2022 VCAA Specialist Mathematics Exam 2 Solutions

Use CAS to save time

SECTION A – Multiple-choice questions

1	2	3	4	5	6	7	8	9	10
B	E	E	B	A	E	D	C	B	E

11	12	13	14	15	16	17	18	19	20
A	A	B	D	E	A	C	B	D	C

Q1 By CAS   B

Q2 $1 - \dfrac{4\sin^2 x}{\tan^2 x + 1} = 1 - \dfrac{4\sin^2 x}{\sec^2 x} = 1 - (2\sin x \cos x)^2$

$= 1 - \sin^2 2x = \cos^2 2x$ E

Q3 Consider $c = 0$ which has 1 horizontal and 1 vertical asymptotes.   E

Q4 b is real .: a and c are complex conjugates .: $|a| = |c|$ B

Q5 $\dfrac{y-1}{x} = \tan \dfrac{3\pi}{4}$.: $y = 1 - x$ and $x < 0$ A

Q6 Circle $|z - 5 - 5i| = 4$ intersects $|z - 5| = 2$ at two points   E

Q7 $u = 1 + e^x$, $\dfrac{du}{dx} = e^x$, $\dfrac{1}{e^x}\dfrac{du}{dx} = 1$, $\dfrac{1}{u-1}\dfrac{du}{dx}$

$\displaystyle\int_0^{\log_e 2} \dfrac{1}{u}\cdot\dfrac{1}{u-1}\dfrac{du}{dx}\, dx = \int_2^3 \dfrac{1}{u(u-1)}\, du = \int_2^3 \left(\dfrac{1}{u-1} - \dfrac{1}{u}\right) du$ D

Q8 $x = 0$, $\dfrac{dy}{dx} = 0$.: either A, B or C

When $x < 0$ and $y > 0$, $\dfrac{dy}{dx} > 0$.: not A

At $(-1, 1)$, $\dfrac{dy}{dx} > 1$.: C   C

Q9 $x_0 = 1$ $y_0 = 2$ $\dfrac{dy}{dx} = 2x^2 = 2$

$x_1 = 1 + h$, $y_1 = 2 + 2h$ $\dfrac{dy}{dx} = 2(1+h)^2$

$x_2 = 1 + 2h$ $y_2 = 2 + 2h + h\times 2(1+h)^2 = 2.976$.: $h = 0.2$ B

Q10 $5x^2 y - 3xy + y^2 = 10$ At $(1, m)$, $5m - 3m + m^2 = 10$

.: $m = -1 \pm \sqrt{11}$. By checking the graph both values give negative gradient.   E

Q11 Let $\tilde{c} = \alpha \tilde{a} + \beta \tilde{b}$,

$-3\tilde{i} + 2\tilde{j} + 5\tilde{k} = (2\alpha + \beta)\tilde{i} + (-3\alpha + 2\beta)\tilde{j} + (\alpha p - \beta q)\tilde{k}$

Equate coefficients: $2\alpha + \beta = -3$, $-3\alpha + 2\beta = 2$, $\alpha p - \beta q = 5$

Solve the first two to obtain $\alpha = -\dfrac{8}{7}$ and $\beta = -\dfrac{5}{7}$

Substitute into the third equation to obtain $8p = 5q - 35$ A

Q12 $\tilde{u}.\tilde{v} = 0$, $-\operatorname{cosec}x \cos x + \sqrt{3} = 0$, $-\dfrac{\cos x}{\sin x} + \sqrt{3} = 0$,

$\tan x = \dfrac{1}{\sqrt{3}}$, $x = \dfrac{\pi}{6}, \dfrac{7\pi}{6}$ A

Q13 $\ddot{r} = \sin t\, \tilde{i} + 2\cos t\, \tilde{j}$, $\dot{r} = -\cos t\, \tilde{i} + 2\sin t\, \tilde{j} + \tilde{c}$

$\dot{r}(0) = 2\tilde{i} + \tilde{j} = -\tilde{i} + \tilde{c}$.: $\tilde{c} = 3\tilde{i} + \tilde{j}$

.: $\dot{r} = (3 - \cos t)\tilde{i} + (1 + 2\sin t)\tilde{j}$ B

Q14 $17^2 = 7^2 + 2as$.: $s = \dfrac{120}{a}$, $\dfrac{1}{2}s = \dfrac{60}{a}$, $v^2 = 7^2 + 2a\left(\dfrac{60}{a}\right)$ and v

is in the same direction as 7 and 17 .: $v = \sqrt{169} = 13$ D

Q15 Newton's second law.   E

Q16 The three forces form a closed triangle when they are put (added) head to tail. $10^2 = 5^2 + 7^2 - 2\times 5 \times 7 \cos(\pi - \theta)$

