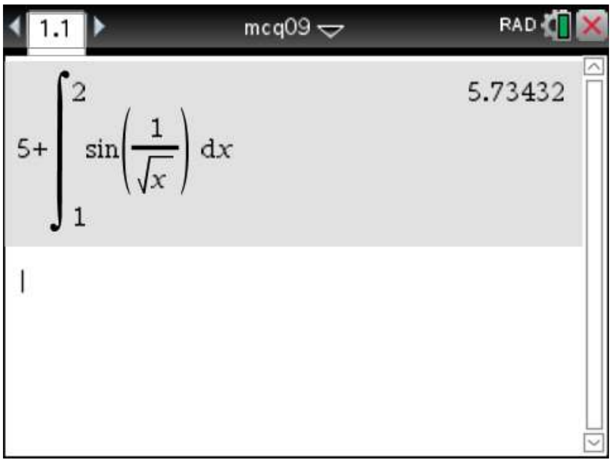
Note that the simplification of the trigonometric terms can be performed by CAS:



Question 9 **Answer is D**

Since $y(2)-y(1)=\int_1^2\sin\left(\frac{1}{\sqrt{x}}\right)dx$ it follows that $y(2)=5+\int_1^2\sin\left(\frac{1}{\sqrt{x}}\right)dx\approx 5.734$.

The integration is performed using CAS:



Question 10 **Answer is D**

The gradient segments are undefined when $y=0$ and when $y=-1$. Therefore, the differential equation which best represents the direction field is $\dfrac{dy}{dx}=\dfrac{x}{y(1+y)}$.

Question 11 **Answer is D**

Let $u=1+\log_e\left(x^2\right)=1+2\log_e\left(x\right)$. Then $\dfrac{du}{dx}=\dfrac{2}{x}$.

When $x=1, u=1+\log_e(1)=1$.

When $x=\sqrt{e}, u=1+\log_e\left(e\right)=2$.

The integral becomes $\dfrac{1}{2}\displaystyle\int_1^2 \frac{1}{\sqrt{u}}\,du$

Question 12 **Answer is E**

Let x be the horizontal distance of the aeroplane from the observer. Then $\tan\theta=\dfrac{1500}{x}$.

First find $\dfrac{dx}{d\theta}$.

Note that $x=\dfrac{1500}{\tan\theta}$ and so

$$\begin{aligned}\frac{dx}{d\theta}&=-\sec^2\theta\cdot\frac{1500}{\tan^2\theta}\\&=-\frac{1500}{\sin^2\theta}\end{aligned}$$

Alternatively, implicit differentiation can be used:

$$\begin{aligned}\sec^2\theta&=-\frac{1500}{x^2}\cdot\frac{dx}{d\theta}\\ \frac{dx}{d\theta}&=-\frac{x^2\sec^2\theta}{1500}\\&=-\left(\frac{1500}{\tan\theta}\right)^2\cdot\frac{\sec^2\theta}{1500}\\&=-\frac{1500}{\sin^2\theta}\end{aligned}$$

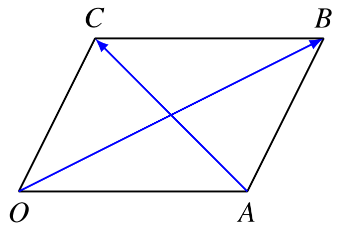
Given that $\frac{dx}{dt} = -90$, we have

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \cdot \frac{dx}{dt}$$
$$= -\frac{\sin^2\theta}{1500} \times -90$$
$$= \frac{3}{50}\sin^2\theta$$

When $\theta = \frac{\pi}{3}$, $\frac{d\theta}{dt} = \frac{9}{200}$.

Question 13 **Answer is A**

A diagram is helpful:



The diagonals are

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$$
$$= -\underset{\sim}{j} + 5\underset{\sim}{k}$$

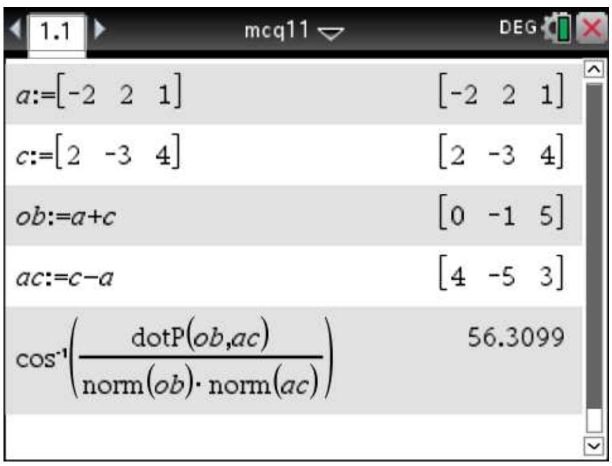
and

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= 4\underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k}$$

Now $\overrightarrow{OB} \cdot \overrightarrow{AC} = 20$, $\left|\overrightarrow{OB}\right| = \sqrt{26}$ and $\left|\overrightarrow{AC}\right| = 5\sqrt{2}$. Therefore the angle between the vectors is

$$\arccos\left(\frac{20}{5\sqrt{26}\sqrt{2}}\right) \approx 56.31^\circ$$

Note that is often easier to do this entirely in CAS:



Question 14 **Answer is B**

If the vectors are linearly dependent then there exist constants α and β such that

$$\alpha\left(\underset{\sim}{i} + p^2\underset{\sim}{j} + 3\underset{\sim}{k}\right) + \beta\left(-2\underset{\sim}{i} + 6\underset{\sim}{j} + 2\underset{\sim}{k}\right) = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$$

This gives rise to equations

$$\begin{aligned} \alpha - 2\beta &= 2 \\ \alpha p^2 + 6\beta &= 1 \\ 3\alpha + 2\beta &= -1 \end{aligned}$$

From the first and third equations we find $\alpha = \frac{1}{4}$ and $\beta = -\frac{7}{8}$. Substituting these values into the second equation and solving for p gives

$$\begin{aligned} \frac{1}{4}p^2 - \frac{21}{4} &= 1 \\ p^2 &= 25 \\ p &= \pm 5 \end{aligned}$$

So, the vectors are linearly independent if $p \in R \setminus \{-5, 5\}$.

Alternatively, the values of p for which the vectors are dependent can be found by considering the determinant of the 3×3 matrix whose rows (or columns) consist of the vectors:

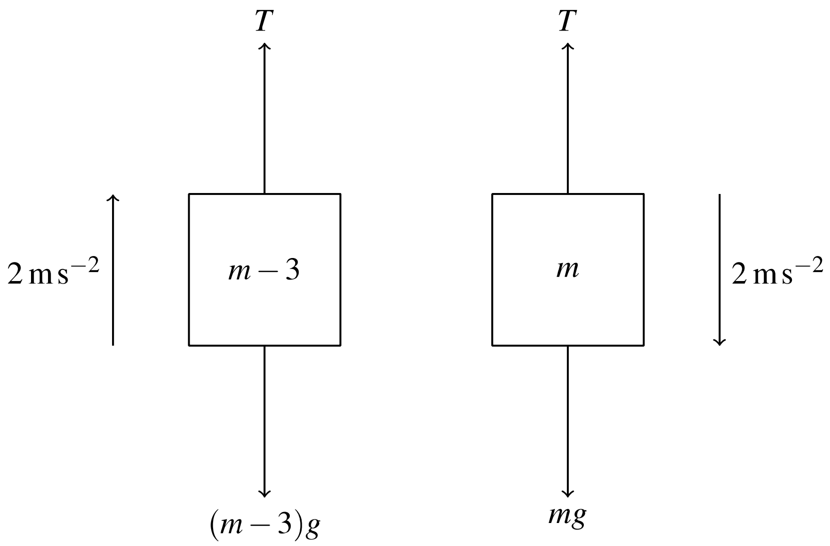
$$\begin{vmatrix} 1 & p^2 & 3 \\ -2 & 6 & 2 \\ 2 & 1 & -1 \end{vmatrix} = 0 \Rightarrow p = \pm 5$$

Again, the vectors are linearly independent if $p \in R \setminus \{-5, 5\}$.

