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SPECIALIST MATHEMATICS 2023

Unit 4
Key Topic Test 1 – Antidifferentiation applications
Technology Free

Recommended writing time: 45 minutes
Total number of marks available: 30 marks

SOLUTIONS

Question 1

a. $Area = 2\int_0^{\pi} \sin^3(x)\ dx$

2 marks

b. $Area = 2\int_0^{\pi} \sin^2(x)\sin(x)\ dx = Area = 2\int_0^{\pi} (1-\cos^2(x))\sin(x)\ dx$

$Let \cos(x) = u$

$\frac{du}{dx} = -\sin(x)$

$Area = 2\int_1^{-1} -(1-u^2)\ du$

$Area = 2\int_{-1}^{1} (1-u^2)\ du$

$Area = 2\left[u - \frac{u^3}{3}\right]_{-1}^{1}$

$Area = 2\left(\left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right)\right) = 2\left(2-\frac{2}{3}\right) = \frac{8}{3}\ sq\ units$

4 marks

Question 2

a. $x\cos(2x) = 0$

$x = 0, \cos(2x) = 0$

$x = 0,\ \ 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = 0, \frac{\pi}{4}, \frac{3\pi}{4}$

$(0,0), \left(\frac{\pi}{4}, 0\right)\ and\ \left(\frac{3\pi}{4}, 0\right)$

2 marks

b. $Area = \int_0^{\frac{\pi}{4}} x\cos(2x)\ dx - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\cos(2x)\ dx + \int_{\frac{3\pi}{4}}^{\pi} x\cos(2x)\ dx$

$\int x\cos(2x)\ dx$

$Let\ u = x\ \ and\ \frac{dv}{dx} = \cos(2x)$

$\frac{du}{dx} = 1\ \ and\ \ v = \frac{\sin(2x)}{2}$

$\int x\cos(2x)\ dx = \frac{x\sin(2x)}{2} - \int \frac{\sin(2x)}{2} dx$

$\int x\cos(2x)\ dx = \frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4}$

$Area = \left[\frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4}\right]_0^{\frac{\pi}{4}} - \left[\frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} + \left[\frac{x\sin(2x)}{2} + \frac{\cos(2x)}{4}\right]_{\frac{3\pi}{4}}^{2\pi}$

$Area = \frac{\pi}{8} - \frac{1}{4} - \left(-\frac{3\pi}{8} - \frac{\pi}{8}\right) + \left(\frac{1}{4} + \frac{3\pi}{8}\right) = \pi$

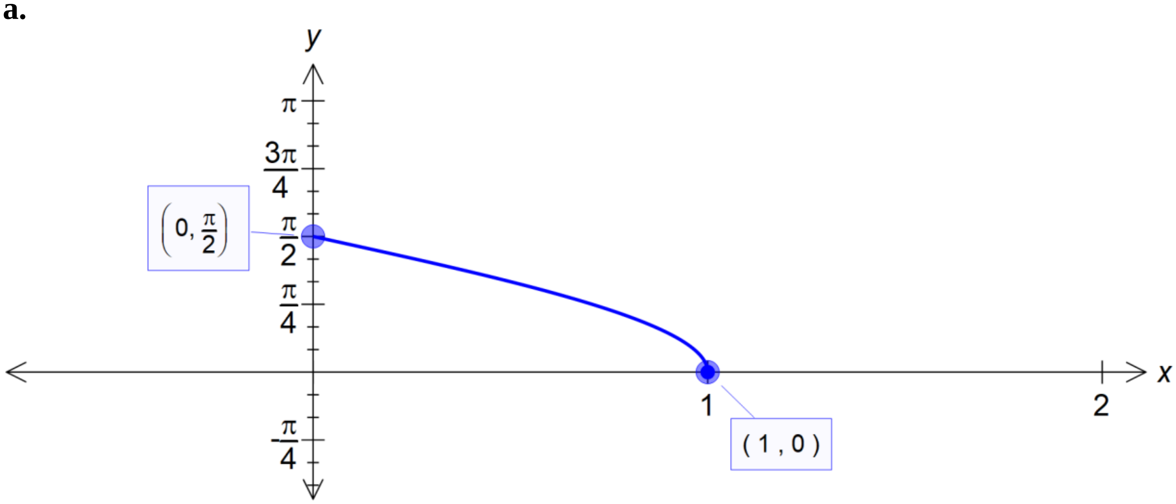
4 marks

c. $Signed\ area = \frac{\pi}{8} - \frac{1}{4} + \left(-\frac{3\pi}{8} - \frac{\pi}{8}\right) + \left(\frac{1}{4} + \frac{3\pi}{8}\right) = 0$

2 marks

Question 3

a.



2 marks

b. $y = \arccos(x) \rightarrow x = \cos(y)$

$$V = \pi \int_0^{\frac{\pi}{2}} (\cos(y))^2 \, dy$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{(1+\cos(2y))}{2} \, dy$$

$$= \pi \left[\frac{y}{2} + \frac{\sin(2y)}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \pi \left(\frac{\pi}{4}\right)$$

$$= \frac{\pi^2}{4}$$

4 marks

Question 4

$$y = \frac{1}{3}(1+x)^{\frac{3}{2}}$$
$$\frac{dy}{dx} = \frac{1}{3} \times \frac{3}{2}(1+x)^{\frac{1}{2}} = \frac{1}{2}\sqrt{1+x}$$
$$Arc\ length = \int_0^4 \sqrt{1 + \left(\frac{1}{2}\sqrt{1+x}\right)^2}\, dx$$
$$Arc\ length = \int_0^4 \sqrt{\frac{5}{4} + \frac{x}{4}}\, dx$$
$$Arc\ length = \left[\frac{2}{3} \times 4\left(\frac{5}{4} + \frac{x}{4}\right)^{\frac{3}{2}}\right]_0^4 = \frac{8}{3}\left(\frac{5}{4} + 1\right)^{\frac{3}{2}} - \frac{8}{3}\left(\frac{5}{4}\right)^{\frac{3}{2}} = 9 - \frac{5\sqrt{5}}{3}$$

4 marks

Question 5

a. $x = \frac{4}{3}(t^2 - 1)$ and $y = 2t^2$

$$\frac{dx}{dt} = \frac{4}{3}(2t) = \frac{8t}{3} \quad and \quad \frac{dy}{dt} = 4t$$
$$\frac{dy}{dx} = \frac{4t}{\frac{8t}{3}} = \frac{3}{2}$$

2 marks

b. $\frac{3x}{4} + 1 = \frac{y}{2} \rightarrow y = \frac{3x}{2} + 2$

$$Surface\ area = 2\pi \int_0^1 y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$Surface\ area = 2\pi \int_0^1 \left(\frac{3x}{2} + 2\right)\sqrt{1 + \frac{9}{4}}\, dx$$
$$Surface\ area = \frac{2\sqrt{13}}{2}\pi \int_0^1 \left(\frac{3x}{2} + 2\right) dx$$
$$Surface\ area = \sqrt{13}\pi \left[\frac{3x^2}{4} + 2x\right]_0^1$$
$$Surface\ area = \sqrt{13}\pi\left(\frac{3}{4} + 2\right) = \frac{11\sqrt{13}}{4}\pi$$

4 marks

END OF KEY TOPIC TEST SOLUTIONS