Neap

Trial Examination 2023

VCE Specialist Mathematics Units 3&4

Written Examination 2

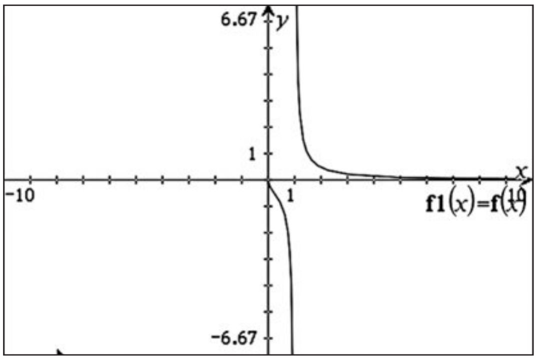
Suggested Solutions

SECTION A – MULTIPLE-CHOICE QUESTIONS

Question	Answer	Question	Answer
1	B	11	A
2	D	12	C
3	E	13	A
4	D	14	C
5	C	15	D
6	A	16	D
7	B	17	E
8	C	18	E
9	B	19	E
10	D	20	D

Question 1 B

Sketching the graph of $f(x)$ using a CAS calculator gives:



Thus, finding the solutions of $f(x)$ when $x \geq 0$ gives:

$$f(x) := \frac{\sqrt{x}}{x^2-1} \qquad \textit{Done}$$

$$\text{solve}\left(\frac{d^2}{dx^2}(f(x))=0,x\right)\Big| x\geq 0$$

$$x=\frac{\sqrt{15\cdot(4\cdot\sqrt{6}-9)}}{15}$$

Since $\dfrac{d^2}{dx^2}$ has one solution in its domain, the graph has one inflection point.

Question 2 D

D is correct. Letting P represent 'n divisible by 10' and Q represent 'n divisible by 5' gives 'if P, then Q'. Therefore, the converse of this statement is 'if Q, then P', or 'if n is divisible by 5, then it is divisible by 10'.

A is incorrect. This statement is 'if not P, then Q'.

B is incorrect. This statement is 'if not P, then not Q'.

C is incorrect. This statement is 'if P, then not Q'.

E is incorrect. This statement is 'if not Q, then not P'.

Question 3 E

Using a CAS calculator gives:

$$\frac{|0-2\cdot 0+3\cdot 0-5|}{\sqrt{1^2+(-2)^2+3^2}} \qquad \frac{5\cdot\sqrt{14}}{14}$$

$$\frac{5\sqrt{14}}{14}=\frac{5}{\sqrt{14}}$$

Question 4 D

Given that $\tan\left(-\frac{3\pi}{4}\right)=1$:

$$\frac{a}{5}=1$$
$$a=5$$

Question 5 C

C cannot be correct and is therefore the required response. Since complex roots occur in conjugate pairs, there must be an odd number of real roots. It is given that $3 - i$ is a root.

A can be correct and is therefore not the required response. $3 - i$ is a root.

B can be correct and is therefore not the required response. By the conjugate root theorem, $3 + i$ is also a root.

D and **E** may be correct and are therefore not the required response. There are two possibilities for the remaining three roots.

- Two of the roots are non-real (conjugate points) and one is real. Hence, $P(z) = 0$ has one real root only. Therefore, **D** might be correct.
- All three roots are real. Hence, $P(z) = 0$ has two non-real roots. Therefore, **E** might be correct.

Question 6 A

Using complementary angles gives:

$$\begin{aligned} z &= \sin(\theta) - i\cos(\theta) \\ &= \cos\left(\frac{\pi}{2}-\theta\right) - i\sin\left(\frac{\pi}{2}-\theta\right) \\ &= \cos\left(\theta-\frac{\pi}{2}\right) + i\sin\left(\theta-\frac{\pi}{2}\right) \end{aligned}$$

$$\begin{aligned} z^6 &= \cos(6\theta-3\pi) + i\sin(6\theta-3\pi) \\ &= \cos(6\theta-\pi) + i\sin(6\theta-\pi) \\ &= \cos(\pi-6\theta) - i\sin(\pi-6\theta) \\ &= -\cos(6\theta) - i\sin(6\theta) \end{aligned}$$

Question 7 B

Using a CAS calculator gives:

$a:=[1 \;\; -3 \;\; 2]$	$[1 \;\; -3 \;\; 2]$
$b:=[6 \;\; -7 \;\; 2]$	$[6 \;\; -7 \;\; 2]$
$\frac{180}{\pi}\cdot\cos^{-1}\left(\frac{\lvert \text{dotP}(a,b)\rvert}{\text{norm}(a)\cdot\text{norm}(b)}\right)$	28.5718

Question 8 C

$u = 1 - x$

$du = -dx$

$x = 1 - u$

Therefore:

$$-\int_1^0 (2-u)\sqrt{u}\,du = \int_0^1 (2u^{\frac{1}{2}} - u^{\frac{3}{2}})du$$

Question 9 B

Considering the second derivative and finding the values of a that will give a positive discriminant gives:

$f(x) := a \cdot x^4 - 3 \cdot x^3 - 5 \cdot x^2$	*Done*
$fdd(x) := \frac{d^2}{dx^2}(f(x))$	*Done*
$fdd(x)$	$12 \cdot a \cdot x^2 - 18 \cdot x - 10$

$\text{solve}((-18)^2 - 4 \cdot 12 \cdot a \cdot -10 > 0, a)$	$a > \frac{-27}{40}$

Question 10 D

Finding the direction vector:

$$\underset{\sim}{v} = (2-1)\underset{\sim}{i} + (1-(-2))\underset{\sim}{j} + (-3-2)\underset{\sim}{k}$$
$$= \underset{\sim}{i} + 3\underset{\sim}{j} - 5\underset{\sim}{k}$$

D does not describe a line that passes through the two points and is therefore the required response. This vector equation does not use a multiple of $\underset{\sim}{v}$ from point (2, 1, –3) or (1, –2, 2).

