



Trial Examination 2020

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

Question 1 (2 marks)

Let $u = \frac{2}{x}$ and $y = \arctan(u)$.

$$\frac{du}{dx} = -\frac{2}{x^2} \text{ and } \frac{dy}{du} = \frac{1}{1+u^2}$$

Using chain rule differentiation, $f'(x) = \left(-\frac{2}{x^2}\right) \frac{1}{\left(1 + \frac{4}{x^2}\right)}$. M1

So $f'(x) = \frac{-2}{x^2 + 4}$ ($a = -2, b = 4$). A1

Question 2 (3 marks)

a. The conjugate root theorem cannot be applied because not all of the coefficients of the equation are real. A1

b. Let the other root be w , where $w \in \mathbb{C}$.

$$(z - (-1 + i))(z - w) = z^2 - (1 - 2i)z + 1 + 5i$$

$$\text{LHS} = z^2 - (-1 + i + w)z + (-1 + i)w \quad \text{M1}$$

Comparing coefficients of z , for example, gives $-1 + i + w = 1 - 2i$.

$$\text{So } w = 2 - 3i.$$

The other root is $2 - 3i$. A1

Note: w can also be found by solving $1 + 5i = (-1 + i)w$.

Question 3 (4 marks)

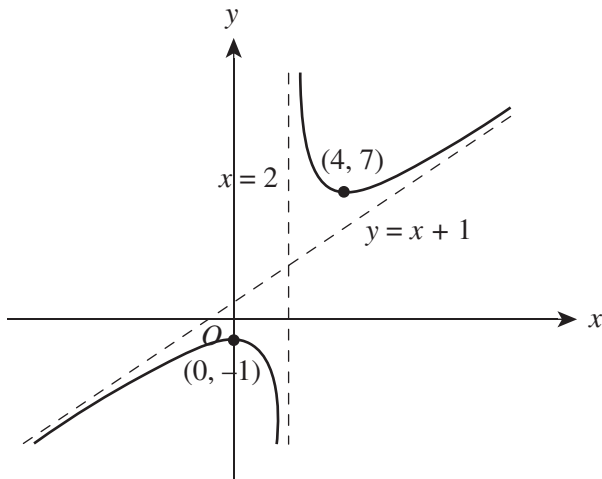
$$y = x + 1 + \frac{4}{x-2}$$

The vertical asymptote is $x = 2$ and the non-vertical asymptote is $y = x + 1$.

$$\frac{dy}{dx} = 1 - \frac{4}{(x-2)^2}$$

$$\frac{dy}{dx} = 0 \text{ occurs for } x = 0, 4.$$

The stationary points are $(0, -1)$, which is also the y-intercept, and $(4, 7)$.



correct shape (two branches and asymptotic behaviour) A1
correct vertical asymptote A1
correct non-vertical asymptote A1
correct stationary points A1

Question 4 (3 marks)**Method 1:**

$$2x + 1 = \pm x$$

A1

Solving $x + 1 = 0$ and $3x + 1 = 0$ for x gives $x = -1, -\frac{1}{3}$.

M1

$$\text{So } -1 < x < -\frac{1}{3}.$$

A1

Method 2:

$$\text{Either } (2x + 1)^2 < x^2 \text{ or } (2x + 1)^2 = x^2.$$

A1

$$\text{Either } 3x^2 + 4x + 1 < 0 \text{ or } 3x^2 + 4x + 1 = 0.$$

Either $(x + 1)(3x + 1) < 0$ or $(x + 1)(3x + 1) = 0$ gives $x = -1, -\frac{1}{3}$.

M1

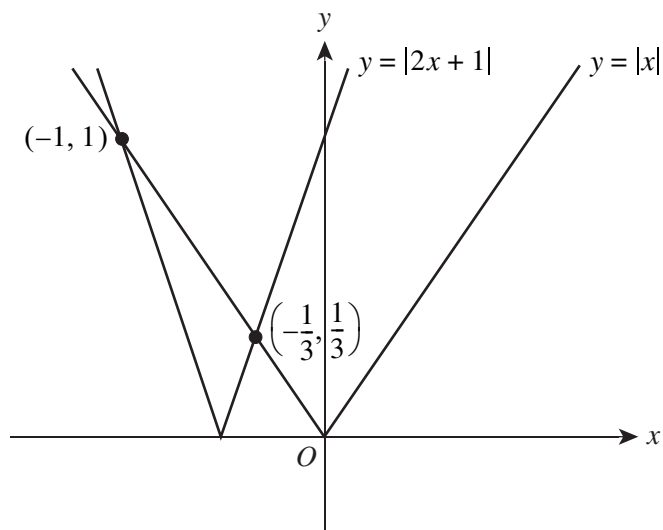
$$\text{So } -1 < x < -\frac{1}{3}.$$

A1

Method 3:

Sketch the graphs of $y = |2x + 1|$ and $y = |x|$.

