The Mathematical Association of Victoria

Trial Examination 2023

SPECIALIST MATHEMATICS

Written Examination 2

STUDENT NAME ____

Reading time: 15 minutes

Writing time: 2 hours

QUESTION & ANSWER BOOK

Structure of Book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 30 pages.
- Formula sheet
- Answer sheet for multiple-choice questions.

Instructions

- Write your **name** in the space provided above on this page.
- Write your **name** on the multiple-choice answer sheet.
- Unless otherwise indicated, the diagrams are **not** drawn to scale.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A - Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

Question 1

The graph of $y = \dfrac{|2x-1|}{x-1}$ has straight line asymptotes

A. $x = 1$, $y = 2$ and $y = -2$

B. $x = -1$, $y = 2$ and $y = -2$

C. $x = 1$ and $y = 2$ only

D. $x = 1$ and $y = -2$ only

E. $x = -1$ and $y = 0$ only

Question 2

The implied domain of the function with rule $f(x) = \dfrac{1}{\arcsin\left(x^2-1\right)+\frac{\pi}{2}}$ is

A. $\left[-\sqrt{2}, \sqrt{2}\right]$

B. $\left(-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right)$

C. $\left[-\sqrt{2}, 1\right) \cup \left(1, \sqrt{2}\right]$

D. $\left[-\sqrt{2}, 0\right) \cup \left(0, \sqrt{2}\right]$

E. $\left(-\sqrt{2}, 1\right) \cup \left(1, \sqrt{2}\right)$

SECTION A – continued

Question 3

If $\tan(2x) = -1$ and $\frac{\pi}{4} < x < \frac{\pi}{2}$ then $\tan(x)$ is equal to

A. $1-\sqrt{2}$

B. $1+\sqrt{2}$

C. $-1+\sqrt{2}$

D. $1+2\sqrt{2}$

E. $-1-\sqrt{2}$

Question 4

Consider the following statement about the natural number n:

If n is divisible by 4 then n is divisible by 2

The converse of this statement is

A. If n is not divisible by 4 then n is not divisible by 2

B. If n is divisible by 2 then n is divisible by 4

C. If n is not divisible by 2 then n is divisible by 4

D. If n is not divisible by 2 then n is not divisible by 4

E. If n is divisible by 4 then n is not divisible by 2

Question 5

When proving by induction that $1^2+2^2+3^2+\ldots+n^2>\dfrac{n^3}{3}$, it is necessary in the inductive step to assume that

A. $1^2+2^2+3^2+\ldots+k^2>\dfrac{k^3}{3}$ and deduce that $1^2+2^2+3^2+\ldots+k^2+(k+1)^2>\dfrac{(k+1)^3}{3}$

B. $1^2+2^2+3^2+\ldots+k^2>\dfrac{k^3}{3}$ and deduce that $1^2+2^2+3^2+\ldots+k^2+(k+1)^2>\dfrac{k^3+1}{3}$

C. $1^2+2^2+3^2+\ldots+k^2>\dfrac{k^3}{3}$ and deduce that $1^2+2^2+3^2+\ldots+k^2+(k+1)^2>\dfrac{k^3}{3}+\dfrac{(k+1)^3}{3}$

D. $1^2+2^2+3^2+\ldots+k^2>\dfrac{k^3}{3}$ and deduce that $1^2+2^2+3^2+\ldots+k^2+(k+1)^2>\dfrac{k^3}{3}+\dfrac{k^3+1}{3}$

E. $1^2+2^2+3^2+\ldots+k^2+(k+1)^2>\dfrac{k^3+1}{3}$ and deduce that $1^2+2^2+3^2+\ldots+k^2+(k+1)^2>\dfrac{(k+1)^3}{3}$

Question 6

The Cartesian equation of the line defined by $|z+4|=|z-2i|$ is

A. $y=\dfrac{1}{2}x-3$

B. $y=\dfrac{1}{2}x-5$

C. $y=-2x-10$

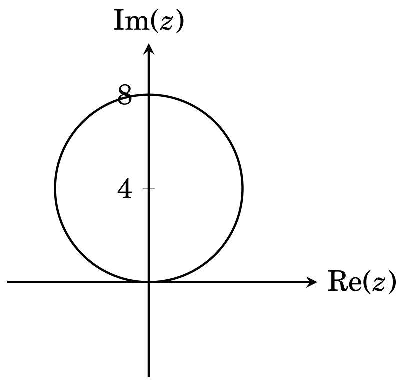
D. $y=-2x-3$

E. $y=-2x-5$

Question 7

Which one of the following shows the set of points $(z+4i)(\overline{z}-4i)=16$, where $z \in C$?