A describes a line that passes through the two points and is therefore not the required response. This vector equation is reached by multiplying $\underset{\sim}{v}$ by t from point (1, –2, 2).

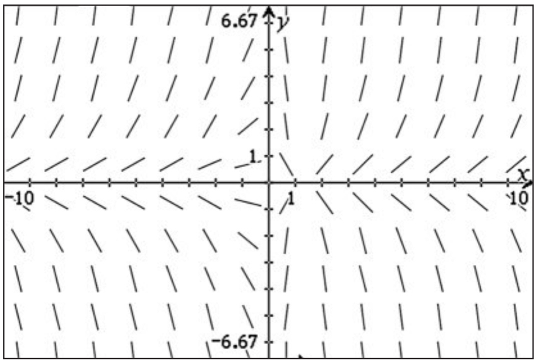
B describes a line that passes through the two points and is therefore not the required response. This vector equation is reached by multiplying $\underset{\sim}{v}$ by $-t$ from point (2, 1, –3).

C describes a line that passes through the two points and is therefore not the required response. This vector equation is reached by multiplying $\underset{\sim}{v}$ by $2t$ from point (1, –2, 2).

E describes a line that passes through the two points and is therefore not the required response. This vector equation is reached by multiplying $\underset{\sim}{v}$ by $-t$ from point (1, –2, 2).

Question 11 A

A is correct. Using a CAS calculator to verify the direction field of $\frac{dy}{dx}=\frac{xy}{x-1}$ gives:



B is incorrect. The direction field shows that $\frac{dy}{dx}>0$ in the second quadrant. However, $\frac{xy}{1-x}<0$.

C and **E** are incorrect. The direction field shows that $\frac{dy}{dx}$ is defined when $y = 1$.

D is incorrect. The direction field shows that $\frac{dy}{dx}=0$ when either x or y is equal to zero. Hence, x and y need to be factors in the numerator.

Question 12 C

$$\frac{dm}{dt}=\text{inflow}-\text{outflow}$$

$$=(5\text{ grams/L}\times 2\text{ L/min})-\left(\frac{1\text{ gram/L}\times m\text{ grams}}{3\text{ L}+(2\text{ L/min}-1\text{ L/min})t\text{ min}}\right)$$

$$=(5\times 2)-\frac{1\times m}{3+(2-1)t}$$

$$=10-\frac{m}{3+t}$$

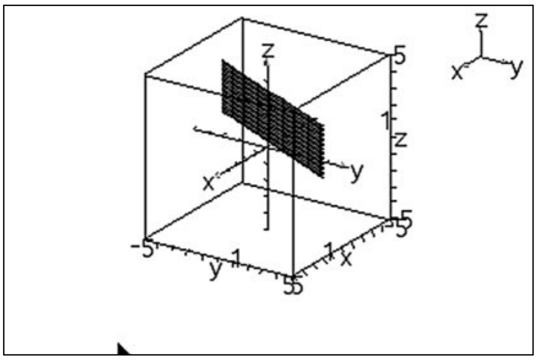
Question 13 C

Using a CAS calculator gives:

$a:=\begin{bmatrix}2 & 3 & p\end{bmatrix}$	$\begin{bmatrix}2 & 3 & p\end{bmatrix}$
$b:=\begin{bmatrix}3 & -2 & 4\end{bmatrix}$	$\begin{bmatrix}3 & -2 & 4\end{bmatrix}$
$c:=\begin{bmatrix}1 & -2 & q\end{bmatrix}$	$\begin{bmatrix}1 & -2 & q\end{bmatrix}$
$\text{solve}(a=m\cdot b+n\cdot c,m,n)$	$m=\frac{7}{4}$ and $n=\frac{-13}{4}$ and $p=\frac{-(13\cdot q-28)}{4}$

Question 14 C

Using a CAS calculator gives:

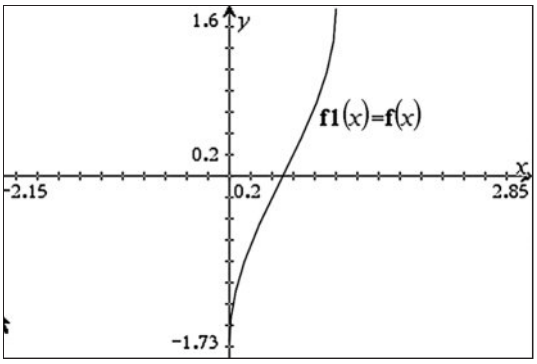


In three dimensions, $y = 2x + 1$ means that for any valid (x, y) pair, the z-coordinate can be any number. The plane will be parallel to the z-axis since (x, y) pairs are independent of z.

