

Trial Examination 2023

VCE Specialist Mathematics Units 3&4

Written Examination 1

Suggested Solutions

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Question 1 (2 marks)	
E(X + 2X - 4Y) = E(X) + 2E(X) - 4E(Y)	
=3+6-20	
=-11	A1
$\operatorname{Var}(X + 2X - 4Y) = \operatorname{Var}(X) + 4\operatorname{Var}(X) + 16\operatorname{Var}(Y)$	
=2+8+64	
=74	A1

Question 2 (3 marks)

a. If p + q is not even, then $p^2 + q^2 + 1$ is not odd. A1 OR If p + q is odd, then $p^2 + q^2 + 1$ is even. A1 b. Letting p + q be odd gives:

$$p^{2} + q^{2} + 1 = (p+q)^{2} - 2pq + 1$$

$$= odd - even + 1$$

$$= (odd + 1) - even$$

$$= even - even$$

$$= even$$
M1

 $V = \frac{4}{3}\pi r^{3} \text{ and } S = 4\pi r^{2}, \text{ where } r \text{ is the radius of the spherical balloon.}$ $\frac{dV}{dt} = 24 \text{ cm}^{3}/\text{s}$ $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ $24 = (4\pi \times 8^{2}) \times \frac{dr}{dt}$ $\frac{dr}{dt} = \frac{24}{256\pi}$ $= \frac{3}{32\pi}$ A1 $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$ $= (8\pi \times 8) \times \frac{3}{32\pi}$ $= 6 \text{ cm}^{2}/\text{s}$ A1

Question 4 (3 marks)

$$z^3 = -64$$

Letting $z = r \operatorname{cis}(\theta)$ gives:
 $r^3 \operatorname{cis}(3\theta) = 64 \operatorname{cis}(\pi)$ M1
When $r^3 = 64$ and $3\theta = \pi + 2k, k \in \mathbb{Z}$:
 $r = 4$ and $\theta = \frac{\pi + 2\pi k}{3}, k \in \mathbb{Z}$ A1
Choosing $k = 0, 1, 2$ gives:
 $z_1 = 4 \operatorname{cis}\left(\frac{\pi}{3}\right), z_2 = 4 \operatorname{cis}(\pi), z_3 = 4 \operatorname{cis}\left(-\frac{\pi}{3}\right)$ A1

Proving for
$$n = 1$$
:
LHS = $1^{3} = 1$
RHS = $\frac{1}{4} \times 1^{2} \times 2^{2} = 1$ M1

Assuming true for n = k:

$$1^{3} + 2^{3} + 3^{3} + \dots + k^{3} = \frac{1}{4}k^{2}(k+1)^{2}$$

Proving true for $n = k + 1$:

$$1^{3} + 2^{3} + 3^{3} + \dots + k^{3} + (k+1)^{3} = \frac{1}{4}(k+1)^{2}(k+2)^{2}$$
 M1

LHS =
$$\frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

= $(k+1)^2\left(\frac{k^2}{4} + k + 1\right)$
= $(k+1)^2\left(\frac{k^2 + 4k + 4}{4}\right)$
= $(k+1)^2\frac{(k+2)^2}{4}$
= $\frac{1}{4}(k+1)^2(k+2)^2$ M1
= RHS

Question 6 (6 marks)

a.

$$\begin{cases}
x = 2\cos(t) \\
y = -\sin(t)
\end{cases}$$

$$\begin{cases}
\cos(t) = \frac{x}{2} \\
\sin(t) = -y
\end{cases}$$

$$\begin{cases}
\cos^{2}(t) = \frac{x^{2}}{4} \\
\sin^{2}(t) = y^{2}
\end{cases}$$
M1

Using the identity $\cos^2(t) + \sin^2(t) = 1$ gives:

$$\frac{x^2}{4} + y^2 = 1$$
 A1

$$\dot{\mathbf{t}}(t) = -2\sin(t)\dot{\mathbf{t}} - \cos(t)\dot{\mathbf{j}} \qquad \text{M1}$$

$$\begin{vmatrix} \dot{\mathbf{t}}(t) \end{vmatrix} = \sqrt{4\sin^2(t) + \cos^2(t)}$$
$$\begin{vmatrix} \dot{\mathbf{t}}(\frac{\pi}{4}) \end{vmatrix} = \sqrt{4 \times \frac{1}{2} + \frac{1}{2}}$$
$$= \sqrt{\frac{5}{2}}$$
A1

c.
$$\ddot{\mathbf{r}}(t) = -2\cos(t)\dot{\mathbf{i}} + \sin(t)\dot{\mathbf{j}}$$
 M1

$$\begin{aligned} \left| \ddot{\mathbf{z}}(t) \right| &= \sqrt{4\cos^2(t) + \sin^2(t)} \\ &= \sqrt{3\cos^2(t) + 1} \end{aligned}$$

The particle's maximum acceleration occurs when $\cos^2(t) = 1$. Therefore:

$$\left| \dot{\underline{r}}(t) \right|_{\max} = \sqrt{(3 \times 1) + 1}$$

$$= 2$$
A1

Question 7 (4 marks)

b.