A.



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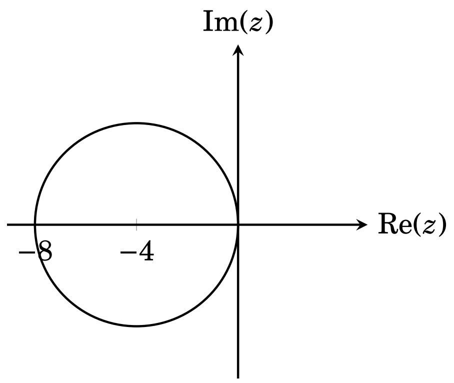

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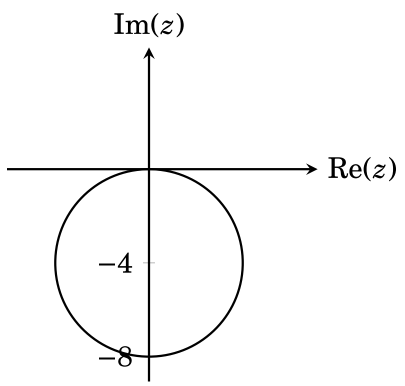

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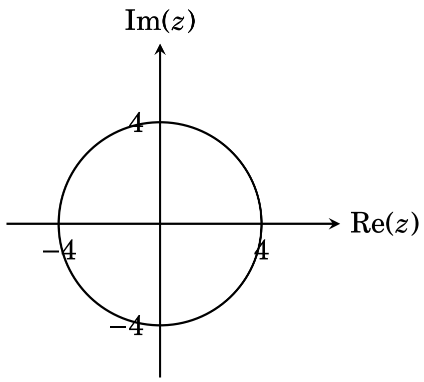

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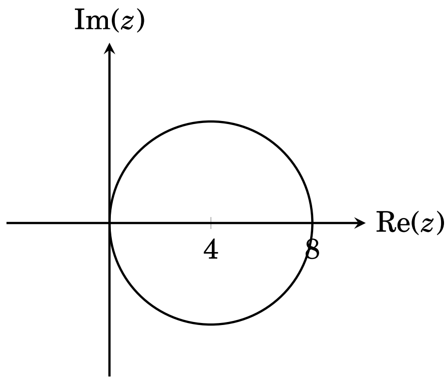
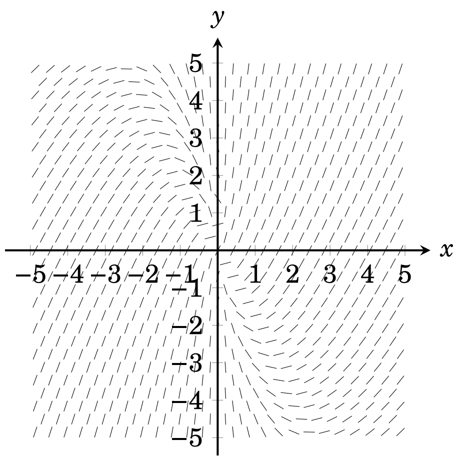


SECTION A – continued
TURN OVER

Question 8

$P(x.y)$ is a point on a curve. The y-intercept of a tangent to the point $P(x, y)$ is equal to twice the x-value at P. Which one of the following slope fields best represents this curve?

A.



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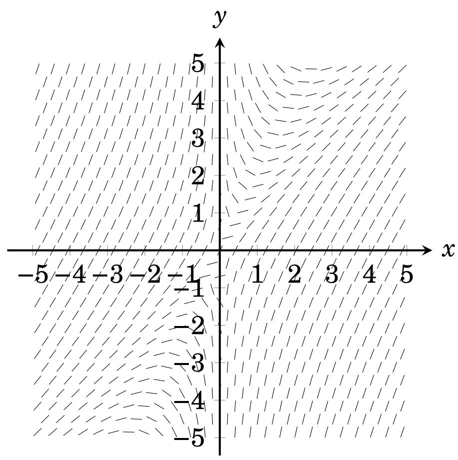

