The Mathematical Association of Victoria

Trial Examination 2021

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes

Writing time: 2 hours

QUESTION & ANSWER BOOK

Structure of Book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 28 pages.
- Formula sheet
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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SECTION A - Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1

The graph of the function with rule $f(x) = 2x + 1 + \frac{1}{(x-a)^2}$ has

A. one asymptote and a local minimum at $(1+a, 0)$

B. one asymptote and a local maximum at $(1-a, 0)$

C. two asymptotes and a local minimum at $(1+a, 4+2a)$

D. two asymptotes and local maximum at $(1+a, 4+2a)$

E. two asymptotes and a local minimum at $(1+a, 4+4a)$

Question 2

If $\cot(2x) = \frac{1}{3}$, $\frac{\pi}{2} < x < \frac{3\pi}{4}$, then the value of $\sec^2(x)$ is

A. $\frac{2}{3}\left(\sqrt{10} - 10\right)$

B. $\frac{2}{3}\left(10 - \sqrt{10}\right)$

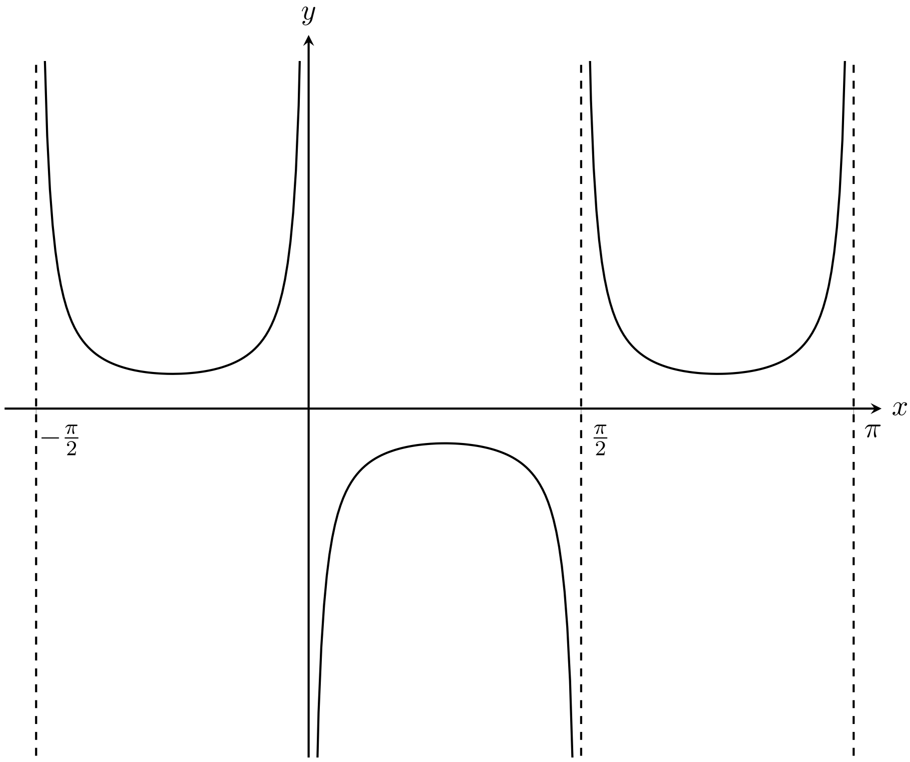
C. $\frac{2}{9}\left(\sqrt{10} - 10\right)$

D. $\frac{2}{9}\left(\sqrt{10} + 10\right)$

E. $\frac{2}{9}\left(10 - \sqrt{10}\right)$

Question 3

The graph of $y = \sec\left(a(x-b)\right)$ for $x \in \left(-\frac{\pi}{2}, \pi\right)$ is shown below:



Possible values of a and b are

A. $a = 2$, $b = \frac{\pi}{4}$

B. $a = 2$, $b = -\frac{\pi}{4}$

C. $a = 2$, $b = -\frac{\pi}{2}$

D. $a = \frac{1}{2}$, $b = \frac{\pi}{4}$

E. $a = \frac{1}{2}$, $b = -\frac{\pi}{4}$

SECTION A – continued

Question 4

If $z = a + bi$, where $a, b \in R$, then the value of $\text{Im}\left(\frac{iz}{\bar{z}}\right)$ is