M1



The graphs intersect at $x = -1, -\frac{1}{3}$.

A1

$$\text{So } -1 < x < -\frac{1}{3}.$$

A1

Question 5 (5 marks)

a. $\vec{AB} = (t - 8)\underline{i} + (t + 6)\underline{j} + (2t - 5)\underline{k}$

M1

$$|\vec{AB}| = \sqrt{(t - 8)^2 + (t + 6)^2 + (2t - 5)^2}$$

$$= \sqrt{(t^2 - 16t + 64) + (t^2 + 12t + 36) + (4t^2 - 20t + 25)}$$

A1

$$= \sqrt{6t^2 - 24t + 125}$$

b. Either $\frac{d}{dt}|\vec{AB}|^2 = 12t - 24$ or $\frac{d}{dt}|\vec{AB}| = \frac{6t - 12}{\sqrt{6t^2 - 24t + 125}}$. M1

Solving $12t - 24 = 0$ for t gives $t = 2$. M1

Substituting $t = 2$ into $|\vec{AB}|$ gives:

$$\begin{aligned} |\vec{AB}| &= \sqrt{6(2)^2 - 24(2) + 125} \\ &= \sqrt{101} \end{aligned} \quad \text{A1}$$

The minimum distance between points A and B is $\sqrt{101}$.

Question 6 (4 marks)

Method 1:

Resolving vertically: $S \cos(\alpha) = W + T \sin(\beta)$ (1) A1

Resolving horizontally: $S \sin(\alpha) = T \cos(\beta)$ (2)

Substituting $S = \frac{T \cos(\beta)}{\sin(\alpha)}$ into (1) gives $\frac{T \cos(\beta)}{\sin(\alpha)} \cos(\alpha) = W + T \sin(\beta)$. M1

$$\frac{T \cos(\beta)}{\tan(\alpha)} = W + T \sin(\beta)$$

$$T \cos(\beta) = W \tan(\alpha) + T \sin(\beta) \tan(\alpha) \quad \text{M1}$$

$$T \cos(\beta) - T \sin(\beta) \tan(\alpha) = W \tan(\alpha)$$

$$T(\cos(\beta) - \sin(\beta) \tan(\alpha)) = W \tan(\alpha) \quad \text{A1}$$

$$\text{So } T = \frac{W \tan(\alpha)}{\cos(\beta) - \sin(\beta) \tan(\alpha)}.$$

Method 2:

Use of the sine rule (Lami's theorem) gives $\frac{W}{\sin(90^\circ + \alpha + \beta)} = \frac{T}{\sin(180^\circ - \alpha)}$. A1

$$\frac{W}{\cos(\alpha + \beta)} = \frac{T}{\sin(\alpha)}$$

$$T \cos(\alpha + \beta) = W \sin(\alpha) \Rightarrow T = \frac{W \sin(\alpha)}{\cos(\alpha + \beta)} \quad \text{M1}$$

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

$$T = \frac{\frac{W \sin(\alpha)}{\cos(\alpha)}}{\frac{\cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)}{\cos(\alpha)}} \quad \text{M1 A1}$$

$$= \frac{W \tan(\alpha)}{\cos(\beta) - \sin(\beta) \tan(\alpha)}$$

Question 7 (4 marks)

Differentiating implicitly with respect to x gives $2y \frac{dy}{dx} = 2 - 2(x+y) \left(1 + \frac{dy}{dx}\right)$. M1

Tangents parallel to the x -axis satisfy the condition $\frac{dy}{dx} = 0$. That is, $0 = 2 - 2(x+y)$.

So this condition satisfies $y = 1 - x$ (or equivalent). A1

Substituting $y = 1 - x$ (or equivalent) into $y^2 = 2x - (x+y)^2$ gives $(1-x)^2 = 2x - (x+(1-x))^2$.

Expanding gives $1 - 2x + x^2 = 2x - 1$.

$$x^2 - 4x + 2 = 0 \quad \text{M1}$$

Solving $x^2 - 4x + 2 = 0$ for x (quadratic formula or completing the square) gives $x = 2 \pm \sqrt{2}$.