Question 15 **Answer is C**

There are two ways to solve this problem.

The first method is to consider the forces acting on each mass. The tension in the string is T.



Consider the forces acting on the mass on the left. This gives:

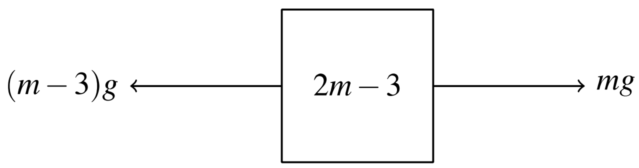
$$T-(m-3)g=2(m-3)$$

Now consider the forces acting on the mass on the right. This gives:

$$mg-T=2m$$

Solving these simultaneously gives $m=\dfrac{3g+6}{4}$.

The second method is to consider the system as a single mass acted on by forces as shown in the diagram below:



The acceleration of the system is $2\,ms^{-2}$ and so

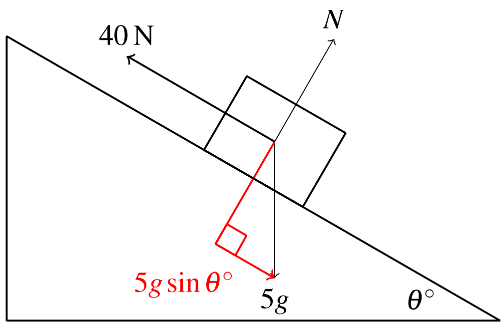
$$2(2m-3)=mg-(m-3)g$$

□ $$4m-6=3g$$

$$m=\frac{3g+6}{4}$$

Question 16 **Answer is A**

Label the forces on the diagram:



Then we see that

$$40 - 5g\sin\theta = 5\times 3.5$$
$$\theta = 27.33^\circ$$

Question 17 **Answer is B**

The force of $6\underset{\sim}{j}$ produces an acceleration of $3\underset{\sim}{j}\,\text{ms}^{-2}$ (using $\underset{\sim}{F} = m\underset{\sim}{a}$ with $m = 2$). Therefore

$\underset{\sim}{a}(t) = 3\underset{\sim}{j}$

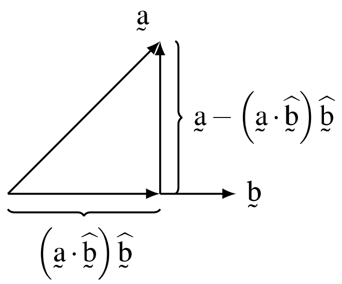
$\underset{\sim}{v}(t) = 5\underset{\sim}{i} + 3t\underset{\sim}{j}$

$\underset{\sim}{v}(4) = 5\underset{\sim}{i} + 12\underset{\sim}{j}$

and so $\left|\underset{\sim}{p}\right| = 2\left|\underset{\sim}{v}(4)\right| = 2\sqrt{25+144} = 26$.

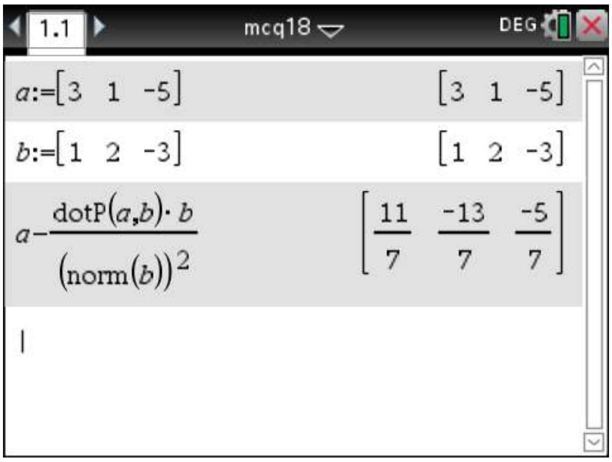
Question 18 **Answer is E**

Refer to the diagram below:



The component of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a}-(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.

This can be found quickly be CAS:



So $\underset{\sim}{a}-(\underset{\sim}{a}\cdot\hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}=\frac{1}{7}\left(11\underset{\sim}{i}-13\underset{\sim}{j}-5\underset{\sim}{k}\right)$

Question 19 **Answer is B**

We have that

$$\frac{dx}{dt} = 3 - x$$
$$\int \frac{dx}{3-x} = \int dt$$
$$-\log_e |3 - x| = t + c$$

Apply the condition $x = 2$ when $t = 1$. This gives

$$-\log_e |3-2| = 1 + c$$
$$0 = 1 + c$$
$$c = -1$$

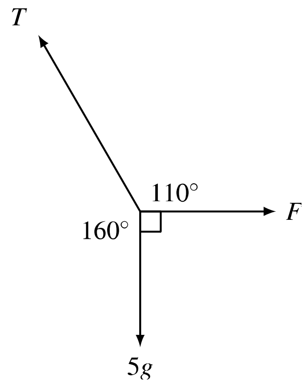
So

$$-\log_e |3-x| = t - 1$$
$$\log_e |3-x| = 1 - t$$
$$3 - x = \pm e^{1-t}$$
$$x = 3 \mp e^{1-t}$$

Consider the condition again: $x = 2$ when $t = 1$. Therefore we need to choose the negative sign and so $x = 3 - e^{1-t}$.

Question 20 **Answer is A**

We have three forces in equilibrium:



Therefore

$$\frac{5g}{\sin 110^\circ} = \frac{F}{\sin 160^\circ} = \frac{T}{\sin 90^\circ}$$

and so $T = 52.145$ and $F = 17.835$.