A. $\frac{a^2 - b^2}{a^2 + b^2}$

B. $-\frac{2ab}{a^2 + b^2}$

C. $\frac{b^2 - a^2}{a^2 + b^2}$

D. $\frac{a^2 + b^2}{a^2 - b^2}$

E. $\frac{2ab}{a^2 - b^2}$

Question 5

The function that describes the ray $\left\{z : \text{Arg}(z + 1 - 2i) = \frac{\pi}{4}\right\}$ is

A. $f : (-1, \infty) \to R,\ f(x) = x + 1$

B. $f : (1, \infty) \to R,\ f(x) = x - 1$

C. $f : (-1, \infty) \to R,\ f(x) = x - 1$

D. $f : [-1, \infty) \to R,\ f(x) = x + 3$

E. $f : (-1, \infty) \to R,\ f(x) = x + 3$

Question 6

If $\left(\sqrt{3} + i\right)^n = -a$ where a is a positive real constant, then $\left(\sqrt{3} + i\right)^{3n+3}$ is equal to

A. $-8a^3$

B. $-16a^3 i$

C. $-8a^3 i$

D. $8a^3 i$

E. $8a^3$

SECTION A – continued
TURN OVER

Question 7

The length of the curve $f(x)=e^{\sqrt{x}}$ between $x=1$ and $x=4$ is given by

A. $\int_1^4 \sqrt{1+\frac{e^x}{4x}}\,dx$

B. $\int_1^4 \sqrt{1+\frac{e^x}{2x}}\,dx$

C. $\int_1^4 \sqrt{1+\frac{e^{\sqrt{x}}}{2\sqrt{x}}}\,dx$

D. $\int_1^4 \sqrt{1+\frac{e^{2\sqrt{x}}}{2x}}\,dx$

E. $\int_1^4 \sqrt{1+\frac{e^{2\sqrt{x}}}{4x}}\,dx$

Question 8

With a suitable substitution, $\int_0^{\frac{\pi}{4}} \cos^3(2x)\,dx$ can be expressed as

A. $2\int_0^1 \left(1-u^2\right)du$

B. $\frac{1}{2}\int_0^1 \left(1-u^2\right)du$

C. $\int_0^1 \left(1-u^2\right)du$

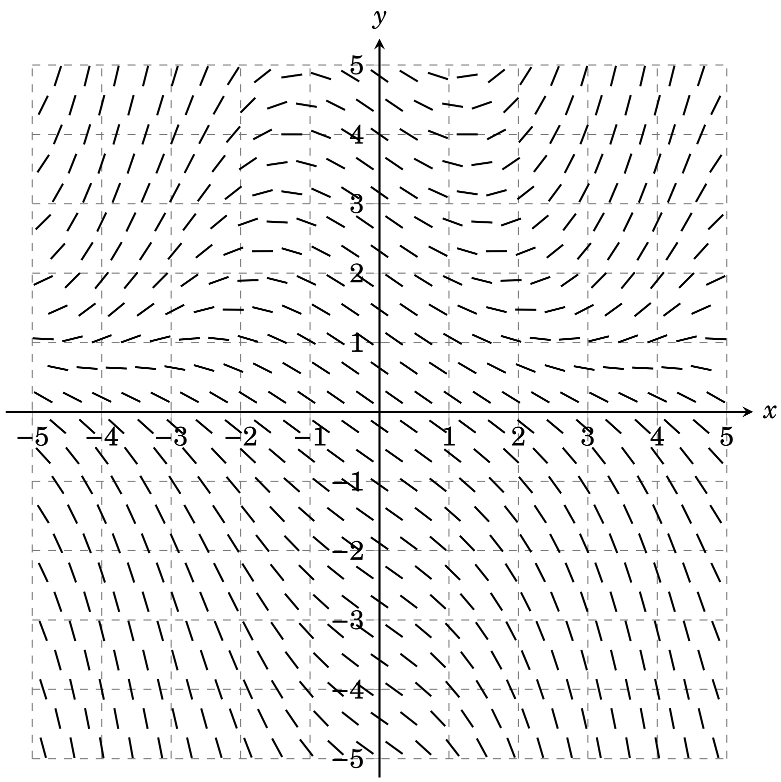
D. $\frac{1}{2}\int_0^{\frac{\pi}{4}} \left(1-u^2\right)du$