If A, B and C are collinear, $\overrightarrow{AB} = \lambda \overrightarrow{AC}$ for some constant λ . a.

$$\overline{AB} = \begin{bmatrix} 3 - (-2) \\ 4 - 1 \\ -2 - 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

$$\overline{AC} = \begin{bmatrix} -5 - (-2) \\ -2 - 1 \\ 1 - 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \lambda \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}$$

$$M1$$

$$5 = -3\lambda$$

$$\lambda = -\frac{5}{3}$$

$$\lambda = -\frac{5}{3}$$
does not satisfy $3 = -3\lambda$.
$$M1$$
Therefore, the points are not collinear.
$$\operatorname{area} = \frac{1}{2} |\overline{AB} \times \overline{AC}|$$

$$|\underline{i} \quad \underline{j} \quad \underline{k}|$$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} z & z & z \\ 5 & 3 & -2 \\ -3 & -3 & 1 \end{vmatrix}$$

$$= -3\underline{i} + \underline{j} - 6\underline{k}$$

$$\operatorname{area} = \frac{1}{2} \begin{vmatrix} -3\underline{i} + \underline{j} - 6\underline{k} \end{vmatrix}$$

$$= \frac{\sqrt{46}}{2}$$
A1

A1

Note: Consequential on answer to Question 7a.

Question 8 (4 marks) When sin(2x) = 0: $2x = 0, \pi, 2\pi$ $x = 0, \frac{\pi}{2}, \pi$

When $\cos(2x) = 0$:

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

Therefore, the smallest non-negative root of f(x) is $\frac{\pi}{4}$.

$$A = \int_{0}^{\frac{\pi}{4}} \sin^{2}(2x) \cos^{3}(2x) dx$$

$$= \int_{0}^{\frac{\pi}{4}} \sin^{2}(2x) \cos^{2}(2x) \cos(2x) dx$$

$$u = \sin(2x) \qquad M1$$

$$du = 2\cos(2x) dx$$

$$\cos^{2}(2x) = 1 - \sin^{2}(2x)$$

$$= 1 - u^{2}$$

$$A = \frac{1}{2} \int_{0}^{1} u^{2} (1 - u^{2}) du \qquad A1$$

$$= \frac{1}{2} \left[\frac{u^{3}}{3} - \frac{u^{5}}{5} \right]_{0}^{1}$$

$$= \frac{1}{15} \qquad A1$$

Question 9 (4 marks)

 $u = x^2 \Rightarrow u' = 2x$ $v' = \sin(x) \Rightarrow v = -\cos(x)$ M1 Using integration by parts gives:

$$I = \int x^{2} \sin(x) dx$$

= $uv - \int u'v dx$
= $-x^{2} \cos(x) + 2 \int x \cos(x) dx$ A1
 $m = x \Rightarrow m' = 1$
 $n' = \cos(x) \Rightarrow n = \sin(x)$ M1
 $I = -x^{2} \cos(x) + 2 \left(x \sin(x) - \int \sin(x) dx \right)$
= $-x^{2} \cos(x) + 2x \sin(x) + 2\cos(x) + c$ A1

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A1

Question 10 (8 marks)

a.



correct shape A1

correct demonstration of zero gradient at x = 1 and x = -1 and no gradient at y = 0 A1 Note: The solution is obtained by using the initial condition y(0) = 2 and drawing a solution curve that is tangential to the direction field.

b.
$$\int y dy = \int (x^2 - 1) dx$$
 M1

$$\frac{y^2}{2} = \frac{x^3}{3} - x + c$$

$$\left\{ \begin{cases} x = 0 \\ y = 2 \Rightarrow c = 2 \end{cases}$$

$$\frac{y^2}{2} = \frac{x^3}{3} - x + c$$

$$\frac{3y^2}{2} = x^3 - 3x + 6$$

$$3y^2 = 2x^3 - 6x + 12$$
 A1
c.
$$\frac{dy}{dx} = \frac{x^2 - 1}{y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$
 M1

$$= \frac{2xy - (x^2 - 1) \frac{dy}{dx}}{y^2}$$

$$= \frac{2xy^2 - (x^2 - 1)^2}{y^3}$$
A1

$$\frac{d^2y}{dx^2} = 0 \Rightarrow 2xy^2 - (x^2 - 1)^2 = 0$$
M1

$$2xy^2 - (x^2 - 1)^2 = 0$$

$$2xy^2 = (x^2 - 1)^2$$

$$y^2 = \frac{(x^2 - 1)^2}{2x}$$

$$y = \frac{|x^2 - 1|}{\sqrt{2x}}$$
M1