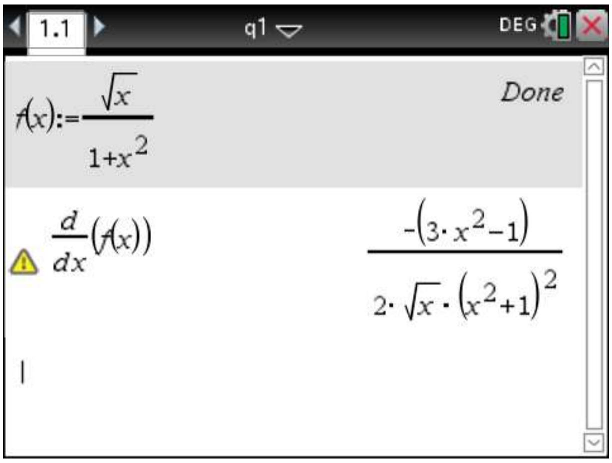
SECTION B

Question 1

a. $D=[0,\infty)$ [A1]

b. $f'(x)=\dfrac{1-3x^2}{2\sqrt{x}\left(x^2+1\right)^2}$ [A1]

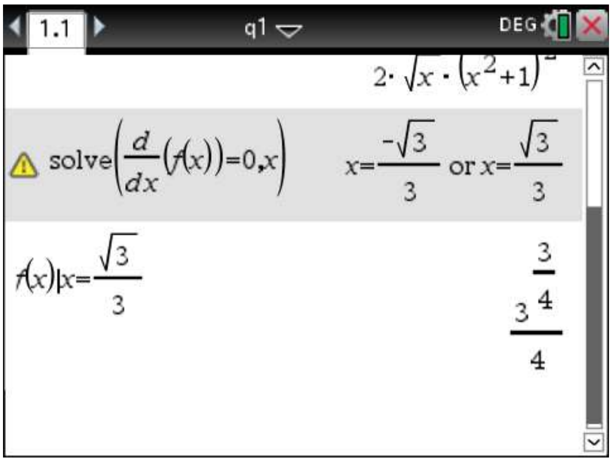
This should be found using CAS:



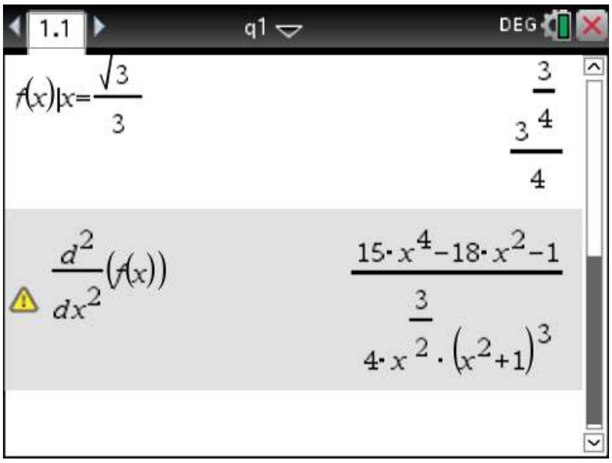
The coordinates of the turning point are:

$\left(\dfrac{\sqrt{3}}{3},\dfrac{3^{3/4}}{4}\right)$ or $\left(\dfrac{1}{\sqrt{3}},\dfrac{3^{3/4}}{4}\right)$ [A1]

Again the coordinates of the turning point should be found using CAS:



c. i. Use CAS to find the second derivative:

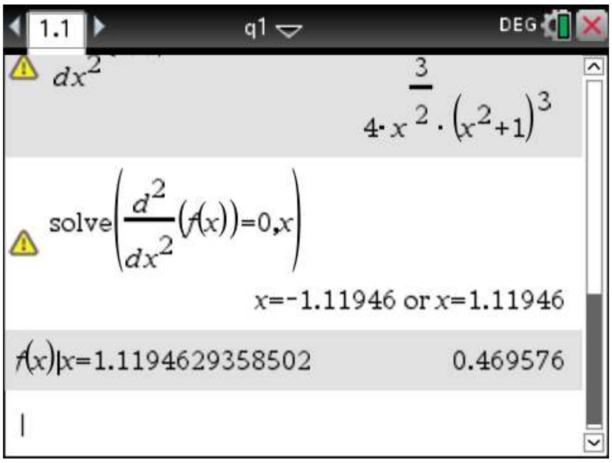


It is found that $\frac{d^2y}{dx^2}=\frac{15x^4-18x^2-1}{4x^{3/2}\left(x^2+1\right)^3}$.

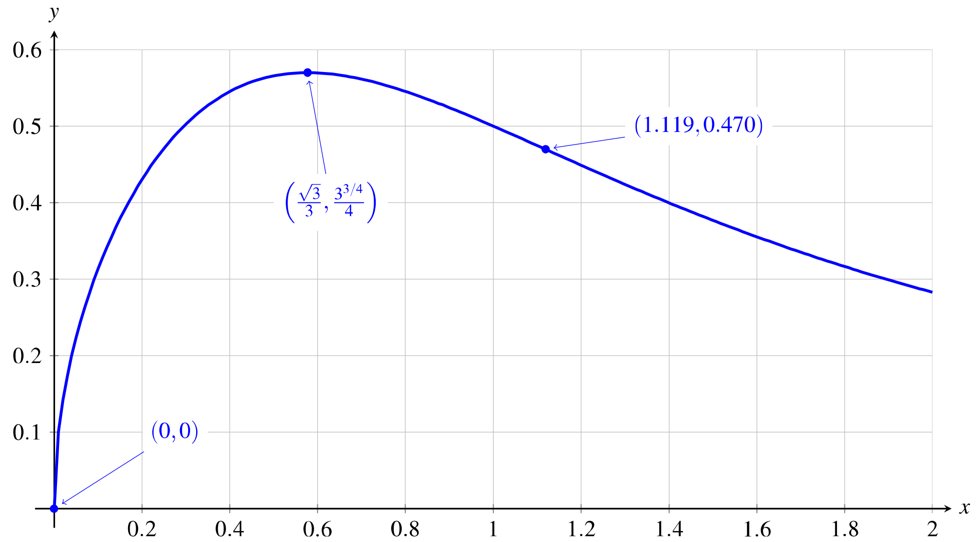
A quartic equation that can be solved to give the x-coordinate of the point of inflection of f is

$15x^4-18x^2-1=0$ [A1]

ii. The coordinates of the point of inflection are $(1.119, 0.470)$.



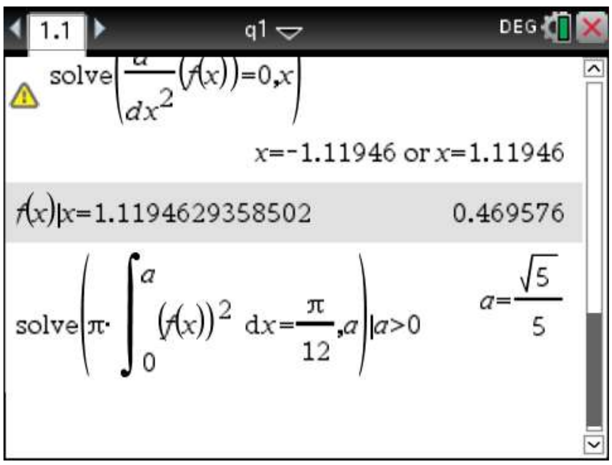
d.