E. $2\int_0^{\frac{\pi}{4}} \left(1-u^2\right)du$

SECTION A – continued

Question 9

The direction field for a certain differential equation is shown below:



The solution curve to the differential equation that passes through the point $(-3,2)$ could also pass through the point

A. $(4,3)$

B. $(4,0)$

C. $(0,-3)$

D. $(2,3)$

E. $(1,1)$

Question 10

A tank contains 10kg of salt dissolved in 100 litres of water. Water containing 0.05 kg of salt per litre is poured into the tank at a rate of 10 litres per minute. At the same time, 5 litres per minute is removed from the tank.

If x kg is the amount of salt in the tank at time t, then a differential equation for x would be

A. $\dfrac{dx}{dt}=\dfrac{1}{20}-\dfrac{x}{100+5t}$

B. $\dfrac{dx}{dt}=\dfrac{1}{20}-\dfrac{x}{20+t}$

C. $\dfrac{dx}{dt}=\dfrac{1}{2}-\dfrac{x}{100+5t}$

D. $\dfrac{dx}{dt}=\dfrac{1}{2}-\dfrac{x}{20+t}$

E. $\dfrac{dx}{dt}=\dfrac{1}{2}-\dfrac{x}{20-t}$

Question 11

Let the vectors $\underset{\sim}{a}=2\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$ and $\underset{\sim}{b}=2\underset{\sim}{i}-\underset{\sim}{j}+2\underset{\sim}{k}$.

Let θ be the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. The value of $\tan\theta$ is

A. $\dfrac{4}{\sqrt{65}}$

B. $\dfrac{9}{\sqrt{65}}$

C. $\dfrac{\sqrt{65}}{4}$

D. $\dfrac{\sqrt{65}}{9}$

E. $\dfrac{\sqrt{65}}{3}$

SECTION A – continued

Question 12

If the vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + m\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ are **linearly independent** then

A. $m \in R$

B. $m = -\frac{8}{21}$

C. $m \in R \setminus \left\{-\frac{8}{21}\right\}$

D. $m \in R \setminus \left\{\frac{8}{21}\right\}$

E. $m = \frac{8}{21}$

Question 13

The magnitude of the force $\underset{\sim}{F} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ that acts perpendicular to the direction of $\underset{\sim}{d} = 2\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ is

A. $\frac{1}{3}$

B. $\frac{1}{\sqrt{3}}$

C. $\frac{\sqrt{17}}{3}$

D. $\frac{17}{9}$

E. $\frac{1}{9}$

Question 14

If $\frac{dy}{dx}=\frac{2y}{x+1}$ and $y_0=1$ when $x_0=1$, then using Euler's formula with step size 0.1, y_3 is equal to

A. $\frac{46}{35}$

B. $\frac{552}{385}$

C. $\frac{121}{35}$

D. $\frac{253}{210}$

E. $\frac{23}{21}$

Question 15

A solution to the differential equation $\frac{dy}{dx}=\frac{\sin(x+y)-\sin(x-y)}{2xy}$ can be obtained from