Question 15 D

The features of the graph need to be investigated before the correct definite integral expression is set. Using a CAS calculator gives:

$f(x):=\sin^{-1}(2\cdot x-1)$	*Done*
$\text{domain}(f(x),x)$	$0\le x\le 1$
$\text{solve}(f(x)=0,x)$	$x=\frac{1}{2}$



Thus, finding the surface area of the solid of revolution gives:

$$2\cdot 2\cdot \pi\cdot \int_{\frac{1}{2}}^{1}\left(f(x)\cdot \sqrt{1+\left(\frac{d}{dx}(f(x))\right)^{2}}\right)dx \qquad 16.0601$$

Question 16 D

Using a CAS calculator gives:

$v(x):=2\cdot\sin(x)$	*Done*
$\frac{1}{2}\cdot\frac{d}{dx}\left((v(x))^2\right)$	$4\cdot\sin(x)\cdot\cos(x)$

$$\begin{aligned}4\sin(x)\cos(x) &= 2\times 2\sin(x)\cos(x)\\ &= 2\sin(2x)\end{aligned}$$

Question 17 E

The given algorithm is used to apply Euler's method to approximate the solution of the differential equation $\frac{dy}{dx}=x^2-3y$.

Entering the following information into a CAS calculator gives:

- function expression: $x^2 - 3y$
- independent variable: x
- dependent variable: y
- initial x-value: 3
- final x-value: $3 + 6 \times 0.2 = 4.2$
- initial y-value: 2
- increment: 0.2

$\text{euler}(x^2-3\cdot y,x,y,\{3,4.2\},2,0.2)$

3.4	3.6	3.8	4.	4.2
.088	3.5472	4.01088	4.49235	4.99694

Question 18 E

Using a CAS calculator gives:

$m:=\frac{95.3+110.6}{2}$	102.95
$z:=-\text{invNorm}\left(\frac{1-0.95}{2},0,1\right)$	1.95996
$\text{solve}\left(m-95.3=\frac{z\cdot s}{\sqrt{80}},s\right)$	$s=34.9107$

Question 19 E

Using a CAS calculator gives:

$$\text{normCdf}\left(-\infty, 34, 32, \frac{4}{\sqrt{6}}\right) \quad 0.889664$$

Question 20 D

D is correct. For a one-tailed test performed at a 5% level of significance, H_0 should be rejected if $p < 0.05$. As $0.04 < 0.05$, H_0 should be rejected.

A is incorrect. As $0.005 < 0.05$, H_0 should be rejected.

B is incorrect. As $0.07 > 0.05$, H_0 should not be rejected.

C is incorrect. As $p > 0.06$ is greater than 0.05, H_0 should not be rejected.

E is incorrect. As $p < 0.04$ is less than 0.05, H_0 should be rejected.

SECTION B

Question 1 (7 marks)

a. **i.** Subtracting the coordinates of A from D gives $\overrightarrow{AD} = -6\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$.

Method 1:

$\underset{\sim}{r} = (2-6t)\underset{\sim}{i} + (-1+t)\underset{\sim}{j} + (6-t)\underset{\sim}{k}$ A1

Method 2:

$\underset{\sim}{r} = (-4-6t)\underset{\sim}{i} + t\underset{\sim}{j} + (5-t)\underset{\sim}{k}$ A1

Note: Students may reach either vector equation depending on the method used.

ii. **Method 1:**

$$2-6t=-10$$
$$t=2$$
$$-1+t=m$$
$$-1+2=m$$
$$m=1$$
$$6-t=n$$
$$6-2=n$$
$$n=4$$

values for m and n A1

Method 2:

$$-4-6t=-10$$
$$t=1$$
$$t=m=1$$
$$5-t=n$$
$$5-1=n$$
$$n=4$$

values for m and n A1

Note: Consequential on answer to ***Question 1a.i.*** *Students should reach the same answer using either method.*

b. The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ can be obtained by subtracting the coordinates of A from B and A from C, respectively.

$\overrightarrow{AB} = 2\underset{\sim}{i} + 9\underset{\sim}{j} - 7\underset{\sim}{k}, \overrightarrow{AC} = -8\underset{\sim}{i} + 4\underset{\sim}{j} - 4\underset{\sim}{k}$ M1

$\overrightarrow{AB} \times \overrightarrow{AC} = -8\underset{\sim}{i} + 64\underset{\sim}{j} + 80\underset{\sim}{k}$ A1

Substituting point A into the vector gives:

$-8(x-2)+64(y+1)+80(z-6)=0$

$-x+8y+10z-50=0$ **OR** $x-8y-10z+50=0$ A1

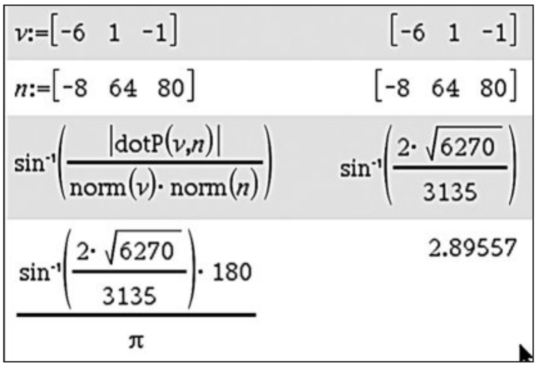
$u:=[2\ \ 9\ \ -7]$	$[2\ \ 9\ \ -7]$
$v:=[-8\ \ 4\ \ -4]$	$[-8\ \ 4\ \ -4]$
crossP(u,v)	$[-8\ \ 64\ \ 80]$
$-8\cdot(x-2)+64\cdot(y+1)+80\cdot(z-6)=0$	$-8\cdot x+64\cdot y+80\cdot z-400=0$

c. $\underset{\sim}{v} = -6\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$\underset{\sim}{n} = -8\underset{\sim}{i} + 64\underset{\sim}{j} + 80\underset{\sim}{k}$