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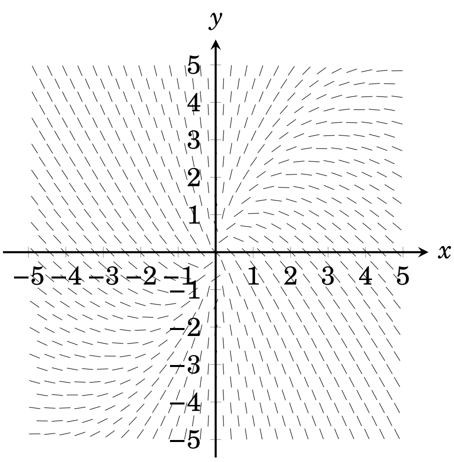

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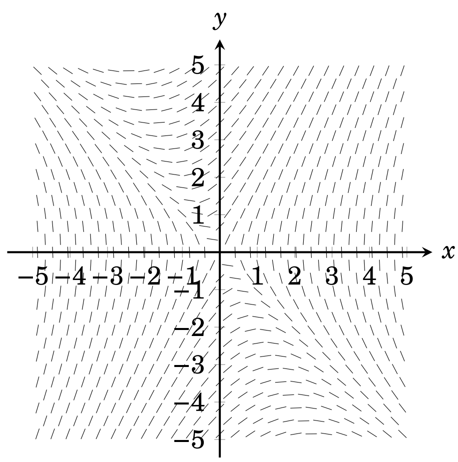

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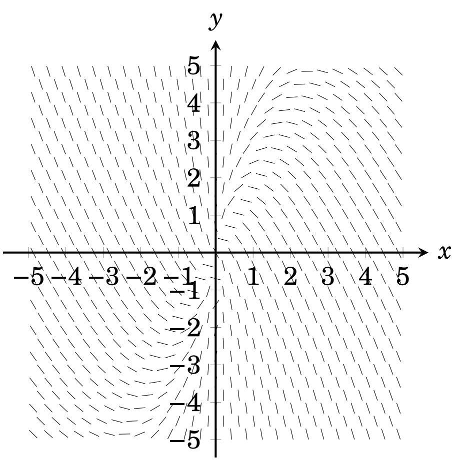


SECTION A – continued

Question 9

Pseudocode for a particular algorithm is shown below.

```
define f(x,y)
     return x/(y*y+1)
a ← 1
b ← 2
n ← 10
h ← (b-a)/n
x ← a
y ← 2.5
i ← 0
for i from 0 to n
     print (x,y)
     i ← i+1
     y ← y + h*f(x,y)
     x ← x+h
end for
```

The print statement when `i=2` is

A. `(1.1,2.5138)`

B. `(1.1,2.5288)`

C. `(1.2,2.5138)`

D. `(1.2,2.5288)`

E. `(1.3,2.5450)`

Question 10

The coordinates of the point of intersection of the line with vector equation $\underset{\sim}{r}(t) = 4\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k} + t\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}\right)$ and the plane with equation $2x - 3y + 4z = 20$ are

A. $\left(5, 0, \frac{5}{2}\right)$

B. $\left(-3, 2, -\frac{3}{2}\right)$

C. $\left(-3, 2, \frac{1}{2}\right)$

D. $\left(3, 1, \frac{7}{2}\right)$

E. $\left(3, 0, \frac{7}{2}\right)$

Question 11

The Cartesian equation of the plane which is perpendicular to the line with Cartesian equation

$$\frac{x-1}{3} = \frac{y+2}{5} = \frac{z+1}{-2}$$

and which passes through the point $P(1,\ 1,\ -3)$ is

A. $3x + 5y - 2z = -8$

B. $3x + 5y - 2z = 14$

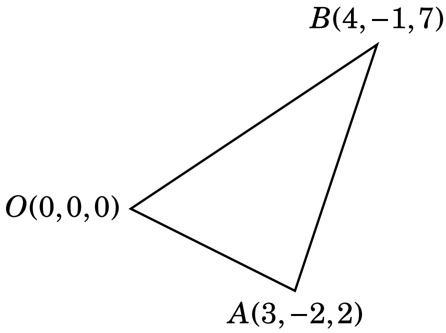
C. $3x - 5y + 2z = 4$

D. $-3x + 5y - 2z = -4$

E. $3x + 5y + 2z = 14$

Question 12

Consider the triangle OAB shown below:



The area of the triangle OAB is equal to

A. $\frac{13}{2}$

B. $26\sqrt{2}$

C. 28

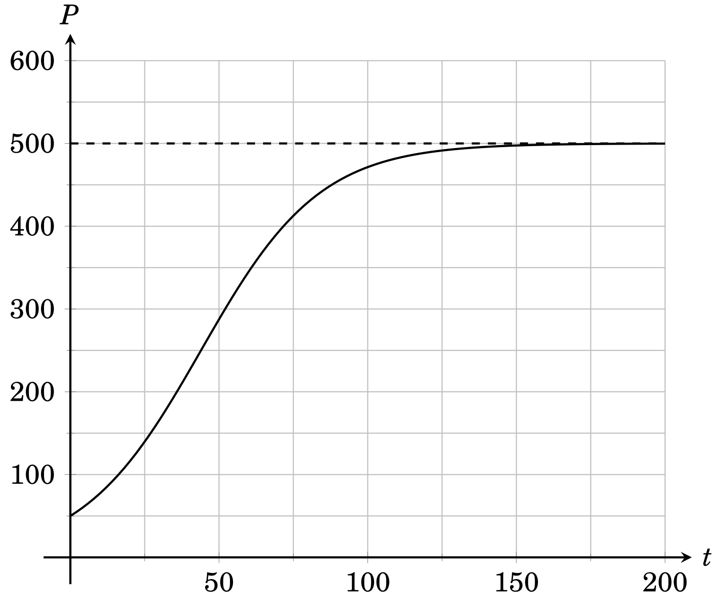
D. $\frac{13}{\sqrt{2}}$

E. $13\sqrt{2}$

Question 13

The graph below shows the solution curve to the logistic differential equation with solution

$P(t) = \dfrac{a}{1+be^{-\frac{1}{20}t}}$, where $a, b \in R$.



The rate of change of P is greatest when t is equal to

A. $20\log_e(6)$

B. $40\log_e(6)$

C. $20\log_e(3)$

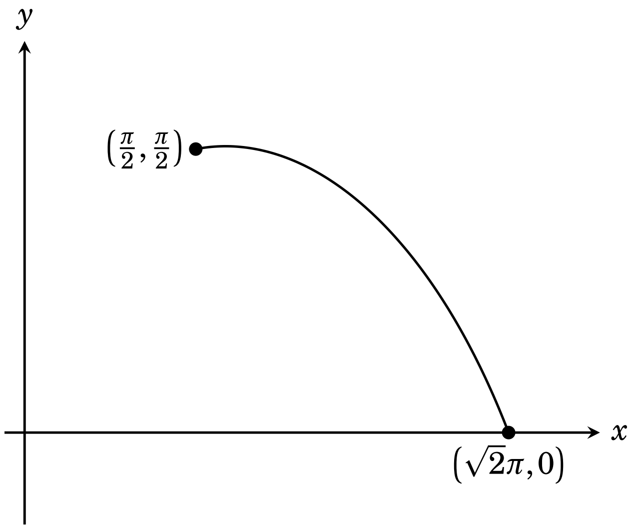
D. $40\log_e(9)$

E. $20\log_e(9)$

Question 14

The position of an object at time t is defined by $\underset{\sim}{r}(t) = 2\sqrt{2}t\sin(t)\underset{\sim}{i} + 2\sqrt{2}t\cos(t)\underset{\sim}{j}$, $t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

The graph of the path followed by the object is shown below.



The area of the surface generated by rotating this graph about the x-axis is found by evaluating

A. $16\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t\sqrt{1+t^2}\cos(t)\,dt$

B. $16\pi\int_{\frac{\pi}{2}}^{\sqrt{2}\pi} t\sqrt{1+t^2}\cos(t)\,dt$

C. $16\pi\int_{0}^{\frac{\pi}{2}} t^2\sqrt{1+t^2}\cos(t)\,dt$

D. $2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t\sqrt{1+2t^2}\cos(t)\,dt$

E. $4\pi\int_{\frac{\pi}{2}}^{\sqrt{2}\pi} t\sqrt{1+4t^2}\cos(t)\,dt$

Question 15

$\int x^5 \sin(3x)\,dx$ is equal to

A. $\frac{x^5}{3}\cos(3x) - \frac{5}{3}\int x^4 \cos(3x)dx$

B. $-\frac{x^5}{3}\cos(3x) + \frac{5}{3}\int x^4 \cos(3x)dx$

C. $-3x^5 \cos(3x) + 15\int x^4 \cos(3x)dx$

D. $-\frac{x^5}{3}\cos(3x) + \frac{5}{3}\int x^4 \sin(3x)dx$

E. $\frac{x^5}{3}\sin(3x) + \frac{5}{3}\int x^4 \cos(3x)dx$

Question 16

A body moves in a straight line so that when its displacement from the origin is x metres, its velocity is v ms^{-1} and its acceleration is a ms^{-2}.