A. $\int \frac{\tan(y)}{y}dy=\int \frac{\cot(x)}{x}dx$

B. $\int \frac{\cos(y)}{y}dy=\int \frac{\cos(x)}{x}dx$

C. $\int \frac{y}{\cos(y)}dy=\int \frac{\sin(x)}{x}dx$

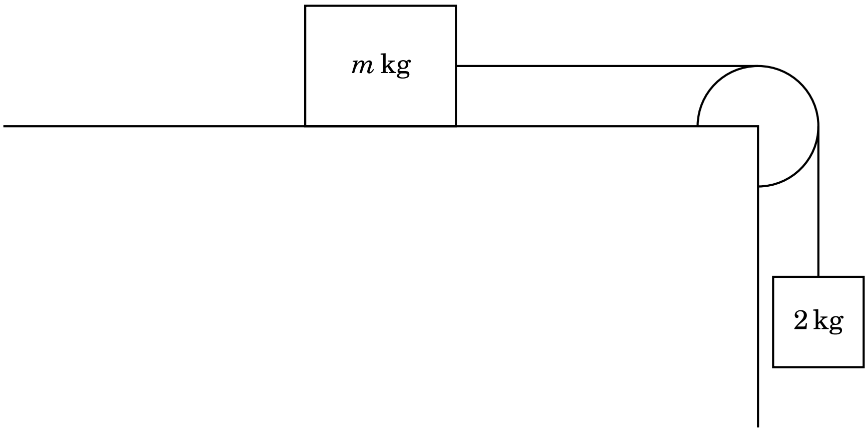
D. $\int \frac{\sin(y)}{y}dy=\int \frac{x}{\cos(x)}dx$

E. $\int \frac{y}{\sin(y)}dy=\int \frac{\cos(x)}{x}dx$

SECTION A – continued

Question 16

A particle of mass m kg is placed on a smooth horizontal surface. It is attached by a light inextensible string passing over a smooth pulley to a particle of mass 2 kg which hangs as shown in the diagram below:



When released, the system experiences an acceleration of 6 ms^{-2}. The value of m in kg is equal to

A. $\frac{g}{3}-2$

B. $2-\frac{g}{3}$

C. $\frac{g}{3}+2$

D. $\frac{2g}{3}$

E. $\frac{3g}{2}$

Question 17

The position vectors of two moving particles are given by

$$\underset{\sim}{r}_A(t) = \left(1+t^2\right)\underset{\sim}{i} + (3t+2)\underset{\sim}{j} \text{ and } \underset{\sim}{r}_B(t) = \left(6t-4\right)\underset{\sim}{i} + \left(t^2-8\right)\underset{\sim}{j}, \text{ where } t \geq 0.$$

The particles will collide at the point with position vector

A. $26\underset{\sim}{i}+17\underset{\sim}{j}$

B. $2\underset{\sim}{i}+5\underset{\sim}{j}$

C. $5\underset{\sim}{i}-4\underset{\sim}{j}$

D. $12\underset{\sim}{i}+23\underset{\sim}{j}$

E. $15\underset{\sim}{i}\text{-}3\underset{\sim}{j}$

SECTION A – continued
TURN OVER

Question 18

The acceleration, a ms^{-2}, of a particle moving in a straight line is given by $a = \frac{v}{\sqrt{v+1}}$, where v is the velocity of the particle at any time $t \geq 0$.

The particle is initially travelling with a velocity of 1 ms^{-1}.

The time taken for the velocity of the particle to increase to 5 ms^{-1} is closest to

A. 1.762

B. 2.966

C. 3.113

D. 5.221

E. 6.311

Question 19

The mean length of the gestation period of a random sample of camels was found to be 385 days. The standard deviation of the gestation period of all camels is known to be 12 days.

A 99% confidence interval for the mean length of the gestation period of all camels calculated from the sample is $(382.81, 387.19)$.

The number of camels in the sample was closest to

A. 36

B. 80

C. 100

D. 200

E. 250

Question 20

When working properly, a machine fills boxes of breakfast cereal with a mean mass of μ=725g and σ =15g. A random sample of 40 boxes of cereal was weighed and it was found that the mean mass of the boxes was $\bar{x} = 731g$. Assuming $H_0 : \mu = 725$, $H_1 : \mu \neq 725$ and $\sigma = 15$, the p-value for the two-sided test is closest to

A. 0.0057

B. 0.0114

C. 0.0165

D. 0.0253

E. 0.9943

END OF SECTION A

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SECTION B

Instructions for Section B
Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1 (10 marks)