$\sin^{-1}\left(\dfrac{|\underset{\sim}{v} \cdot \underset{\sim}{n}|}{|\underset{\sim}{v}||\underset{\sim}{n}|}\right) = \sin^{-1}\left(\dfrac{2\sqrt{6270}}{3135}\right)$ M1

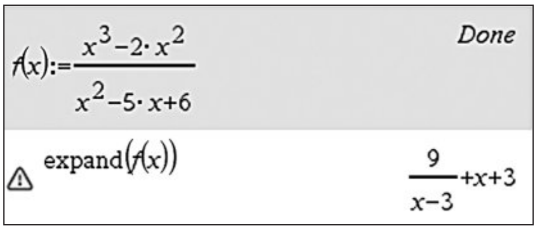
$\approx 2.9°$ A1



Note: Consequential on answers to ***Questions 1a.i.*** *and* ***1b.***

Question 2 (9 marks)

a.



$x = 3$ A1

$y = x + 3$ A1

Note: Deduct a maximum of 1 mark for any additional asymptotes stated.

b. $f(x) = 9(x-3)^{-1} + x - 3$

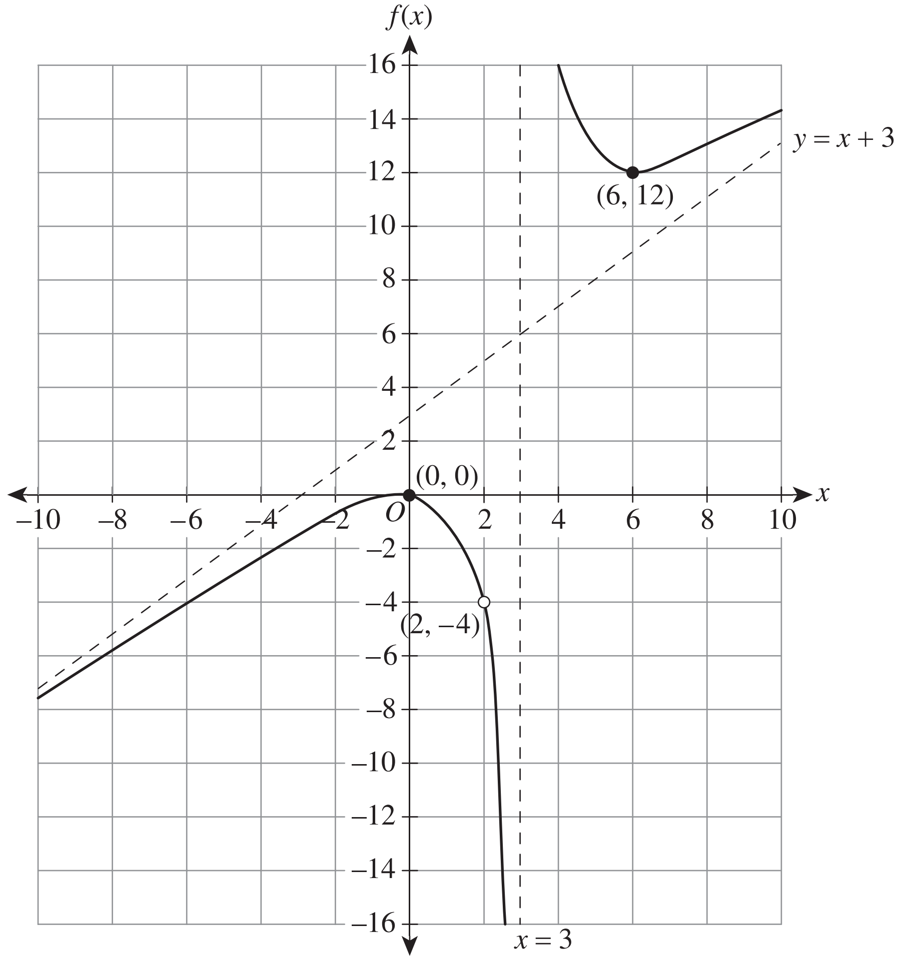
$f'(x) = -9(x-3)^{-2} + 1$ M1

$f''(x) = 18(x-3)^{-3}$

$= \dfrac{18}{(x-3)^3}$ M1

$f''(x) = 0$ has no solutions. Therefore, $f(x)$ has no points of inflection. M1

c.



correct shape, including asymptotical behaviour A1
correct asymptotes and key points A1

d. $g(x) = \dfrac{x^2(x+p)}{x^2 + qx + r}$

Relationship 1: $(x + p)$ is a common factor.