Given that $a = \sqrt{x}$ and that $v = 2$ when $x = 1$, the speed in metres per second when $x = 16$ is

A. $2\sqrt{22}$

B. 44

C. $2\sqrt{11}$

D. 22

E. $8\sqrt{11}$

Question 17

The position vector of a particle at time t is given by $\underset{\sim}{r}(t) = 3t^2\underset{\sim}{i} + e^{2t}\underset{\sim}{j} - 3t\underset{\sim}{k}$, $t \geq 0$, where time is measured in seconds and distance in metres.

The initial speed of the particle in metres per second is

A. 13

B. $\sqrt{13}$

C. $\sqrt{5}$

D. 5

E. 7

Question 18

The masses of boxes of NutriFlakes are normally distributed with a standard deviation of 20 g.

A 95% confidence interval for the mean mass of NutriFlake boxes was calculated from a random sample of n boxes and found to be $(712.46, 723.54)$, correct to two decimal places.

The value of n is

A. 40

B. 45

C. 50

D. 55

E. 60

Question 19

The masses of boxes of CocoPuffs are normally distributed with a mean of 650 g and a standard deviation of s g.

It is suspected that the boxes are being underfilled and so a **two-sided** statistical test at the 5% level of significance is performed using a random sample of 30 boxes. The null and alternative hypotheses for this test are

$$H_0 : \mu = 650$$

$$H_1 : \mu \neq 650$$

Th value of the standard deviation s is closest to

A. 10

B. 14

C. 15

D. 18

E. 82

Question 20

At a particular fruit shop, the masses of apples are normally distributed with a mean of 80 g and a standard deviation of 5 g and the masses of mandarins are normally distributed with a mean of 60 g and a standard deviation of 8 g.

The masses of apples and mandarins are independent.

Let X represent the random variable which is the difference in the mean mass of a bag of four mandarins and the mean mass of a bag of three apples.

The mean and standard deviation respectively of X are

A. 0, $\sqrt{331}$

B. 0, $\sqrt{181}$

C. 0, $\sqrt{481}$

D. 480, $\sqrt{331}$

E. 480, $\sqrt{481}$

END OF SECTION A

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SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$.

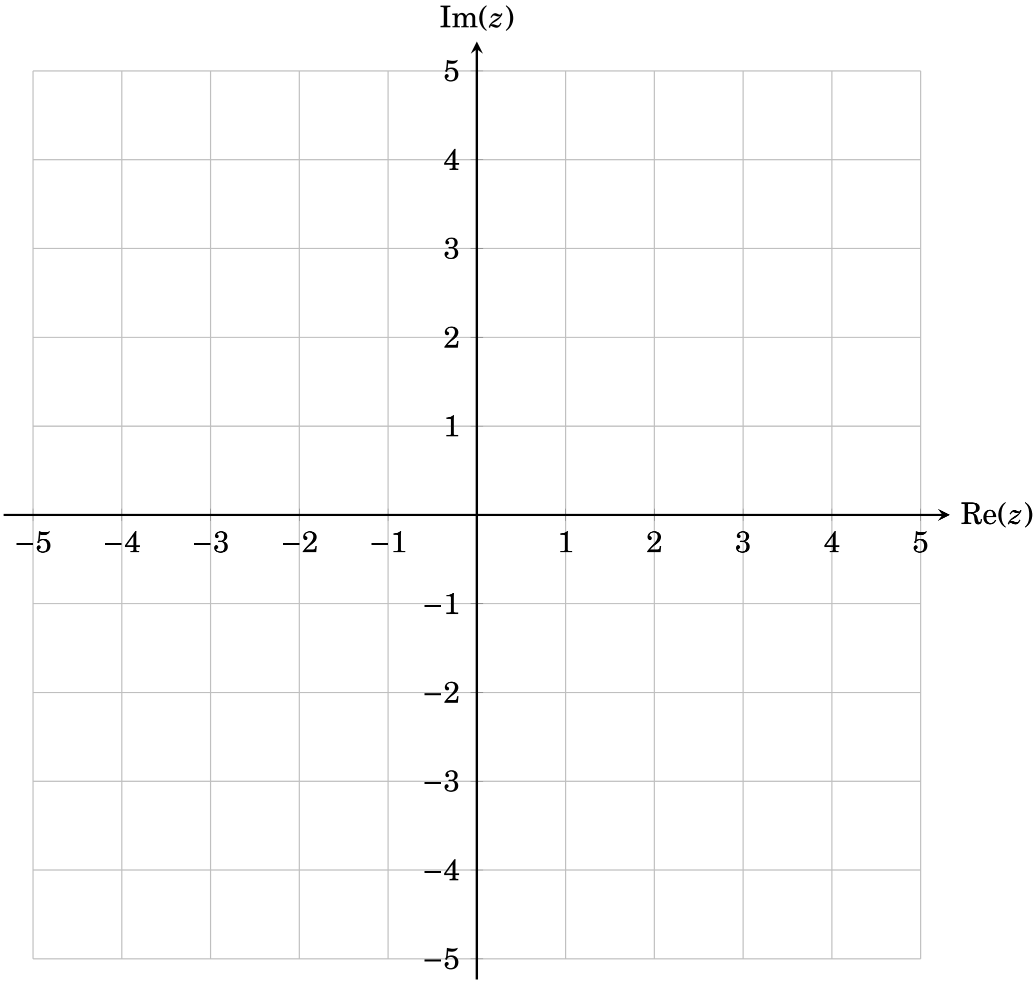
Question 1 (8 marks)