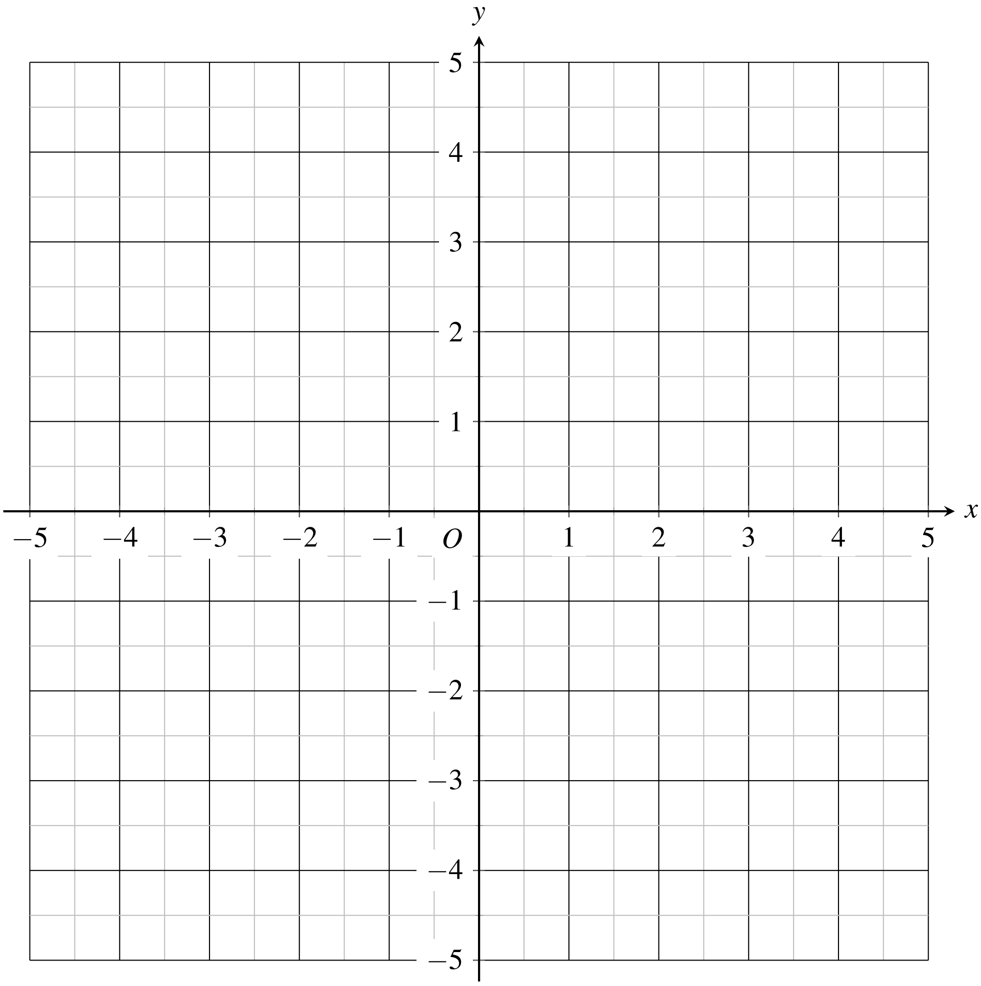
Consider the function f with rule $f(x) = \dfrac{4(x-1)}{x^2-4}$.

a. Find the coordinates of the point of inflection of f, correct to three decimal places. 2 marks

b. State the equations of all the asymptotes of the graph of f. 2 marks

SECTION B – Question 1 - continued

c. Sketch the graph of $y = f(x)$ on the axes provided below, labelling all axis intercepts with their coordinates. Label the point of inflection with its coordinates, correct to three decimal places. 3 marks



The region in the first quadrant bounded by the coordinate axes and the graph of f is rotated about the x-axis to form a solid of revolution V.

d. **i.** Write down a definite integral in terms of x that gives the volume V. 2 marks

ii. Find the volume V. Give your answer in the form $\frac{\pi}{a}\left(b - c\log_e(c)\right)$, where a, b and c are integers. 1 mark

Question 2 (9 marks)

The positions of two particles, relative to the origin O, are given by

$$\underset{\sim}{r}_A(t) = (t-1)\underset{\sim}{i} + \left(\frac{t-1}{2}\right)\underset{\sim}{j}$$

$$\underset{\sim}{r}_B(t) = \left(3\sin(t)+2\right)\underset{\sim}{i} + 2\cos(t)\underset{\sim}{j}$$

where $t \geq 0$.

