2022 VCAA Specialist Mathematics Exam 1 Solutions
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Q1a $\ p(z)=(z+ai)^2+b=z^2+2aiz-a^2+b=z^2+6iz-25$

$\therefore a=3$ and $b=-16$ $\ \therefore p(z)=(z+3i)^2-16$

Q1b $\ p(z)=(z+3i)^2-16=0$ $\ \therefore (z+3i)^2=16$, $z=\pm4-3i$

Q2 $\ \dfrac{dy}{dx}=-x\sqrt{4-y^2}$ and $y(2)=0$; $\ \displaystyle\int\dfrac{-1}{\sqrt{4-y^2}}dy=\int xdx$

$\therefore \cos^{-1}\left(\dfrac{y}{2}\right)=\dfrac{x^2}{2}+c$ and $y(2)=0$ $\ \therefore c=\dfrac{\pi}{2}-2$,

$y=2\cos\left(\dfrac{x^2}{2}+\dfrac{\pi}{2}-2\right)$

Q3a For 1 cup, $\mu=10$, $\sigma=1.5$, $\text{Var}(X)=1.5^2=2.25$

For 4 cups, $\mu=4\times10=40$, $\text{Var}(X)=4\times2.25=9$, $\sigma=\sqrt{9}=3$

$34=40-2\times3=\mu-2\sigma$ $\ \therefore \Pr(X>34)=1-\dfrac{0.05}{2}=9.75\approx9.8$

Q3b Note: Misleading statement "Following the service, the mean time taken to dispense 25 cups of coffee is found to be 9 s"?
$n=25$, $\bar{x}=9$, $\sigma=1.5$
95% confidence interval for μ is

$\left(9-1.96\times\dfrac{1.5}{\sqrt{25}},\ 9-1.96\times\dfrac{1.5}{\sqrt{25}}\right)\approx(8.4,9.6)$

Q4 $\ \dfrac{3x^2+4x+12}{x(x^2+4)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}$, A, B and C are found to be 3, 0

and 4 respectively. $\ \displaystyle\int\dfrac{3x^2+4x+12}{x(x^2+4)}dx=\int\dfrac{3}{x}+\dfrac{4}{x^2+4}dx$

$=3\ln|x|+2\tan^{-1}\left(\dfrac{x}{2}\right)+c$

Q5a $\ t=0$, $v=0$, $\tan\theta=\dfrac{1}{3}$ $\ \therefore \sin\theta=\dfrac{1}{\sqrt{10}}$, $\cos\theta=\dfrac{3}{\sqrt{10}}$

$a=\dfrac{10g\sin\theta}{10}=\dfrac{g}{\sqrt{10}}$ constant acceleration $\ \therefore v=at=\dfrac{2g}{\sqrt{10}}$ m s^{-1}

Q5b $\ R=10g\sin\theta=\sqrt{10}g$

Q6a $\ \tilde{a}.\tilde{b}=ab\cos\theta$, $8=21\cos\theta$, $\cos\theta=\dfrac{8}{21}$ and $0<\theta<\dfrac{\pi}{2}$

Q6bi $\ \overrightarrow{OP}=x\tilde{i}+y\tilde{j}$, $\overrightarrow{OQ}=\overrightarrow{OP}-2a\tilde{i}=(x-2a)\tilde{i}+y\tilde{j}$

Q6bii $\ \overrightarrow{OP}.\overrightarrow{QP}=x(x-2a)+y^2=x(x-2a)+a^2-(x-a)^2=0$
$\therefore \overrightarrow{OP}\perp\overrightarrow{QP}$

Q7 By implicit differentiation: $\ \dfrac{d}{dx}x\cos(x+y)=0$

$x\left(-\sin(x+y)\times\left(1+\dfrac{dy}{dx}\right)\right)+\cos(x+y)=0$

$x\sin(x+y)\dfrac{dy}{dx}=\cos(x+y)-x\sin(x+y)$ $\ \therefore \dfrac{dy}{dx}=\dfrac{1}{x\tan(x+y)}-1$

At $\left(\dfrac{\pi}{24},\dfrac{7\pi}{24}\right)$, $\dfrac{dy}{dx}=\dfrac{1}{\frac{\pi}{24}\tan\left(\frac{\pi}{24}+\frac{7\pi}{24}\right)}-1=\dfrac{1}{\frac{\pi}{24}\tan\frac{\pi}{3}}-1=\dfrac{8\sqrt{3}-\pi}{\pi}$

Q8 $\ a=-4x$, $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)=-4x$, $\dfrac{1}{2}v^2=-2x^2+c$

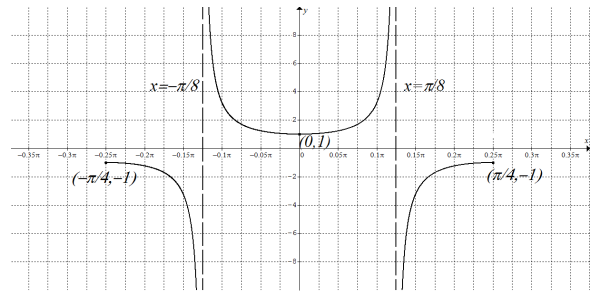$v=-2$ at $x=0$ $\ \therefore \dfrac{1}{2}v^2=-2x^2+2$, $v^2=4(1-x^2)$, $v=-2\sqrt{1-x^2}$

Q9 $\ u=\sin2x$, $\dfrac{1}{2}\dfrac{du}{dx}=\cos2x$

$f(x)=\displaystyle\int\dfrac{\cos2x}{\sin^32x}dx=\int\dfrac{1}{2}u^{-3}du=\dfrac{1}{-4u^2}+c$ and given $f\left(\dfrac{\pi}{8}\right)=\dfrac{3}{4}$

$\therefore c=\dfrac{5}{4}$ $\ \therefore f(x)=\dfrac{5}{4}-\dfrac{1}{4\sin^22x}$

Q10a

Q10b $\ V=\displaystyle\int_{-\frac{\pi}{24}}^{\frac{\pi}{48}}\pi\sec^2(4x)dx=\left[\dfrac{\pi}{4}\tan(4x)\right]_{\frac{-\pi}{24}}^{\frac{\pi}{48}}$

$=\dfrac{\pi}{4}\left(\tan\left(\dfrac{\pi}{12}\right)-\tan\left(-\dfrac{\pi}{6}\right)\right)=\dfrac{\pi}{4}\left(\tan\left(\dfrac{\pi}{12}\right)+\tan\left(\dfrac{\pi}{6}\right)\right)$

Note: $\tan\dfrac{\pi}{6}=\dfrac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$, let $x=\tan\dfrac{\pi}{12}>0$ $\ \therefore \dfrac{1}{\sqrt{3}}=\dfrac{2x}{1-x^2}$

Solve to obtain $x=\tan\dfrac{\pi}{12}=2-\sqrt{3}$

$\therefore V=\dfrac{\pi}{4}\left(2-\sqrt{3}+\dfrac{1}{\sqrt{3}}\right)=\dfrac{\pi}{4}\left(\dfrac{2\sqrt{3}-2}{\sqrt{3}}\right)=\dfrac{(3-\sqrt{3})\pi}{6}$

Please inform mathline@itute.com re conceptual and/or mathematical errors.