a. One of the solutions of the equation $z^3 + 10z^2 + 36z + 48 = 0$ is $z = -3 + \sqrt{3}i$.
Write down the other solutions. 1 mark

b. The solutions of the equation $z^3 + 10z^2 + 36z + 48 = 0$ all lie on the circumference of the circle with equation $|z - a| = r$, where $a \in R$.
Determine the values of a and r. 2 marks

SECTION B - Question 1 – continued

c. On the Argand diagram below, sketch the circle obtained in **part b.** and label the points that represent the solutions of the equation $z^3 + 10z^2 + 36z + 48 = 0$. 2 marks



d. $z = -3 + \sqrt{3}i$ lies on the ray $\text{Arg}(z) = \alpha$.

Write down the value of α and sketch the ray on the Argand diagram in **part c.** 1 mark

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e. Determine the area of the region $\{z : |z - a| \le r\} \cap \{z : \text{Arg}\, z \ge \alpha\}$. 2 marks

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SECTION B - continued
TURN OVER

Question 2 (9 marks)

The plane Π_1 has Cartesian equation $3x - 3y + 2z = 25$. The line l_1 passes through the point $A(3,2,-1)$ and is perpendicular to Π_1.

The plane Π_2 contains the points $D(-2,-1,1)$, $E(-7,3,-1)$ and $F(3,1,5)$.

a. Write down a vector equation of the line l_1. 1 mark

b. Find a Cartesian equation for the plane Π_2. 2 marks

c. **i.** Show that both planes Π_1 and Π_2 contain the point $P(4,-1,5)$. 1 mark

ii. Find a Cartesian equation for the line l_2 contained in both Π_1 and Π_2. 2 marks

d. Determine the shortest distance between the skew lines l_1 and l_2. Give your answer correct to two decimal places. 3 marks

Question 3 (9 marks)

A parachutist falls from a helicopter hovering at a height of 1500 m and immediately opens her parachute. She falls directly towards the earth with an acceleration of $g-\frac{5}{4}v\ \text{ms}^{-2}$.

a. Write down the limiting (terminal) velocity of the parachutist. Give your answer in metres per second. 1 mark

b. Find an expression for the velocity of the parachutist t seconds after she falls from the helicopter. 2 marks

c. How long after falling from the helicopter does the parachutist have a velocity of $7.5\ \text{ms}^{-1}$? Give your answer in seconds, correct to two decimal places. 1 mark

SECTION B - Question 3 - continued

d. Determine how far the parachutist has fallen when her velocity is $v = \frac{3g}{5}$ ms^{-1}. Give your answer in metres, correct to two decimal places. 2 marks

The instant the parachutist falls from the helicopter, the helicopter starts to move in a straight line and at a constant height of 1500 m. It accelerates at a constant rate to reach a speed of 50 ms^{-1} in 30 seconds. The helicopter then continues at a constant speed of 50 ms^{-1}.

e. Find the distance from the parachutist to the helicopter at the instant that she reaches the ground. Give your answer correct to the nearest metre. 3 marks

Question 4 (12 marks)

At the start of the year 1850, the population of polar bears in the Arctic Circle at any time t was modelled according to the logistic equation

$$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right)$$

where P is the number of polar bears measured in thousands and t is measured in years from the year 1850.