$\underset{\sim}{i}$ is a unit vector in the horizontal direction and $\underset{\sim}{j}$ is a unit vector in the vertical direction.

Distances are measured in metres and time in measured in seconds.

a. Write down the Cartesian equations of both particles. 2 marks

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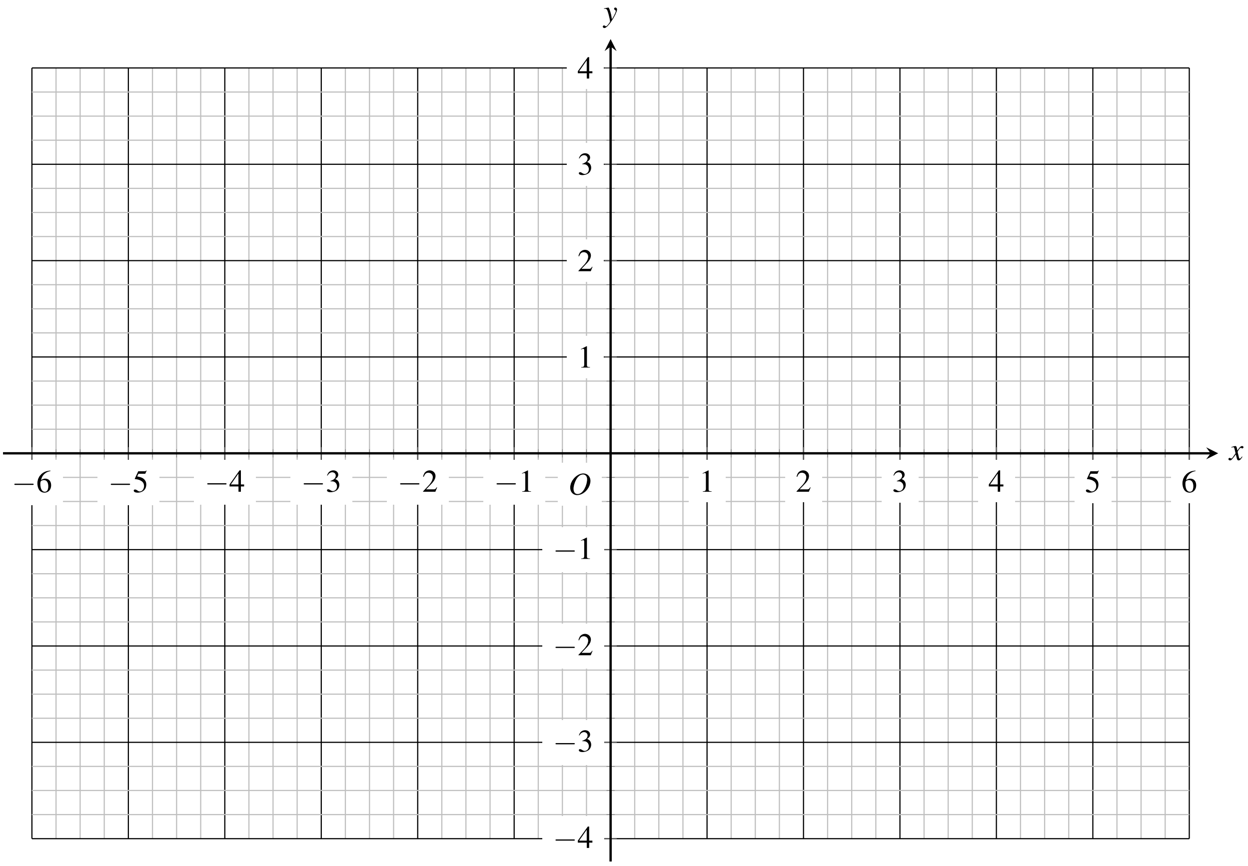
b. Sketch the paths of each particle on the axes below. Label the starting point of each particle with its coordinates and indicate the direction of each particle with an arrow. 3 marks