.: $100 = 25 + 49 + 70\cos\theta$ A

Q17 $s = \dfrac{1}{2}(u+v)t$, $30 = \dfrac{1}{2}(3+v)6$.: $v = 7$

.: $\Delta p = m\times(v-u) = 7(7-3) = 28$ C

Q18 For T, $\mu = 30$, $\operatorname{Var}(T) = 2.5^2 = 6.25$. For $T_d = T_1 - T_2$,

$\operatorname{E}(T_d) = 30 - 30 = 0$, $\operatorname{Var}(T_d) = 6.25 + 6.25 = 12.5$.: $\operatorname{sd}(T_d) = \sqrt{12.5}$

$\Pr(T_d < -6 \cup T_d > 6) = 1 - \Pr(-6 < T_d < 6) \approx 0.0897$ B

Q19 $\operatorname{E}(C) = \operatorname{E}(0.3m + 0.5) = 0.3\operatorname{E}(m) + 0.5 = 2.6$

$\operatorname{Var}(C) = 0.3^2 \operatorname{Var}(m) = 0.0009$, $\operatorname{sd}(C) = \sqrt{0.0009} = 0.03$

For a sample of 100, $\operatorname{E}(\overline{C}) = 2.6$, $\operatorname{sd}(\overline{C}) = \dfrac{0.03}{\sqrt{100}} = 0.003$

95% confidence interval
$= (2.6 - 1.96\times 0.003, \ 2.6 + 1.96\times 0.003) = (2.594, 2.606)$ D

Q20 Two 2-kg masses: $X = X_1 + X_2$, $\operatorname{E}(X) = 1.980 + 1.980 = 3.960$

$\operatorname{Var}(X) = 0.015^2 + 0.015^2 = 0.00045$, $\operatorname{sd}(X) = \sqrt{0.00045}$

One 4-kg mass Y: $\operatorname{E}(Y) = 3.940$, $\operatorname{sd}(Y) = 0.002$,

$\operatorname{Var}(Y) = 0.002^2 = 0.000004$
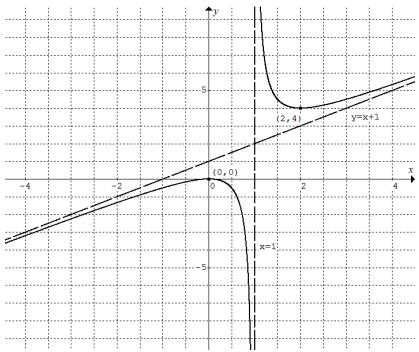
Let $W = X - Y$, $\operatorname{E}(W) = 3.960 - 3.940 = 0.020$

$\operatorname{Var}(W) = 0.00045 + (-1)^2 \times 0.000004 = 0.000454$

$\operatorname{sd}(W) = \sqrt{0.000454}$.: $\Pr(W > 0) \approx 0.826$ C

SECTION B

Q1a $f(x) = \dfrac{x^2}{x-1} = \dfrac{(x+1)(x-1)+1}{x-1} = x + 1 + \dfrac{1}{x-1}$

Asymptotes: $x = 1$ and $y = x + 1$

Q1b

Q1ci $f(x) = \dfrac{x^2}{x-k} = \dfrac{(x+k)(x-k)+k^2}{x-k} = x+k+\dfrac{k^2}{x-k}$

Asymptotes: $x=k$ and $y=x+k$

Q1cii $f'(x) = 1 - \left(\dfrac{k}{x-k}\right)^2 = 0$

$\therefore x=2k$ and $y=4k$ OR $x=0$ and $y=0$

Distance $= \sqrt{(2k)^2+(4k)^2} = \sqrt{20k^2} = 2\sqrt{5}\,|k|$

Q1di Let $x+3 = \left|\dfrac{x^2}{x-1}\right|$, $(x+3)^2 = \left(\dfrac{x^2}{x-1}\right)^2$,

$(2x-3)(2x^2+2x-3)=0$, $x=\dfrac{3}{2}$, $x=\dfrac{-1\pm\sqrt{7}}{2}$

$V = \displaystyle\int_{\frac{-\sqrt{7}-1}{2}}^{\frac{\sqrt{7}-1}{2}} \left(\pi(x+3)^2 - \pi\left|\dfrac{x^2}{x-1}\right|^2\right)dx = \pi\int_{\frac{-\sqrt{7}-1}{2}}^{\frac{\sqrt{7}-1}{2}} \left((x+3)^2 - \left(\dfrac{x^2}{x-1}\right)^2\right)dx$