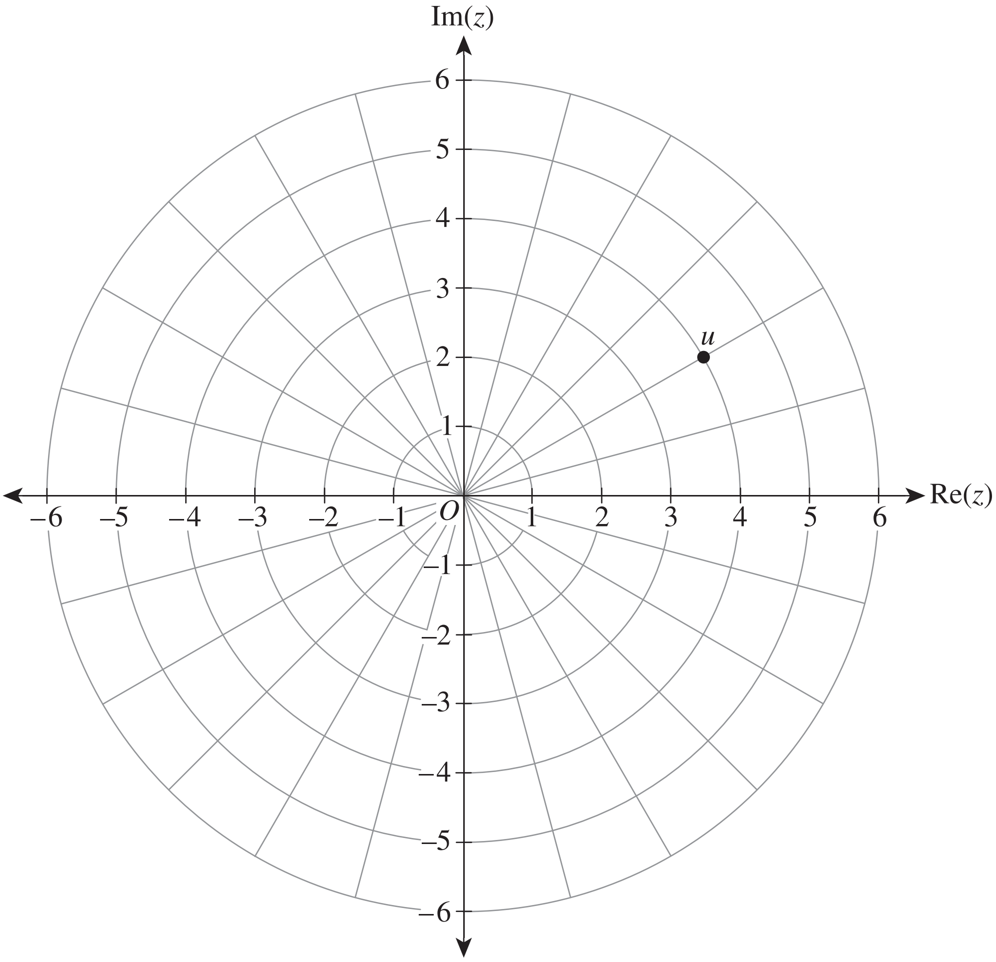
$\therefore\ p^2 - pq + r = 0$ A1

Relationship 2: The denominator is a perfect square.

$\therefore\ q^2 - 4r = 0$ A1

Question 3 (12 marks)

a.



correct point A1

b. By the conjugate root theorem, the other solution is $2\sqrt{3}-2i$. A1

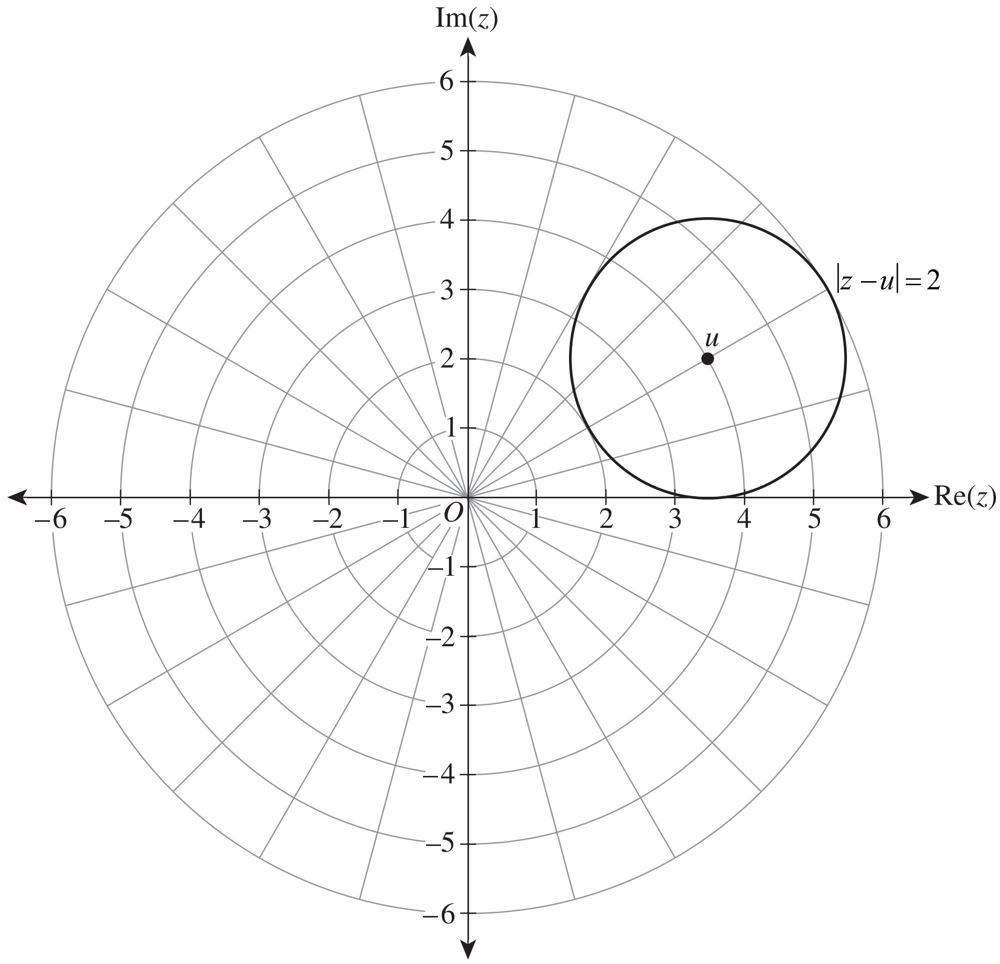
A quadratic equation with roots z_1 and z_2 can be written as $z^2-(z_1+z_2)z+z_1z_2=0$.