a. **i.** State, in thousands per year, the growth rate of the polar bears. 1 mark

ii. State the maximum number of polar bears that the model predicts the Arctic Circle can sustain in the long term. 1 mark

The population of polar bears is changing at its maximum rate in a particular year after the year 1850.

iii. State the polar bear population when it is changing at its maximum rate. 1 mark

It is estimated that the population of polar bears in the Arctic Circle in the year 1872 was 28,500.

iv. Find, correct to the nearest whole number, the population of polar bears in the Arctic Circle predicted by the model in the year 1850. 1 mark

From the beginning of the year 1900, n thousand polar bears were removed from the Arctic Circle each year due to large scale hunting.

b. **i.** Modify the above differential equation to include this change. 1 mark

ii. Find the value of n such that the equilibrium population of polar bears is 18,000. 2 marks

It is estimated that in the year 1900 the population of polar bears in the Arctic Circle was 29,976.

iii. If 1,290 polar bears are being removed from the Arctic Circle each year due to large scale hunting, determine the year in which the polar bear population first drops below 12,000. 1 mark

SECTION B - Question 4 - continued
TURN OVER

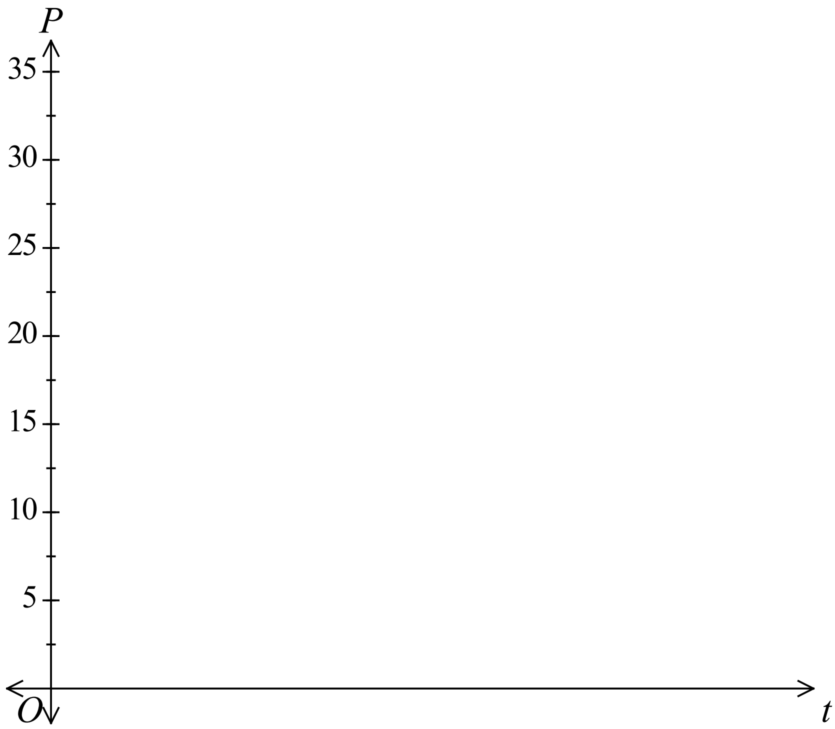
The large-scale hunting of polar bears raised international concern for the future of the species. In the year 1960, controls and quotas were introduced. The population of polar bears in the Arctic Circle at any time t is now modelled according to the differential equation

$$\frac{dP}{dt} = 0.15P\left(1 - \frac{P}{30}\right) - 0.2\log_e\left(\frac{P}{4}\right)$$

where P is the number of polar bears measured in thousands and t is measured in years from 1960.

It was estimated that in the year 1960 the population of polar bears in the Arctic Circle was 11,720.

c. Sketch the graph of $P = P(t)$ on the set of axes below, labelling any asymptotes with their equation and any point of inflection with its coordinates. Give your answers correct to two decimal places. 4 marks



Working space

SECTION B – continued

Question 5 (12 marks)