For example, $x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(2)}}{2} = 2 \pm \sqrt{2}$.

Substituting $x = 2 \pm \sqrt{2}$ into $y = 1 - x$ gives $y = -1 \mp \sqrt{2}$.

So the equations of the tangents are $y = -1 + \sqrt{2}$, $y = -1 - \sqrt{2}$. A1

Question 8 (4 marks)

Let the length be L , where $L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$.

$$\frac{dx}{dt} = 2t - \frac{2}{t} \text{ and } \frac{dy}{dt} = 4$$

$$L = \int_1^2 \sqrt{\left(2t - \frac{2}{t}\right)^2 + (4)^2} dt \quad \text{M1}$$

$$= \int_1^2 \sqrt{4t^2 + \frac{4}{t^2} + 8} dt$$

$$= 2 \int_1^2 \sqrt{\left(t + \frac{1}{t}\right)^2} dt$$

$$= 2 \int_1^2 \left(t + \frac{1}{t}\right) dt \quad \text{A1}$$

$$= 2 \left[\frac{t^2}{2} + \log_e(t) \right]_1^2 \quad \text{M1}$$

$$= 2 \left(2 + \log_e(2) - \frac{1}{2} \right)$$

$$= 3 + 2\log_e(2) \quad \text{A1}$$

Question 9 (4 marks)

$$\arctan(2x) + \arctan(x) = \arctan(3)$$

$$\tan(\arctan(2x) + \arctan(x)) = \tan(\arctan(3))$$

$$\frac{\tan(\arctan(2x)) + \tan(\arctan(x))}{1 - (\tan(\arctan(2x))\tan(\arctan(x)))} = 3 \quad \text{M1}$$

$$\frac{2x + x}{1 - 2x^2} = 3 \quad \text{A1}$$

$$3(2x^2 + x - 1) = 0 \Rightarrow 3(x + 1)(2x - 1) = 0$$

Solving for x gives $x = -1, \frac{1}{2}$.

When $x = -1$, $\arctan(-2) + \arctan(-1) < 0$ and $\arctan(3) > 0$. A1

Hence we reject $x = -1$.

So $x = \frac{1}{2}$. A1

Question 10 (7 marks)

- a. Attempt product rule differentiation on both components to find $\dot{\underline{r}}(t)$. M1

Product rule on the \underline{i} component:

$$\frac{d}{dt}(e^t \cos(t)) = -e^t \sin(t) + e^t \cos(t)$$

Product rule on the \underline{j} component:

$$\frac{d}{dt}(e^t \sin(t)) = e^t \cos(t) + e^t \sin(t)$$

$$\dot{\underline{r}}(t) = (e^t \cos(t) - e^t \sin(t))\underline{i} + (e^t \cos(t) + e^t \sin(t))\underline{j} \quad \text{A1}$$

$$\dot{\underline{r}}(0) = \underline{i} + \underline{j} \quad \text{A1}$$

- b. $|\underline{r}(t)| = e^t \sqrt{\cos^2(t) + \sin^2(t)}$
 $= e^t$ A1

$$|\dot{\underline{r}}(t)| = e^t \sqrt{(\cos(t) - \sin(t))^2 + (\sin(t) + \cos(t))^2}$$

$$= e^t \sqrt{2\cos^2(t) + 2\sin^2(t)}$$

$$= \sqrt{2}e^t \quad \text{A1}$$

Attempt to find $\underline{r}(t) \cdot \dot{\underline{r}}(t)$. M1

$$\underline{r}(t) \cdot \dot{\underline{r}}(t) = e^{2t}(\cos(t)(\cos(t) - \sin(t)) + \sin(t)(\sin(t) + \cos(t)))$$

$$= e^{2t}(\cos^2(t) + \sin^2(t))$$

$$= e^{2t}$$

$$\text{Use of } \cos(\theta) = \frac{\underline{r}(t) \cdot \dot{\underline{r}}(t)}{|\underline{r}(t)||\dot{\underline{r}}(t)|} \text{ gives } \cos(\theta) = \frac{e^{2t}}{e^t \times \sqrt{2}e^t}.$$

$$\text{So } \cos(\theta) = \frac{1}{\sqrt{2}}. \quad \text{A1}$$

Hence, $\theta = \frac{\pi}{4}$ and $\underline{r}(t)$ always makes an angle of $\frac{\pi}{4}$ with $\dot{\underline{r}}(t)$.