[A1]
Smooth curve
[A1]
Point of inflection correctly positioned and labeled
[A1]
Stationary point correctly positioned and labeled

e. **i.** $\pi\int_0^a \frac{x}{\left(1+x^2\right)^2}\,dx = \frac{\pi}{12}$ [A1]

ii. $a = \frac{\sqrt{5}}{5}$ (or $a = \frac{1}{\sqrt{5}}$) [A1]



Question 2

a. $(x-2)^2 + y^2 = 4$ [A1]

b. **i.** Use the quadratic formula or complete the square:

$$z = \frac{1}{2}\left[6 \pm \sqrt{36 - 4 \cdot 1 \cdot 12}\right]$$
$$= \frac{1}{2}\left[6 \pm \sqrt{-12}\right]$$
$$= \frac{1}{2}\left[6 \pm 2\sqrt{3}i\right]$$
$$= 3 \pm \sqrt{3}i$$

[A1]

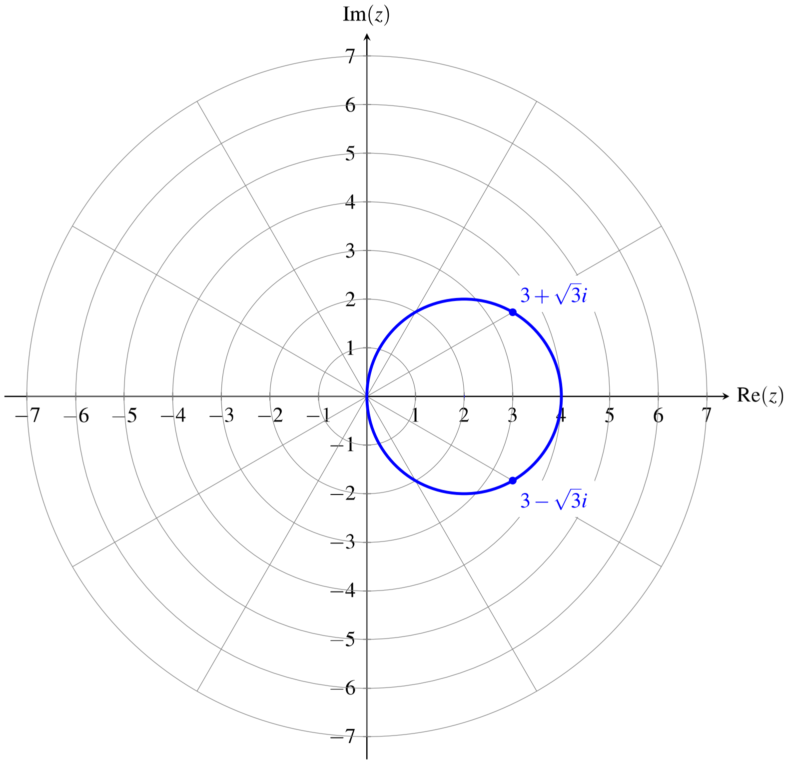
ii.

$$\left|3 + \sqrt{3}i - 2\right| = \left|1 + \sqrt{3}i\right|$$
$$= \sqrt{1+3}$$
$$= 2$$

[A1]

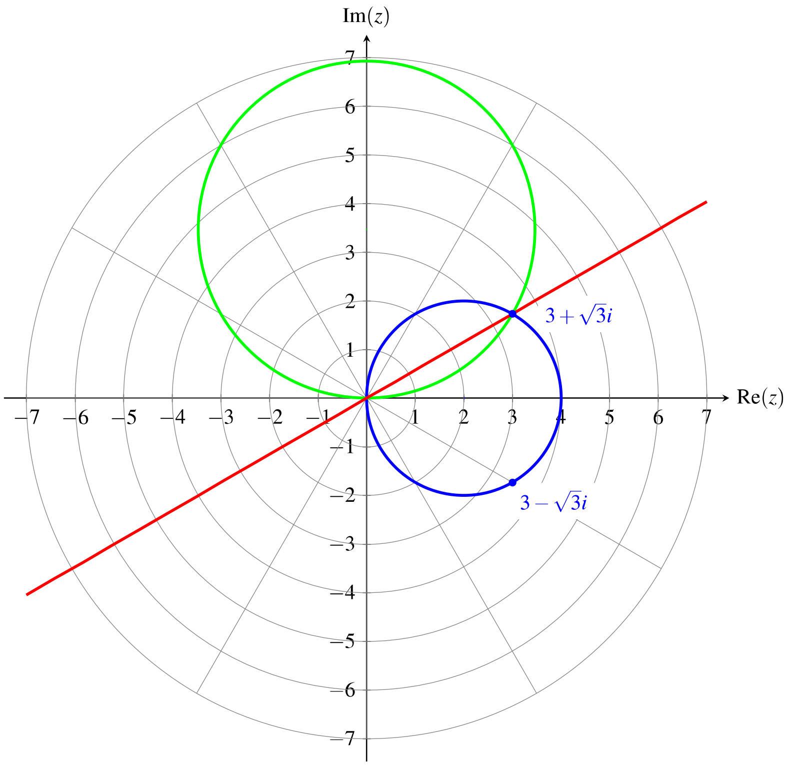
Therefore $z_1 = 3 + \sqrt{3}i$ lies on the circle C_1.

c. The circle C_1 and the solutions of $z^2 - 6z + 12 = 0$ are shown below:



[A1]
Correct circle
[A1]
Points correctly placed

d. The circle C_2 and the line l are shown below:



[A1]
Circle and line

e. The line l is the perpendicular bisector of the line segment through $z = 2$ and $z = \alpha = 1 + \sqrt{3}i$.

Equivalently, $\alpha = 2\,\text{cis}\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = 2\,\text{cis}\left(\frac{\pi}{3}\right) = 1 + \sqrt{3}i$

So $\alpha = 1 + \sqrt{3}i$. [A1]

f. Denote by A_1 the area of the segment bounded by the circle C_1 and l. Denote by A_2 the area of the segment bounded by C_2 and l.
Then

$$A_1 = \frac{1}{2}\cdot 2^2\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right)$$

$$A_2 = \frac{1}{2}\cdot\left(2\sqrt{3}\right)^2\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right)$$

So the area is

$$A_1 + A_2 = 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) + 6\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$$ [A1]

$$\approx 3.544$$

Question 3

a. Separate and integrate:

$$\int \frac{dx}{x-20} = \int -k\,dt$$

$$\log_e|x-20| = -kt + c$$

When $t=0$, $x=12$ and so $\log_e 8 = c$.

When $t=20$, $x=18$:

$$\log_e 2 = -20k + \log_e 8$$

$$k = -\frac{1}{20}\left(\log_e 2 - \log_e 8\right)$$

$$= -\frac{1}{20}\log_e \frac{1}{4}$$ [A1]

$$= \frac{1}{20}\log_e 4$$

b. Left hand side:

$$\frac{dx}{dt} = 2e^{-\frac{1}{10}t} - \frac{1}{5}te^{-\frac{1}{10}t} + \frac{4}{5}e^{-\frac{1}{10}t}$$

$$= \frac{14}{5}e^{-\frac{1}{10}t} - \frac{1}{5}te^{-\frac{1}{10}t}$$

Right hand side:

$$-\frac{1}{10}(x-20) + 2e^{-\frac{1}{10}t} = -\frac{1}{10}\left(2te^{-\frac{1}{10}t} - 8e^{-\frac{1}{10}t}\right) + 2e^{-\frac{1}{10}t}$$