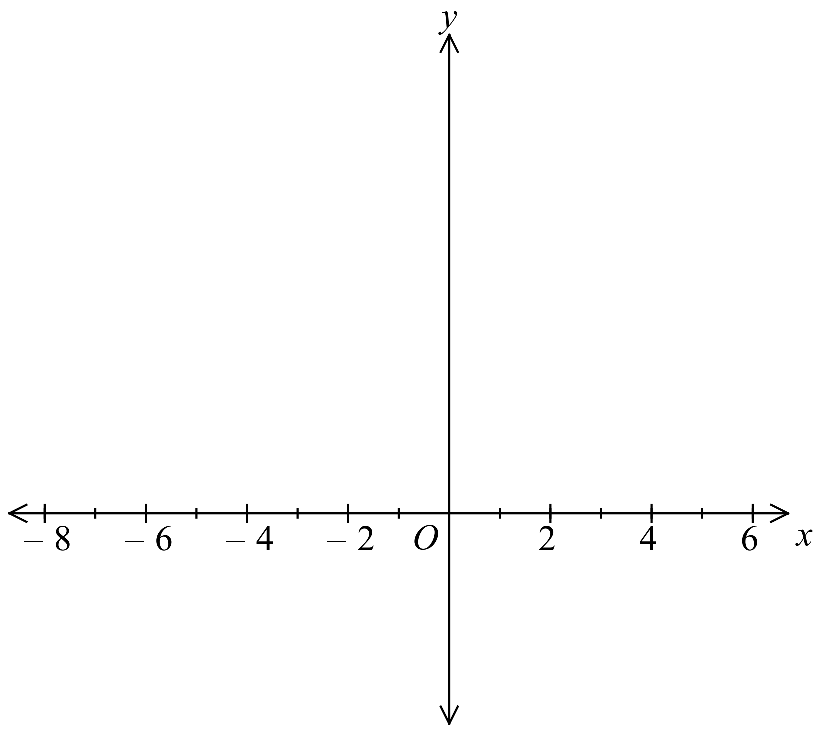
Consider the function

$f_k : D_k \to R$, $f_k(x) = \dfrac{x^2 - 1}{e^x - x - k}$ where $k \in R$ and D_k is the maximal domain of f_k.

a. Let $k = 2$.

i. f_2 has a diagonal asymptote with equation $y = mx + c$. Find the values of m and c. 2 marks

ii. Sketch the graph of $y = f_2(x)$ on the set of axes below, showing all asymptotes. Label all vertical and horizontal asymptotes with their equation and label all axial intercepts with their coordinates. Give all approximate values correct to four decimal places. 3 marks



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b. **i.** State the equation(s) of all horizontal asymptotes of f_k. 1 mark

ii. Find the value(s) of k for which f_k has two vertical asymptotes. 3 marks

SECTION B - Question 5 - continued

c. **i.** Find the values of k for which f_k has a stationary point at $x = 0$. 2 marks

ii. Find the value(s) of k for which f_k has a point of inflection at $x = 0$.
If no such value of k exists you should briefly explain why. 1 mark

Question 6 (10 marks)

Starving John's is a fast food restaurant chain that has recently started selling barista-made coffee. The coffee machines used by *Starving John's* dispense volumes of coffee that are normally distributed with a mean of 240 ml and a standard deviation of 8 ml.

a. Two cups of coffee are dispensed. The volumes of coffee dispensed in each cup are independent. Find, correct to four decimal places, the probability that the volumes of coffee in each cup differ by more than 5 ml. 2 marks

The coffee machines have the option of adding milk to a cup of coffee, where the volume of milk dispensed is also normally distributed with a mean of 10 ml and a standard deviation of 2 ml.

The volume of coffee and volume of milk dispensed by a coffee machine are independent random variables.

b. Find, correct to four decimal places, the probability that the volume of a cup of coffee with milk is less than 245 ml. 2 marks

SECTION B - Question 6 - continued

c. When there is 600 ml of milk left in the coffee machine, the machine can make a maximum of n cups of coffee with milk with a probability of 0.999. Find the value of n. 2 marks

James is a district manager of *Starving John's* restaurants. He is concerned that the coffee machine at one of the restaurants in his district is, on average, dispensing less than 240 ml of coffee. James decides to conduct a statistical test at the 5% level of significance.

A random sample of 15 cups of coffee (with no milk) is dispensed and it is found that the total amount of coffee served in this sample is 3555 ml.

d. **i.** Write down suitable hypotheses H_0 and H_1 for this test. 1 mark

ii. Find the p value for this test, correct to four decimal places. 1 mark

iii. Does the total amount of coffee served in this sample support James' concern? Justify your answer. 1 mark

iv. What is the smallest total amount of coffee (with no milk) served in a sample of 15 cups for H_0 not to be rejected? Give your answer correct to the nearest ml. 1 mark

END OF QUESTION AND ANSWER BOOK