Therefore:

$$z^2-\left(2\sqrt{3}+2i+2\sqrt{3}-2i\right)z+\left(2\sqrt{3}+2i\right)\left(2\sqrt{3}-2i\right)=0$$

$$z^2-4\sqrt{3}z+16=0$$ A1

c. $|z-u|=2$ represents a circle with centre u and a radius of 2.



any circle with centre at point u A1

correct shape such that the circle is tangent to the grid at least three times A1

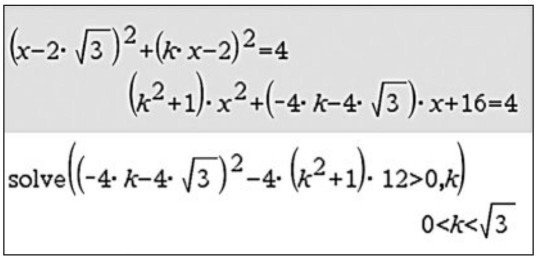
d. The equation of the circle is $\left(x-2\sqrt{3}\right)^2+(y-2)^2=4$. A1

Substituting $y=kx$ into the equation of the circle gives:

$\left(x-2\sqrt{3}\right)^2+(kx-2)^2=4$

As this will have two solutions, $\Delta>0$. M1

$0<k<\sqrt{3}$ A1



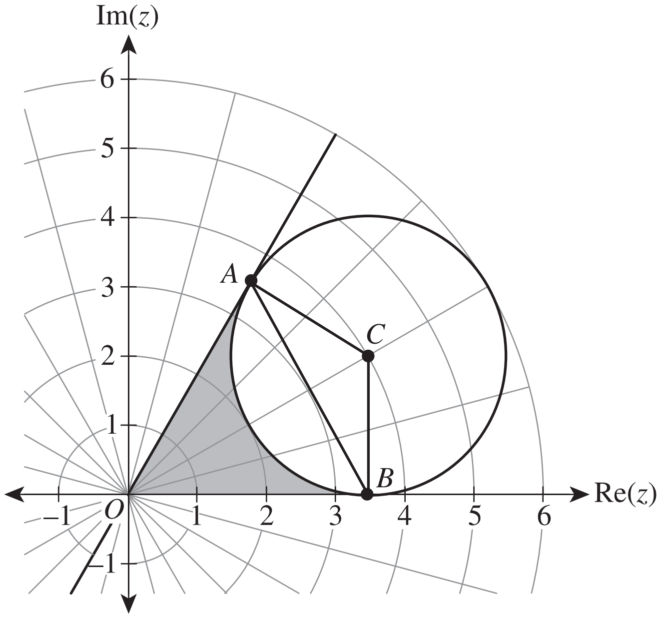
$\left(x-2\cdot\sqrt{3}\right)^2+(k\cdot x-2)^2=4$

$\left(k^2+1\right)\cdot x^2+\left(-4\cdot k-4\cdot\sqrt{3}\right)\cdot x+16=4$

solve$\left(\left(-4\cdot k-4\cdot\sqrt{3}\right)^2-4\cdot\left(k^2+1\right)\cdot 12>0,k\right)$

$0<k<\sqrt{3}$

e. The area to be found can be illustrated as follows.



To find points A and B, it is necessary to look for the intersection of the circle with lines $y = \sqrt{3x}$ and $y = 0$, respectively.

Using a CAS calculator to find the x-coordinates gives:

$\text{solve}\left(\left(x-2\cdot\sqrt{3}\right)^2+\left(\sqrt{3}\cdot x-2\right)^2=4,x\right)$	$x=\sqrt{3}$
$\text{solve}\left(\left(x-2\cdot\sqrt{3}\right)^2+\left(0\cdot x-2\right)^2=4,x\right)$	$x=2\cdot\sqrt{3}$

$A = \left(\sqrt{3}, 3\right),\ B = \left(2\sqrt{3}, 0\right)$ A1

Finding $\angle ABC$ gives:

$\angle O = 60°,\ \angle OAC = \angle OBC = 90°$

$\therefore \angle ACB = 120° = \dfrac{2\pi}{3}$ A1

Therefore:

$OC = 4,\ AB = 2\sqrt{3}$

$$\begin{aligned}\text{area}_{OABC} &= \frac{1}{2} \times OC \times AB \\ &= 4\sqrt{3}\end{aligned}$$ A1

$$\begin{aligned}\text{area}_{\overset{\frown}{CAB}} &= \frac{1}{2} \times \angle ACB \times \text{radius}^2 \\ &= \frac{4\pi}{3}\end{aligned}$$

$\text{area} = 4\sqrt{3} - \dfrac{4\pi}{3}$ A1

Note: A diagram is not required to obtain full marks.

Question 4 (12 marks)

a. To calculate the arc length, it is necessary to find the start and end values of x by substituting the start and end values of y.

When $y = 0$:

$$0 = \log_e\left(x^2 - 4\right)$$

$$x = \sqrt{5}$$

When $y = 3$:

$$3 = \log_e\left(x^2 - 4\right)$$

$$x = \sqrt{e^3 + 4}$$

correct domain A1

Note: From the domain restrictions of the graph, the positive x-values should be chosen.