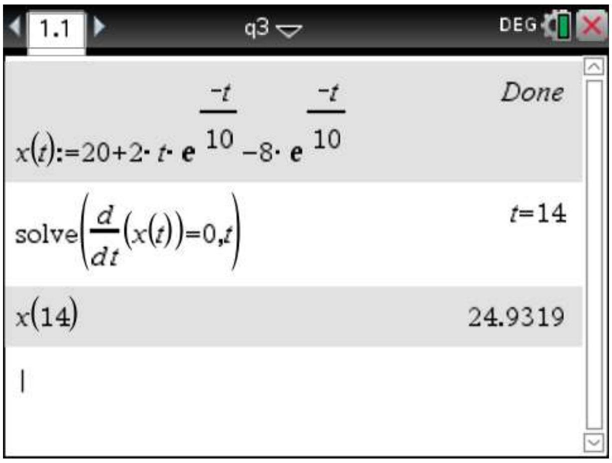
$$= -\frac{1}{5}te^{-\frac{1}{10}t} + \frac{4}{5}e^{-\frac{1}{10}t} + 2e^{-\frac{1}{10}t}$$

$$= \frac{14}{5}e^{-\frac{1}{10}t} - \frac{1}{5}te^{-\frac{1}{10}t}$$

So $x(t) = 20 + 2te^{-\frac{1}{10}t} - 8e^{-\frac{1}{10}t}$ satisfies the differential equation. [A1]

Check the initial condition: $x(0) = 20 - 8 - 12$. [A1]

c. The maximum temperature is 24.932° which occurs when $t = 14$. [A2]



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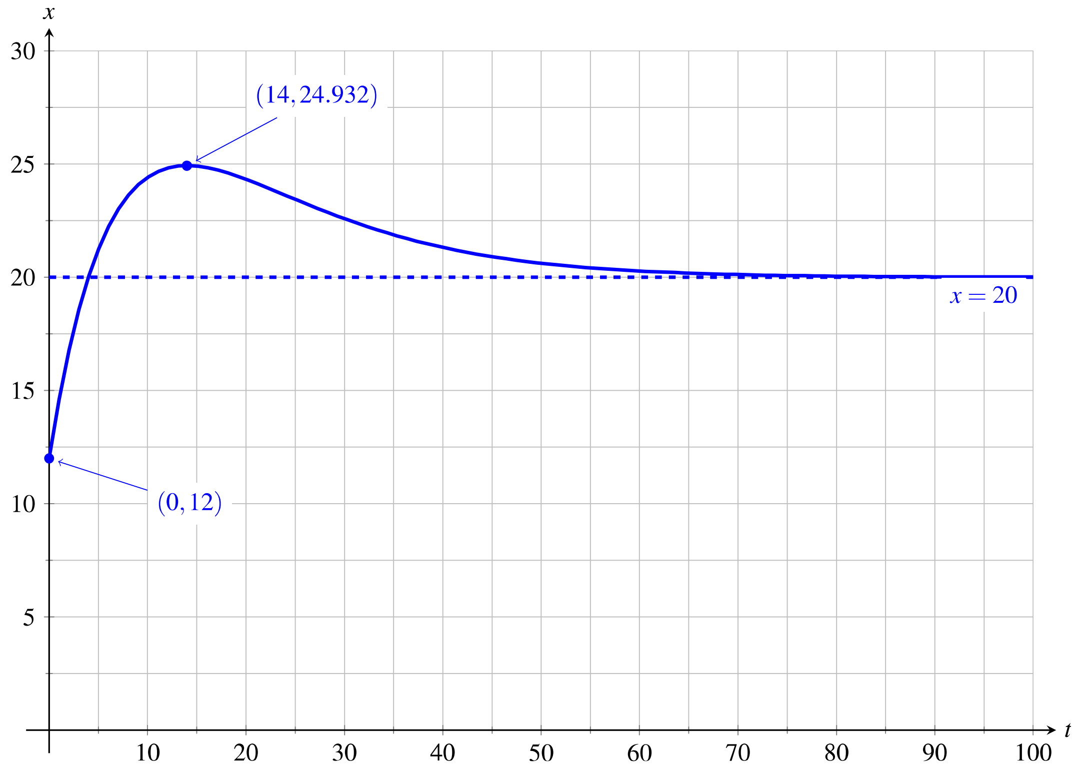


[A1]
Smooth, correct curve
[A1]
Asymptote $x = 20$
[A1]
Stationary point and initial point labeled

e. i.

$$\frac{dx}{dt} = -\frac{1}{10}(x-20) + \frac{1}{2}e^{-\frac{\sqrt{x}}{10}}$$

$$\Rightarrow t = \int_{12}^{r} \frac{1}{-\frac{1}{10}(x-20) + \frac{1}{2}e^{-\frac{\sqrt{x}}{10}}}\,dx$$ [A1]

ii.

$$\int_{12}^{18} \frac{1}{-\frac{1}{10}(x-20) + \frac{1}{2}e^{-\frac{\sqrt{x}}{10}}}\,dx \approx 7.507$$ [A1]

Question 4

Note: "*Show that ….*" questions usually state a result that will be useful or needed in a later part of the question.

In a "*Show that ….*" question a student must clearly demonstrate that s/he could have obtained the given result without it being stated. Some relevant comments from past VCAA Examination Reports regarding "*Show that ….*" questions include:

"There were some unconvincing arguments, often due to insufficient steps being shown."
[2015 Examination Report Question 8 part a.]

"As often happens in a 'show that' type question, some students were unable to do any convincing [working], yet still managed to obtain the result stated."
[2012 Examination Report Question 9 part c.]

A good way to ensure that all necessary working is given when answering a "*Show that ….*" question is to treat it as asking "*Find ….*".

a.

Treat this question as asking "***Find*** the position vector of the midpoint *M* of side *BC* of this triangle".

$\underset{\sim}{m} = \overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$

$= \overrightarrow{OB} + \frac{1}{2}\left(\overrightarrow{BC}\right)$ *

$= \overrightarrow{OB} + \frac{1}{2}\left(\overrightarrow{BO} + \overrightarrow{OC}\right)$

$= \overrightarrow{OB} + \frac{1}{2}\left(-\overrightarrow{OB} + \overrightarrow{OC}\right)$ *

$= \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}$ *

$= \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OC}\right)$ *

$= \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$, which was to be shown. *

[M1]

All lines marked * are required.

b.

This is a "*Show that* …." question therefore it should be treated as asking

"***Given*** $\underset{\sim}{r}_A = \underset{\sim}{a} + \overrightarrow{AM}\,t$, ***find*** $\underset{\sim}{r}_A$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$".

$\underset{\sim}{r}_A = \underset{\sim}{a} + \overrightarrow{AM}\,t$.

Substitute $\overrightarrow{AM} = \overrightarrow{AO} + \overrightarrow{OM} = -\underset{\sim}{a} + \underbrace{\frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)}_{\text{From part a}}$: **[M1]**

$\underset{\sim}{r}_A = \underset{\sim}{a} + t\left(-\underset{\sim}{a} + \frac{1}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)\right)$

$= \underset{\sim}{a} - \underset{\sim}{a}\,t + \frac{t}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$ $= (1-t)\underset{\sim}{a} + \frac{t}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$, which was to be shown. **[M1]**

c.

This is a "*Show that* …." question therefore it should be treated as asking

"***Given*** $\underset{\sim}{r}_A = (1-t)\underset{\sim}{a} + \frac{t}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$, ***find*** $\underset{\sim}{r}_A$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$".

$\underset{\sim}{r}_A = (1-t)\underset{\sim}{a} + \frac{t}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right)$

$= (1-t)\underset{\sim}{a} + \frac{t}{2}\left(\underset{\sim}{b} + \underset{\sim}{c}\right) + \frac{t}{2}\underset{\sim}{a} - \frac{t}{2}\underset{\sim}{a}$ *

$= (1-t)\underset{\sim}{a} - \frac{t}{2}\underset{\sim}{a} + \frac{t}{2}\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}\right)$ *

$= \left(1 - \frac{3}{2}t\right)\underset{\sim}{a} + \frac{t}{2}\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}\right)$, which was to be shown. *

[M1]

All lines marked * are required.

d.