SECTION B – Question 2 – continued

TURN OVER

c. Show that the particles do not collide. 2 marks

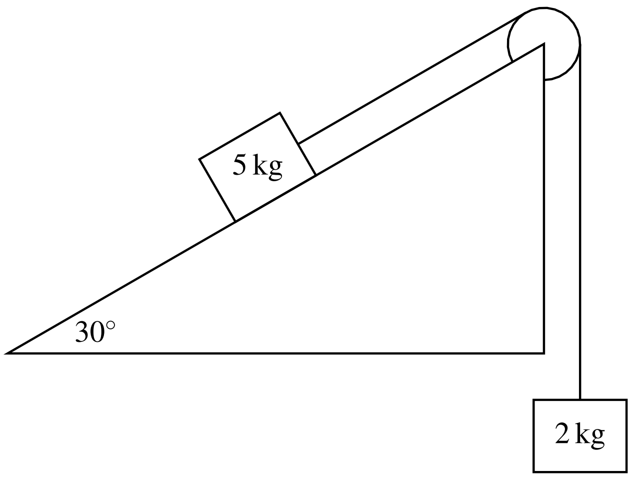
d. **i.** Find the value of t, in seconds, correct to three decimal places, when the particles are closest to each other. 1 mark

ii. Find the minimum distance between the particles, in metres, correct to two decimal places. 1 mark

Question 3 (9 marks)

A 5 kg mass on a smooth plane inclined at 30° to the horizontal is connected by a light inextensible string passing over a smooth pulley to a mass of 2 kg as shown in the diagram below.

The smooth plane is 10 m in length and the 5 kg mass is initially at rest at the top of the plane.



a. On the diagram above, label the forces that act on each mass. 1 mark

b. Determine the acceleration of the 5 kg mass down the plane in ms^{-2}. 2 marks

c. Determine the tension in the string in newtons connecting the two masses. 1 mark

SECTION B – Question 3 - continued
TURN OVER

d. Show that the velocity of the 5 kg mass after it has travelled 5 m down the plane is $\sqrt{7}$ ms^{-1}. 1 mark

After the 5 kg mass has travelled 5 m down the plane, the string connecting the two masses breaks. The motion of the mass is opposed by a force of $0.2v^2$, where v is the velocity of the 5 kg mass when it is a distance of x m from the point where the string broke.

e. **i.** Write down an equation involving an integral that could be solved to find the velocity of the 5 kg mass when it reaches the bottom of the inclined plane. 3 marks

ii. Find the velocity of the 5 kg mass when it reaches the bottom of the plane. Give your answer in ms^{-1} and correct to two decimal places. 1 mark

Question 4 (10 marks)

A tank initially contains 10 L of pure water. A solution containing docosahexaenoic acid (DHA) flows into the tank at a rate of 20 L per minute and the mixture in the tank is kept well stirred. At the same time, 10 L per minute of the mixture flows out of the tank. The solution flowing into the tank contains $e^{-0.2t}$ grams of DHA per litre of water after t minutes.

Let x grams be the amount of DHA in the tank after t minutes.

a. Write down, in terms of x and t, an expression for the **concentration** of DHA in the tank in grams per litre. 1 mark

b. Hence show that $\frac{dx}{dt}+\frac{x}{1+t}=20e^{-0.2t}$. 1 mark

c. Find the time at which the amount of DHA in the tank is equal to 30 grams and **decreasing**. Give your answer in minutes, correct to two decimal places. 2 marks

SECTION B – Question 4 - continued

TURN OVER

d. Using Euler's method with a step size of 20 seconds, find an estimate for the amount of DHA in the tank after 3 minutes. Give your answer in grams, correct to two decimal places. 1 mark