Therefore:

$$\int_{\sqrt{5}}^{\sqrt{e^3+4}} \sqrt{1 + [f'(x)]^2}\, dx = 4.1775$$ A1

$\text{solve}(f(x)=0, x)$	$x=-\sqrt{5}$ or $x=\sqrt{5}$
$\text{solve}(f(x)=3, x)$	$x=-\sqrt{e^3+4}$ or $x=\sqrt{e^3+4}$
$\displaystyle\int_{\sqrt{5}}^{\sqrt{e^3+4}} \sqrt{1+\left(\frac{d}{dx}(f(x))\right)^2}\, dx$	4.1775

b.

$$y = \log_e\left(x^2 - 4\right)$$

$$x^2 - 4 = e^y$$

$$x^2 = e^y + 4$$

Therefore:

$$V = \pi\int_0^h x^2 dy$$

$$= \pi\int_0^h \left(e^y + 4\right) dy$$ A1

$$= \pi\left(e^h + 4h - 1\right)$$ A1

$\pi \cdot \displaystyle\int_0^h \left(e^y+4\right) dy$	$\left(e^h + 4 \cdot h - 1\right) \cdot \pi$

c. The container's maximum volume is achieved when the water is at its maximum height; that is, $h = 3$.

Therefore, half of the maximum volume is:

$$V(h) = \frac{1}{2} \times V(3) \qquad \text{M1}$$

$$= \frac{1}{2} \times \left(\pi\left(e^3 + (4 \times 3) - 1\right)\right)$$

$$h = 2.10 \text{ m} \qquad \text{A1}$$

$v(h) := \left(e^h + 4 \cdot h - 1\right) \cdot \pi$ Done

⚠ $\text{solve}\left(v(h) = \frac{1}{2} \cdot v(3), h\right)$ $h = 2.09808$

*Note: Consequential on answer to **Question 4b.***

d. $$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} \qquad \text{M1}$$

$$\frac{dV}{dt} = -2\sqrt{h}$$

$$\frac{dV}{dh} = \pi\left(e^h + 4\right)$$

$$-2\sqrt{h} = \pi\left(e^h + 4\right) \times \frac{dh}{dt} \qquad \text{A1}$$

$$\frac{dh}{dt} = -\frac{2\sqrt{h}}{\pi\left(e^h + 4\right)}$$

$$\frac{dt}{dh} = -\frac{\pi\left(e^h + 4\right)}{2\sqrt{h}}$$

$$t = \int_3^0 -\frac{\pi\left(e^h + 4\right)}{2\sqrt{h}}\,dh \qquad \text{M1}$$

$$= 44.7 \text{ min} \qquad \text{A1}$$

$\text{solve}\left(\int_0^t 1\,dt = \int_3^0 \frac{-\pi \cdot \left(e^h + 4\right)}{2 \cdot \sqrt{h}}\,dh, t\right)$

$t = 44.7403$

e. Expressing the rate, r, as a function of h gives:

$$r(h) = \frac{-2\sqrt{h}}{\pi\left(e^{h}+4\right)}$$

Differentiating $r(h)$ to find the maximum gives:

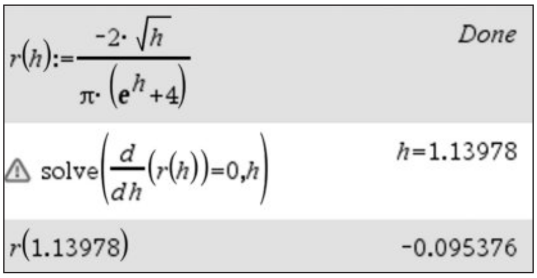
$$\frac{d}{dh}r(h) = 0$$

$$h = 1.14 \text{ m}$$ A1

Substituting $h = 1.14$ into $r(h)$ gives:

$r(1.14) = -0.10$

Therefore, the maximum rate is 0.10 m/min. A1

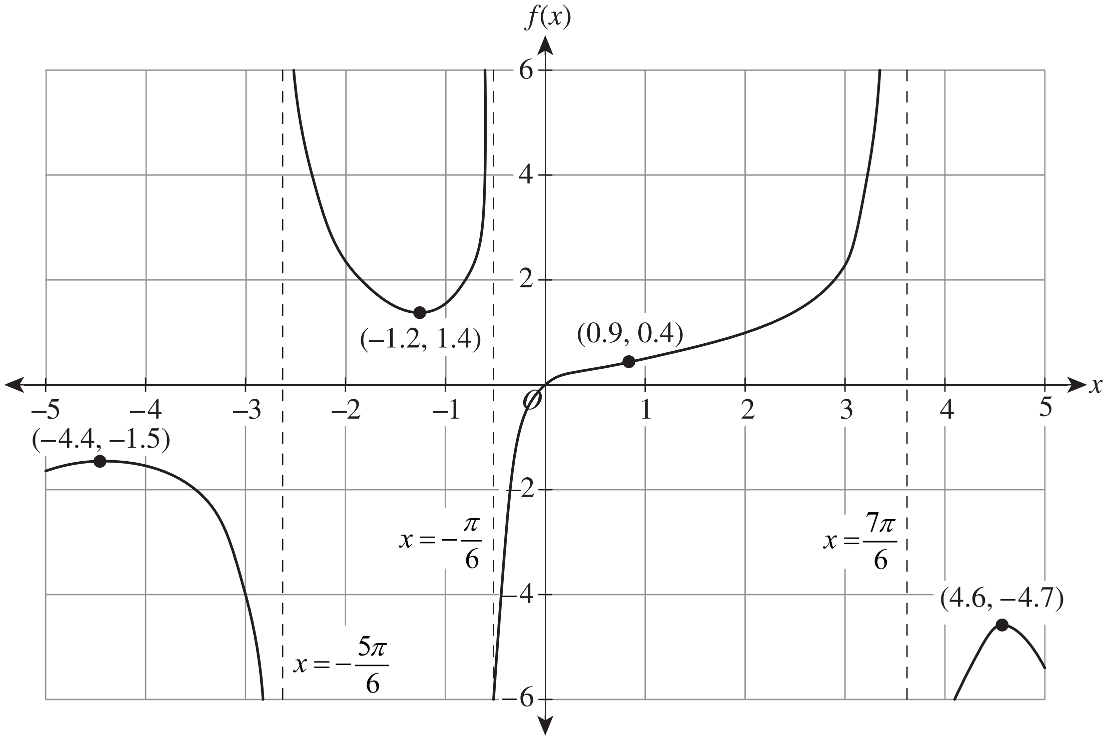


Question 5 (11 marks)

a. The graph has vertical asymptotes when $2\sin(x) + 1 = 0$.

$$x = -\frac{\pi}{6} + 2\pi k,\ x = \frac{7\pi}{6} + 2\pi k,\ k \in Z$$ A1

b.



accurate sketch A1
correct shape A1
correct key points A1
correct asymptotes A1

c. $f(x) = p(x)$

$$\frac{1}{2\sin(x)+1}x = p(x)$$

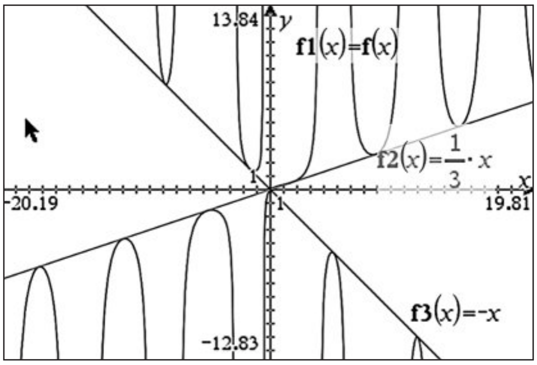
Letting $m = \dfrac{1}{2\sin(x)+1}$ and using $-1 \le 2\sin(x) + 1 \le 3$ gives:

$m \le -1$ or $m \ge \dfrac{1}{3}$ M1

If $f(x) = p(x)$ has only $x = 0$ as a solution, then m and p have no intersection.

Hence, $-1 < p < \dfrac{1}{3}$. A1

Sketching the graph using a CAS calculator gives:



d. **i.** If the denominator has no zeroes, then $g(x)$ has no asymptotes.

Since $-1 \le \sin(x) \le 1$ and $k > 0$:

$$-k \le k\sin(x) \le k$$

$$-k+1 \le k\sin(x)+1 \le k+1$$

For no asymptotes:

$$-k+1 > 0$$

$$k < 1$$ A1

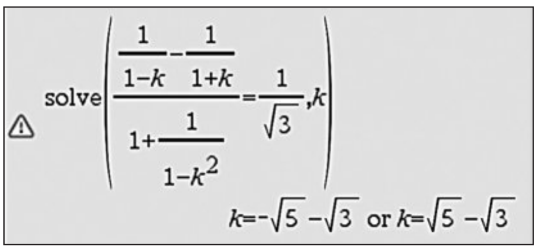
ii. The equations of the tangent lines are $y = \dfrac{1}{1-k}x$ and $y = \dfrac{1}{1+k}x$. A1

Given that the angle between the tangent lines is 30°:

$$\frac{\frac{1}{1-k} - \frac{1}{1+k}}{1 + \frac{1}{1-k^2}} = \tan(30°)$$

$$= \frac{1}{\sqrt{3}}$$ M1

As $k > 0$, $k = \sqrt{5} - \sqrt{3}$. A1



Question 6 (9 marks)

a. $\overline{X} \sim N\left(3200, \left(\dfrac{430}{\sqrt{7}}\right)^2\right)$ M1

$\Pr\left(\overline{X} > 3500\right) = 0.032$ A1

$\text{normCdf}\left(3500, \infty, 3200, \dfrac{430}{\sqrt{7}}\right) \quad 0.032455$

b. $T \sim N\left(7 \times 3200, \left(430 \times \sqrt{7}\right)^2\right)$ M1

$\Pr(T > 22\ 500) = 0.465$ A1

$\text{normCdf}\left(22500, \infty, 7 \cdot 3200, 430 \cdot \sqrt{7}\right)$

0.464979

c. Bakery: $B \sim N\left(3200, 430^2\right)$

Grocery store: $G \sim N\left(3600, 840^2\right)$

$G - B \sim N\left(400, 430^2 + 840^2\right)$ M1

$\Pr(G - B > 0) = 0.664$ A1

$\text{normCdf}\left(0, \infty, 400, \sqrt{430^2 + 840^2}\right) \quad 0.664173$

d. **i**. H_0: $\mu = 3600$

H_1: $\mu < 3600$

both hypotheses A1

ii. $p\text{-value} = \Pr(G < 3440)$

$= 0.1971...$ A1

Since $p\text{-value} \nless 0.05$, H_0 should be accepted. A1

$\text{normCdf}\left(-\infty, 3440, 3600, \dfrac{840}{\sqrt{20}}\right) \quad 0.197153$