A similar argument to that used in **parts a.**, **b.** and **c.** can be used to find the position vectors of points lying on the line passing through B and the midpoint of AC (by symmetry, interchange $\underset{\sim}{a}$ and $\underset{\sim}{b}$) and points lying on the line passing through C and the midpoint of AB (by symmetry, interchange $\underset{\sim}{a}$ and $\underset{\sim}{c}$):

i.

Answer: $\underset{\sim}{r}_B = \left(1-\frac{3}{2}u\right)\underset{\sim}{b} + \frac{u}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right),\ u \in R$. **[A1]**

ii.

Answer: $\underset{\sim}{r}_C = \left(1-\frac{3}{2}v\right)\underset{\sim}{c} + \frac{v}{2}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right),\ v \in R$. **[A1]**

Only penalise once if the parameter t is re-used in each position vector.

e.

A median of a triangle is a line segment joining a vertex to the midpoint of the opposite side.

So the three position vectors found in the **parts c.** and **d.** define all the points lying on the three medians of the triangle ABC.

If the medians of a triangle intersect in a point, it will be at the point that is common to all three lines.

By inspection, the point with position vector such that either

$$1-\frac{3}{2}t = 0 \quad \Rightarrow t = \frac{2}{3}.$$

$$1-\frac{3}{2}u = 0 \quad \Rightarrow u = \frac{2}{3}.$$

$$1-\frac{3}{2}v = 0 \quad \Rightarrow v = \frac{2}{3}.$$

[A1] for any of the above

is common to all three lines:

$$\underset{\sim}{r}_A = \underset{\sim}{r}_B = \underset{\sim}{r}_C = \frac{1}{3}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right).$$

Therefore, the point with position vector $\frac{1}{3}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)$ is common to all three medians and so the three medians intersect in a point.

Answer: $\frac{1}{3}\left(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}\right)$. **[A1]**

Remark: The intersection point of the three medians of a triangle is called the *centroid*.

Question 5

- Forces acting on the 5 kg object:

• Weight force of size $5g$ in the downwards vertical direction.

Component of weight force parallel to plane: $5g\sin(\alpha)$.

Component of weight force perpendicular to plane: $5g\cos(\alpha)$. • Tension

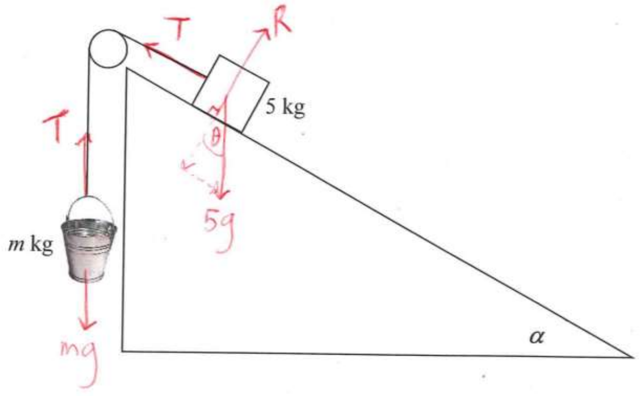
force of size T up the plane.

• Normal reaction force of size R perpendicular to the plane.

- Forces acting on the bucket (m kg object):

• Weight force of size mg downwards.

• Tension force of size T upwards.



a.

In a "*Show that* …." question a student must clearly demonstrate that s/he could have obtained the given result without it being stated.

A good way to ensure that all necessary working is given when answering a "*Show that* …." question is to treat it as asking "*Find* …."

Treat this question as asking "***Find*** in terms of α all values of m such that the 5 kg mass moves down the plane aftre it is released".

Method 1:

Consider the motion of the combined-mass system.

$F_{net} = ma$

$(m+5)a = \left(5\sin(\alpha) - m\right)g$ **[M1]**

The 5 kg object and hence the combined-mass system moves down the plane if $a > 0$:

$\left(5\sin(\alpha) - m\right)g > 0 \quad \Rightarrow 5\sin(\alpha) - m > 0 \quad \Rightarrow m < 5\sin(\alpha)$. **[M1]**

Method 2:

- Forces acting on the bucket (*m* kg mass) (taking upwards [direction of motion] as the positive direction):

$F_{net} = ma$. …. (1)

$F_{net} = T - mg$. …. (2)

Equate equations (1) and (2):

$ma = T - mg \quad \Rightarrow T = ma + mg$. …. (3)

- Forces acting on the 5 kg mass in the direction parallel to the plane (taking down the plane [direction of motion] as the positive direction):

$F_{net} = ma = 5a$. …. (4)

$F_{net} = \underbrace{5g\sin(\alpha)}_{\text{Component of the weight force parallel to the plane}} - T$. …. (5)

Equate equations (4) and (5):

$5a = 5g\sin(\alpha) - T$. …. (6)

Substitute $T = ma + mg$ into equation (6):

$5a = 5g\sin(\alpha) - (ma + mg)$ **[M1]**

$\Rightarrow 5a = 5g\sin(\alpha) - ma - mg \quad \Rightarrow a(5+m) = 5g\sin(\alpha) - mg$.

- The 5 kg object moves down the plane if $a > 0$:

$5g\sin(\alpha) - mg > 0 \quad \Rightarrow 5g\sin(\alpha) > mg \quad \Rightarrow m < 5\sin(\alpha)$. **[M1]**

b.

- Forces acting on the bucket (m kg mass) (taking downwards [direction of motion] as the positive direction):

$F_{net} = m(0.5) = 0.5m$. …. (1)

$F_{net} = mg - T$. …. (2)

Equate equations (1) and (2):

$0.5m = mg - T$. …. (3)

- Forces acting on the 5 kg mass in the direction parallel to the plane (taking up the plane [direction of motion] as the positive direction):

$F_{net} = ma = 5(0.5) = 2.5$. …. (4)

$F_{net} = T - \underbrace{5g\sin(\alpha)}_{\text{Component of the weight force parallel to the plane}}$. …. (5)

Substitute $\tan(\alpha) = \frac{12}{5} \Rightarrow \sin(\alpha) = \frac{12}{13}$ into equation (5):

$F_{net} = T - \frac{60}{13}g$. …. (6)

Equate equations (4) and (6):

$2.5 = T - \frac{60}{13}g$. …. (7) **[M1]**

Use a CAS to solve, correct to two decimal places, equations (3) and (6) simultaneously for m.

Answer: $m = 5.13$. **[A1]**

c. i.

Method 1:

- Consider the motion of the combined-mass system.
- Mass (kg) of bucket at time *t*: $6 - 0.1t$.

Note: The initial mass of the bucket is $m = 6$.

$F_{net} = ma$

$\Rightarrow (5 + [6 - 0.1t])a = (6 - 0.1t)g - 5g\sin(\alpha)$.