Q1dii By CAS $V=51.42$
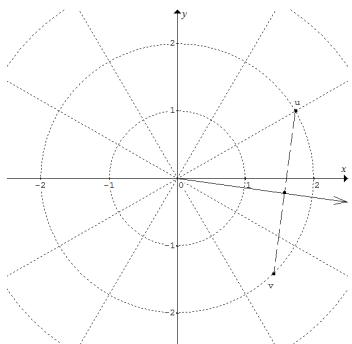
Q2ai $u\,v = (a+i)(b-\sqrt{2}\,i)$
$= \left(ab+\sqrt{2}\right) + \left(b-\sqrt{2}\,a\right)i = \left(\sqrt{2}+\sqrt{6}\right)+\left(\sqrt{2}-\sqrt{6}\right)i$
$\therefore ab=\sqrt{6}$ and $b-\sqrt{2}\,a = \sqrt{2}-\sqrt{6}$
Eliminate b: $a^2 + \left(1-\sqrt{3}\right)a - \sqrt{3} = 0$

Q2aii $a^2 + \left(1-\sqrt{3}\right)a - \sqrt{3} = \left(a-\sqrt{3}\right)(a+1) = 0$
$\therefore a=-1$ and $b=-\sqrt{6}$

Q2b $u = \sqrt{3}+i = 2\text{cis}\dfrac{\pi}{6}$, $v = \sqrt{2}-\sqrt{2}\,i = 2\text{cis}\left(-\dfrac{\pi}{4}\right)$

Q2c $\theta = \dfrac{\text{Arg}(u)+\text{Arg}(v)}{2} = \dfrac{1}{2}\left(\dfrac{\pi}{6}-\dfrac{\pi}{4}\right) = -\dfrac{\pi}{24}$

Q2d $\angle uOv = \dfrac{\pi}{6}+\dfrac{\pi}{4} = \dfrac{5\pi}{12}$
Minor segment area = minor sector area – triangular area
$= \dfrac{5\pi}{6} - \dfrac{1}{2}\times 2\times 2\times\sin\dfrac{5\pi}{12} \approx 0.69$

Q3ai $\displaystyle\int \dfrac{1}{2}e^x\,dx = \int\dfrac{1}{1+4t^2}\,dt$
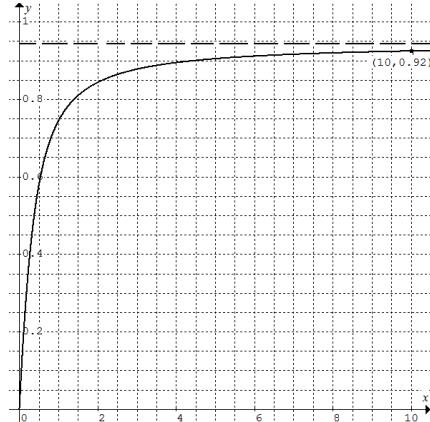
Q3aii $\dfrac{1}{2}\displaystyle\int e^x\,dx = \dfrac{1}{2}\int\dfrac{\frac{1}{2}}{\frac{1}{4}+t^2}\,dt \;\therefore\; e^x = \tan^{-1}(2t)+c$

$x=0$, $t=0$ $\therefore e^x = \tan^{-1}(2t)+1$

Q3bi As $t\to\infty$, $\tan^{-1}(2t)\to\dfrac{\pi}{2}$, $x\to\log_e\left(\dfrac{\pi}{2}+1\right)$

$\therefore$ asymptote is $x = \log_e\left(\dfrac{\pi}{2}+1\right)$

Q3bii $x(10) = \log_e\left(\tan^{-1}20+1\right) \approx 0.92$

Q3c At $t=3$, $x = \log_e\left(\tan^{-1}6+1\right)$, $\dfrac{dx}{dt} = \dfrac{2e^{-x}}{1+4t^2} \approx 0.02$ m s^{-1}

Q3d First particle: $x_1 = \log_e\left(\tan^{-1}(2\times6)+1\right) = \log_e\left(\tan^{-1}12+1\right)$
Second particle: $x_2 = \log_e\left(\tan^{-1}(3\times6-6)+1\right) = \log_e\left(\tan^{-1}12+1\right) = x_1$

Q3e Same distance at $t=6$. First particle: $\dfrac{dx}{dt} = \dfrac{\frac{2}{1+4t^2}}{\tan^{-1}2t+1}$

Second particle: $\dfrac{dx}{dt} = \dfrac{\frac{3}{1+(3t-6)^2}}{\tan^{-1}(3t-6)+1}$

Ratio is $\dfrac{v_1}{v_2} = \dfrac{2}{3}$ when the two particles are at the same distance from O.