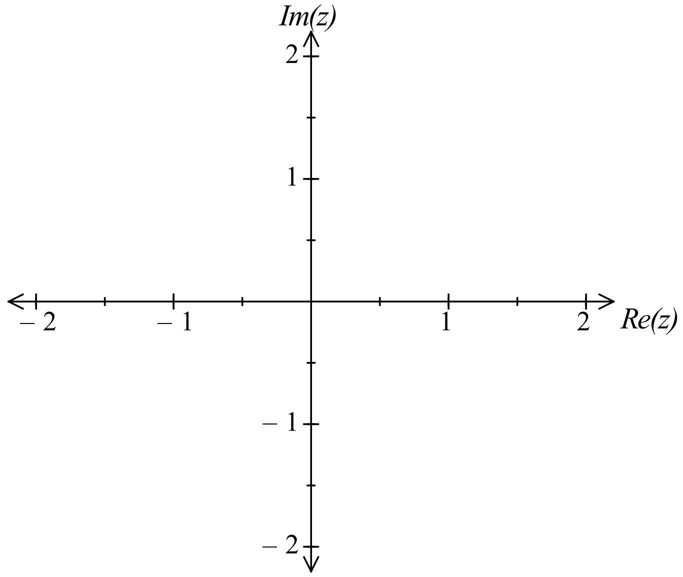
e. Find, correct to three decimal places, the rate of change with respect to t of DHA in the tank after 4 minutes. 2 marks

f. Find the amount of DHA that has **flowed out** of the tank over the first 8 minutes. Give your answer in grams, correct to one decimal place. 3 marks

SECTION B - continued

Question 5 (10 marks)
Consider the relation $|z-1+i|=1$ where $z \in C$.

a. Draw a graph of this relation. 1 mark



b. Find in the form $a+ib$, where $a, b \in R$, the value of z satisfying $|z-1+i|=1$ which has the largest possible

i. modulus. 2 marks

ii. principal argument. 1 mark

SECTION B – Question 5 - continued
TURN OVER

c. Find in the form $a+ib$ where, $a,b \in R$, all values of z satisfying $|z-1+i|=1$ that have a:

i. modulus of $\sqrt{3}$. 2 marks

ii. principal argument of $\arctan(-2)$. 2 marks

d. $|z-1+i|=1$ can be written in the form $\sqrt{2}\left|z-(1+\sqrt{2})+ai\right|=|2z-b+2i|$ where $a,b\in R$.
Find the values of *a* and *b*. 2 marks

SECTION B – continued
TURN OVER

Question 6 (12 marks)

The mass, in grams, of peaches grown in Wakanda is normally distributed with a mean of 125 and a standard deviation of 20.

a. Peaches in Wakanda are sold in paper bags. Unfortunately, despite being one of the world's most technologically advanced nations, the paper bags used in Wakanda rip if the total mass of peaches inside them is greater than 2 kg.

Find the maximum number of peaches that a paper bag can hold without ripping with a probability greater than 0.9. 2 marks

b. T'Challa suspects that the mean mass of peaches grown in Wakanda is not 125 grams and decides to conduct a statistical test at the 2% level of significance. He collects a random sample of thirty peaches from a Wakandan peach farm and calculates the mean mass of peaches in the sample to be 123.7 grams.

i. Write down suitable hypotheses H_0 and H_1 for this test. 1 mark

ii. State the probability of rejecting H_0 when it is true. 1 mark

iii. Let $\bar{x}$ be the mean mass, in grams, of Wakandan peaches in a sample of size 30 and let $\left(C_1^*,\ C_2^*\right)$ be the interval such that H_0 is accepted at the 2% level of significance when $\bar{x} \in \left(C_1^*, C_2^*\right)$. Find C_1^* and C_2^*, correct to two decimal places. 4 marks

iv. State an approximate 98% confidence interval that T'Challa would calculate for the mean mass of peaches grown in Wakanda. Give the endpoints of the interval correct to two decimal places. 1 mark

SECTION B – Question 6 - continued

TURN OVER

v. State whether H_0 should be rejected or not rejected by T'Challa at the 2% level of significance and justify your answer. 1 mark

vi. Find the probability of **not** rejecting H_0 for this test if the mean mass of peaches grown in Wakanda is actually 114 grams. Give your answer correct to four decimal places. 2 marks

END OF QUESTION AND ANSWER BOOK