Substitute $\sin(\alpha) = \dfrac{12}{13}$:

$(11 - 0.1t)a = (6 - 0.1t)g - \dfrac{60g}{13}$. **[M1]**

- Substitute $a = 0.2$

(bucket is moving **downwards**) and solve for *t* correct to two decimal places (use a CAS):

Answer 1: $t = 11.84$. **[A1]**

- Substitute $a = -0.2$ **[M1]**

(bucket is moving **upwards**) and solve for *t* correct to two decimal places (use a CAS):

Answer 2: $t = 15.77$. **[A1]**

Method 2:

- The 5 kg mass moves down the plane (and therefore the bucket moves upwards) if $m < 5\sin(\alpha)$ (from **part a.**). Substitute $\sin(\alpha) = \frac{12}{13}$:

The bucket moves upwards if $m < \frac{60}{13}$.

- The initial mass of the bucket is $m = 6 > \frac{60}{13}$ therefore the bucket initially moves downwards.

- Mass (kg) of bucket at time *t*: $6 - 0.1t$.

Note: The initial mass of the bucket is $m = 6$.

- The bucket moves downwards until

$$m = \frac{60}{13} \quad \Rightarrow 6 - 0.1t = \frac{60}{13} \quad \Rightarrow t = \frac{180}{13}.$$

When $t = \frac{180}{13}$ the direction of motion of the bucket changes *instantaneously* from downwards to upwards (*assuming an inextensible string*).

Therefore, the bucket moves upwards (from rest) for $t > \frac{180}{13}$ and the 5 kg mass moves down the plane.

Therefore, there are two cases to consider.

Case 1: Bucket is moving downwards.

- Forces acting on the bucket (taking downwards [direction of motion] as the positive direction):

$F_{net} = ma = (6 - 0.1t)a$. …. (1)

$F_{net} = mg - T = (6 - 0.1t)g - T$. …. (2)

Equate equations (1) and (2):

$(6 - 0.1t)a = (6 - 0.1t)g - T$. …. (3)

Substitute $a = 0.2$ into equation (3):

$(6 - 0.1t)(0.2) = (6 - 0.1t)g - T$. …. (4) **[M1]**

• Forces acting on the 5 kg mass in the direction parallel to the plane (taking up the plane [direction of motion] as the positive direction):

Adapted from **part b.** equations (4) and (6) (since the 5 kg mass is moving up the plane):

$$5(0.2) = T - \frac{60}{13}g \quad \Rightarrow 1 = T - \frac{60}{13}g . \qquad \ldots. (5)$$

• Use a CAS to solve, correct to two decimal places, equations (4) and (5) simultaneously for t: $t = 11.84$.

Answer 1: $t = 11.84$. **[A1]**

Case 2: Bucket is moving upwards.

• New motion so 're-set' time: Take $t = 0$ to be when the bucket begins moving upwards.

Mass (kg) of bucket at time t: $\frac{60}{13} - 0.1t$. **[M1]**

Note: The mass of the bucket is $m = \frac{60}{13}$ when it begins moving upwards.

• Forces acting on the bucket (taking upwards [direction of motion] as the positive direction):

$$F_{net} = ma = \left(\frac{60}{13} - 0.1t\right)a . \qquad \ldots. (1)$$

$$F_{net} = T - mg = T - \left(\frac{60}{13} - 0.1t\right)g . \qquad \ldots. (2)$$

Equate equations (1) and (2):

$$\left(\frac{60}{13} - 0.1t\right)a = T - \left(\frac{60}{13} - 0.1t\right)g . \qquad \ldots. (3)$$ **[M1]**

Substitute $a = 0.2$ into equation (3):

$$\left(\frac{60}{13} - 0.1t\right)(0.2) = T - \left(\frac{60}{13} - 0.1t\right)g . \qquad \ldots. (4)$$

- Forces acting on the 5 kg mass in the direction parallel to the plane (taking down the plane [direction of motion] as the positive direction):

Adapted from **part a.** equation (6) (substitute $a = 0.2$):

$1 = \frac{60}{13} g - T$. …. (5)

- Use a CAS to solve equations (4) and (5) simultaneously for *t*:

$t = \frac{25}{13}$.

Therefore, the second time at which $a = 0.2$ is $t = \frac{180}{13} + \frac{25}{13}$.

Answer 2: $t = 15.77$. **[A1]**

c. ii.

From **part c.i.**: The bucket moves downwards for $t < \frac{180}{13}$.

Therefore, for the first 6 seconds the bucket is moving downwards and the 5 kg object is moving up the plane.

Therefore, the equations of motion are

Bucket: $(6-0.1t)a = (6-0.1t)g - T$. .... (1)

(from **part c.i. case 1** equation (3)):

5 kg mass: $5a = T - \frac{60}{13}g \quad \Rightarrow T = 5a + \frac{60}{13}g$. .... (2)

(adapted from **part b.** equations (4) and (6)):

Substitute equation (2) into equation (1):

$$(6-0.1t)a = (6-0.1t)g - \left(5a + \frac{60}{13}g\right).$$ **[M1]**

Use a CAS to solve for *a*:

$$a = \frac{\left(\frac{18}{13} - 0.1t\right)g}{11 - 0.1t}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{\left(\frac{18}{13} - 0.1t\right)g}{11 - 0.1t} \quad .... (1)$$ **[M1]**

subject to the boundary conditions $v = \frac{dx}{dt} = 0$ and $x = 0$ at $t = 0$.

There is no direction change therefore the distance travelled is equal to the value of *x* when $t = 6$.

Option 1: Use a CAS to directly solve differential equation (1) and then substitute $t = 6$ into the solution.

Note: This option may not work on a standard CAS device but will work for a CAS software such as *Mathematica*:

```
In[18]:= DSolve[{x''[t] == 9.8 ((18 / 13) - 0.1*t) / (11 - 0.1*t), x'[0] == 0, x[0] == 0},
           x[t], t]
Out[18]= {{x[t] → 487223. - 5371.61 t + 4.9 t^2 - 103654. Log[110. - 1. t] + 942.308 t Log[110. - 1. t]}}

In[5]:= f[t_] := 487222.8686850217 - 5371.606498535163 t + 4.9 t^2 - 103653.84615384617 Log[110 - t] +
           942.3076923076925 t Log[110 - t]

In[6]:= f[6]
Out[6]= 19.3216
```

Option 2: Use a CAS to solve differential equation (1) in two stages and then substitute $t = 6$ into the solution.

Note: This option should be used if Option 1 fails with a CAS calculator.

Stage 1: Solve $\dfrac{dv}{dt} = \dfrac{\left(\dfrac{18}{13} - 0.1t\right)g}{11 - 0.1t}$ subject to the boundary condition $v = 0$ at $t = 0$.

$v = 9.8t + 942.308\log_e(110 - t) - 4429.3$.

Stage 2: Use the integral solution to solve $\dfrac{dx}{dt} = 9.8t + 942.308\log_e(110 - t) - 4429.3$ subject to the boundary condition $x = 0$ at $t = 0$ at $t = 6$.

$x = \int_0^6 9.8t + 942.308\log_e(110 - t) - 4429.3 \, dt \qquad = 19.32$.

Answer: 19.32 metres. **[A1]**

Question 6

a. i.

Answer: $\frac{dw}{dx} = -\sin(x) + i\cos(x) \quad = i\left(i\sin(x) + \cos(x)\right) \quad = iw$. **[A1]**

a. ii.