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Q4a $\dot{r} = \dfrac{\pi}{8}\cos\dfrac{\pi t}{4}\,\tilde{i} + 2\tilde{j}$, $\dot{r}(0) = \dfrac{\pi}{8}\tilde{i} + 2\tilde{j}$, $\tan\theta = \dfrac{\frac{\pi}{8}}{2}$, $\theta \approx 11.1°$

Q4bi Speed $= \left|\dot{r}(0)\right| = \sqrt{\left(\dfrac{\pi}{8}\right)^2 + 2^2} \approx 2.04 \text{ m s}^{-1}$

Q4bii $\left|\dot{r}\right| = \sqrt{\left(\dfrac{\pi}{8}\right)^2 \cos^2\dfrac{\pi t}{4} + 4}$

Min. speed $= 2 \text{ m s}^{-1}$ when $\cos^2\dfrac{\pi t}{4} = 0$, $t = 2$

Q4c Let $\tilde{d} = \tilde{r} - 7\tilde{j} = \dfrac{1}{2}\sin\dfrac{\pi t}{4}\,\tilde{i} + (2t - 7)\tilde{j}$

$\left|\tilde{d}\right| = \sqrt{\dfrac{1}{4}\sin^2\dfrac{\pi t}{4} + (2t - 7)^2}$, by CAS $\min\left|\tilde{d}\right| = 0.188 \text{ m}$

Q4d Distance $= \displaystyle\int_0^4 \sqrt{\left(\dfrac{\pi}{8}\cos\dfrac{\pi t}{4}\right)^2 + 2^2}\, dt \approx 8.077 \text{ m}$

Q5a $a = \dfrac{R}{m} = \dfrac{10\sqrt{2}\cos 45° + 20\sqrt{3}\cos 30° - 5k}{5}$

$= \dfrac{10 + 30 - 5k}{5} = 8 - k \text{ m s}^{-2}$

Q5b Note: The question did not define $t = 0$.
$\Delta p = m\Delta v = 5\times(2 - 0.5) = 7.5 \text{ kg m s}^{-1}$

Q5c $v - u = at$, $2 - 0.5 = (8 - k)5$, $k = 7.7$

Q5d $v^2 = u^2 + 2as$, $2^2 = 0.5^2 + 2\times(8 - 7.7)\times s$, $s = 6.25$
Distance from O to P is 6.25 m

Q5e $a = \dfrac{R}{m} = \dfrac{-5g\sin\theta - 38.5}{5} = -g\sin\theta - 7.7$
$s = 0.2$, $u = 1.95$, $v = 0$
$v^2 = u^2 + 2as$, $0 = 1.95^2 + 2\times(-9.8\sin\theta - 7.7)\times 0.2$, $\theta \approx 10.6°$

Q6a H_0 is $\mu = 15$, H_1 is $\mu < 15$

Q6b $\mu = 15$, $\sigma = 0.25$

Sample: $n = 64$, $\bar{x} = 14.94$, $\text{sd}(\overline{X}) = \dfrac{0.25}{\sqrt{64}} = 0.03125$

$p = \Pr(\overline{X} < 14.94 \mid \mu = 15) \approx 0.027$

Q6c $p < 0.05$ ∴ good evidence against H_0 at 5% level of significance for the one-tailed test ∴ the supplier's claim is not supported.

Q6d $\Pr(\overline{X} < a \mid \mu = 15) > 0.05$, $\Pr\left(Z < \dfrac{a - 15}{0.03125}\right) > 0.05$,

$a \approx 14.95$

Q6e $\mu = 406$, $\sigma = 5$, $\text{Var}(X) = 25$
Let $Y = X_1 - X_2$ be the difference
∴ $\mu_Y = \mu_{X_1} - \mu_{X_2} = 0$,
$\text{Var}(Y) = 1^2\text{Var}(X_1) + (-1)^2\text{Var}(X_2) = 25 + 25 = 50$, $\sigma_Y = \sqrt{50}$
$\Pr(-3 \le Y \le 3) \approx 0.329$

Q6f $375 \text{ mL} \equiv 375\times1.04 = 390$ grams
Let X_1 and X_2 be the masses of a can and soft drink respectively, and
$Y = X_1 + X_2$ the mass of a can of soft drink.
Statistics for
Y: $\mu = 406$, $\text{Var}(Y) = 5^2 = 25$
X_1: $\mu = 15$, $\text{Var}(X_1) = 0.25^2 = 0.0625$
∴ for $X_2 = Y - X_1$, $\mu = 406 - 15 = 391$,
$\text{Var}(X_2) = 25 + 0.0625 = 25.0625$
∴ $\Pr(X_2 < 390) \approx 0.421$

> *Please inform mathline@itute.com re conceptual and/or mathematical errors*

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