- $\frac{dw}{dx} = iw \quad \Rightarrow \frac{dx}{dw} = \frac{1}{iw}$

$\Rightarrow x = \frac{1}{i}\int \frac{1}{w}\, dw$ *

$= \frac{1}{i}\log_e |w| + C$

$\Rightarrow e^{ix-iC} = |w| \quad \Rightarrow |w| = e^{ix}e^{-iC}$

$\Rightarrow w = Ae^{ix}$. *

Lines marked *: **[A1]**

- Substitute $w = 1$ when $x = 0$: $A = 1$. *

Therefore $w = e^{ix}$. *

Substitute $w = \cos(x) + i\sin(x)$: *

Lines marked *: **[A1]**

$e^{ix} = \cos(x) + i\sin(x)$.

b.
Answer:

$e^{ix} = \cos(x) + i\sin(x)$. .... (1)

$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$. .... (2)

Equation (1) + equation (2):

$e^{ix} - e^{-ix} \quad = \left(\cos(x) + i\sin(x)\right) - \left(\cos(x) - i\sin(x)\right) \quad = 2i\sin(x)$

$\Rightarrow \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$.

[A1]

c. i.

Answer:

$\sin(x) = p = \dfrac{e^{ix} - e^{-ix}}{2i} \quad \Rightarrow 2ip = e^{ix} - e^{-ix} \quad \Rightarrow 2ipe^{ix} = \left(e^{ix}\right)^2 - 1$ **[A1]**

$\Rightarrow \left(e^{ix}\right)^2 - 2ipe^{ix} - 1 = 0$.

Accept any other valid approach.

c. ii.

- $\left(e^{ix}\right)^2 - 2ipe^{ix} - 1 = 0$ is a quadratic equation in the variable e^{ix}.

Substitute $a = 1$, $b = -2ip$ and $c = -1$ into the quadratic formula:

$e^{ix} = \dfrac{2ip \pm \sqrt{-4p^2 + 4}}{2}$ **[M1]**

$= \dfrac{2ip \pm 2\sqrt{1 - p^2}}{2} \quad = ip \pm \sqrt{1 - p^2}$.

- Substitute $\sin^{-1}(0) = 0 \Rightarrow p = 0$ and $x = 0$: $1 = 0 \pm \sqrt{1}$. *

Therefore, the negative root solution is rejected: *

$e^{ix} = ip + \sqrt{1 - p^2}$

$\Rightarrow ix = \log_e\left(ip + \sqrt{1 - p^2}\right)$ *

$\Rightarrow x = \dfrac{1}{i}\log_e\left(ip + \sqrt{1 - p^2}\right) \quad = -i\log_e\left(\sqrt{1 - p^2} + ip\right)$.

Substitute $p = \sin(x) \Rightarrow x = \sin^{-1}(p)$: *

$\sin^{-1}(p) = -i\log_e\left(\sqrt{1 - p^2} + ip\right)$.

Lines marked *: **[A1]**

c. iii.

$\log_e(z) = \log_e\left(r\text{cis}(\theta)\right) = \log_e(r) + \log_e\left(\text{cis}(\theta)\right)$. *

Substitute $\text{cis}(\theta) = \cos(\theta) + i\sin(\theta) = e^{i\theta}$:

$\log_e(z) = \log_e(r) + \log_e\left(e^{i\theta}\right) = \log_e(r) + i\theta$. *

Lines marked *: **[A1]**

d.

"Hence show …. " means that the previous results must be used to show the given result:

Substitute $p = 1$ into $\sin^{-1}(p) = -i\log_e\left(\sqrt{1-p^2} + ip\right)$:

$\sin^{-1}(1) = -i\log_e(i)$. *

$i = \text{cis}\left(\frac{\pi}{2}\right)$ therefore $\log_e(i) = \log_e(1) + i\frac{\pi}{2} = i\frac{\pi}{2}$. *

Therefore:

$\sin^{-1}(1) = -i\left(i\frac{\pi}{2}\right) = \frac{\pi}{2}$. *

Lines marked *: **[A1]**

e.

Let $z = \sqrt{1-p^2} + ip = r\text{cis}(\theta)$.

Case 1: $p \in [-1, 1]$.

$1 - p^2 \geq 0 \quad \Rightarrow \sqrt{1-p^2} \in R$ *

therefore $|z| = r = \sqrt{\left(1-p^2\right) + p^2} = 1$ *

therefore $\log_e\left(ip + \sqrt{1-p^2}\right) = \log_e(1) + i\theta = i\theta$ *

therefore $-i\log_e\left(ip + \sqrt{1-p^2}\right) = -i(i\theta) = \theta$ *

which is real.

Lines marked *: **[A1]**

Case 2: $p > 1$ or $p < -1$.

$1 - p^2 < 0 \quad \Rightarrow \sqrt{1-p^2} \notin R$.

$p^2 - 1 > 0 \quad \Rightarrow \sqrt{1-p^2} = \sqrt{-\left(p^2-1\right)} = i\sqrt{p^2-1}$

therefore $z = i\left(\sqrt{p^2-1} + p\right)$ *

therefore $|z| = r = \left|\sqrt{p^2-1} + p\right| \neq 1$ *

therefore $\log_e\left(ip + \sqrt{1-p^2}\right) = \log_e(r) + i\theta$ *

therefore $-i\log_e\left(ip + \sqrt{1-p^2}\right) = -i\left(\log_e(r) + i\theta\right) = \theta - i\log_e(r)$ *

which is not real since the imaginary part $\log_e(r) \neq 0$.

Lines marked *: **[A1]**

Note: It can be shown that $\sin^{-1}(p) = -i\log_e\left(ip + \sqrt{1-p^2}\right)$ when $p > 1$ or $p < -1$.

For example:

- Substitute $p = -2$ into $\sin^{-1}(p) = -i\log_e\left(ip + \sqrt{1-p^2}\right)$:

$\sin^{-1}(-2) = -i\log_e\left(-2i + \sqrt{1-4}\right)$

$= -i\log_e\left(-2i + \sqrt{-3}\right) = -i\log_e\left(-2i + i\sqrt{3}\right)$

$= -i\log_e\left(\left(\sqrt{3}-2\right)i\right)$.

- $z = \left(\sqrt{3}-2\right)i = -\left(2-\sqrt{3}\right)i$

therefore $z = r\text{cis}(\theta)$ where $r = \left(2-\sqrt{3}\right)$ and $\theta = -\dfrac{\pi}{2}$:

$\left(\sqrt{3}-2\right)i = \left(2-\sqrt{3}\right)\text{cis}\left(-\dfrac{\pi}{2}\right)$.

Therefore $\log_e\left(\left(\sqrt{3}-2\right)i\right) = \log_e\left(2-\sqrt{3}\right) - i\dfrac{\pi}{2}$.

- Therefore:

$$\sin^{-1}(-2) = -i\left(\log_e\left(2-\sqrt{3}\right) - i\frac{\pi}{2}\right)$$

$$= -i\log_e\left(2-\sqrt{3}\right) - \frac{\pi}{2}.$$

Input:

$\sin^{-1}(-2)$

Exact Result:

$-\sin^{-1}(2)$

(result in radians)

Decimal approximation:

$-1.5707963267948966192313216916397514420985846996875529104\ldots + 1.3169578969248167086250463473079684440269819714675164797\ldots\, i$

(result in radians)

END OF SOLUTIONS