2023
VCE
Specialist
Mathematics
Year 12
Sample questions

Detailed Answers

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Sample Solutions Specialist Mathematics: Written Examination 1

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Question 1

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^n}=1-\frac{1}{2^n}$, where $n \in N$

a. base case when $n=1$ $LHS=\frac{1}{2}$ $RHS=1-\frac{1}{2^1}=1-\frac{1}{2}=\frac{1}{2}=LHS$

The statement is true when $n=1$

b. Assume the statement is true when $n=k$ so

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^k}=1-\frac{1}{2^k}$

c. Need to show that is true when $n=k+1$ that is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^k}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}$

Consider $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^k}+\frac{1}{2^{k+1}}$

$=1-\frac{1}{2^k}+\frac{1}{2^{k+1}}$

$=1-\frac{2}{2^{k+1}}+\frac{1}{2^{k+1}}=1-\frac{1}{2^{k+1}}$

Since it is true when $n=1$ and assuming it is true when $n=k$ then it is true when $n=k+1$, so by the principle of mathematical induction, it is true for $n \in N$.

Question 2

a. $2^n > n^2$ for $n \ge n_0$, where $n \in N$.

n	1	2	3	4	5	6
2^n	2	4	8	16	32	64
n^2	1	4	9	16	25	36

So that $2^n > n^2$ when $n \ge 5$ so that $n_0 = 5$

b. The base case when $n=5$ it is true

Assume $2^k > k^2$ for $k \ge 5$, to show that $2^{k+1} > (k+1)^2 = k^2+2k+1$ now consider

$2^{k+1} = 2 \times 2^k > 2k^2$

$= k^2 + k^2$

$\ge k^2 + 5k$ since when $k \ge 5$, $k^2 \ge 5k$

$= k^2+2k+3k$

$\ge k^2+2k+15$ since when $k \ge 5$, $3k \ge 15$

$> k^2+2k+1 = (k+1)^2$ shown

Since it is true when $n=5$ and assuming it is true when $n=k$ then it is true when $n=k+1$, so by the principle of mathematical induction, it is true for $n \ge 5$, $n \in N$.

Question 3

To prove $9^n - 5^n$ is divisible by 4 for all $n \in N$.
base case $n = 1$ $LHS = 9^1 - 5^1 = 4$ this is divisible by 4.
The statement is true when $n = 1$
Assume it is true when $n = k$ so that $9^k - 5^k$ is divisible by 4, so that we can write
$9^k - 5^k = 4a$ where $a \in N$. Now consider

$$\begin{aligned} 9^{k+1} - 5^{k+1} &= 9 \times 9^k - 5 \times 5^k \\ &= 9 \times 9^k - 9 \times 5^k + 4 \times 5^k \\ &= 9\left(9^k - 5^k\right) + 4 \times 5^k \\ &= 9 \times 4a + 4 \times 5^k \\ &= 4\left(9a + 5^k\right) \\ &= 4b \quad \text{where } b \in N \text{ so it is divisible by 4.} \end{aligned}$$

Since it is true when $n = 1$ and assuming it is true when $n = k$ then it is true when $n = k+1$, so by the principle of mathematical induction, it is true for $n \in N$.

Question 4

Let *p*: "*n* is odd" and *q*: "$n^3 + 1$ is even", to prove $p \to q$
Using a proof by contradiction, assume that $\neg q$ is true, that is $n^3 + 1$ is odd
and *p* is true, since *p* is true, *n* is odd, we then let $n = 2k + 1$ where $k \in N$. Now consider

$$\begin{aligned} n^3 + 1 &= (2k+1)^3 + 1 \\ &= \left(8k^3 + 12k^2 + 6k + 1\right) + 1 \\ &= 8k^3 + 12k^2 + 6k + 2 \\ &= 2\left(4k^3 + 6k^2 + 3k + 1\right) \\ &= 2m \end{aligned}$$

so $n^3 + 1$ is even, but this is a contradiction since we assumed $n^3 + 1$ is odd,
Since our original assumption is false it follows that $p \to q$ it true.

Question 5

To prove $\sqrt{3} + \sqrt{5} > \sqrt{11}$, we use a proof by contradiction, assume the negation, that is
$\sqrt{3} + \sqrt{5} \le \sqrt{11}$ now square both sides, since if $a > b$ then $a^2 > b^2$ if $a > 0$ and $b > 0$

$$\left(\sqrt{3} + \sqrt{5}\right)^2 \le \left(\sqrt{11}\right)^2$$

$$3 + 2\sqrt{15} + 5 \le 11$$

$$8 + 2\sqrt{15} \le 11$$

$2\sqrt{15} \le 3$ but $3 < \sqrt{15} < 4$ so that $2\sqrt{15} > 3$, we have arrived at a contradiction, and therefore the statement $\sqrt{3} + \sqrt{5} \le \sqrt{11}$ must be false, therefore $\sqrt{3} + \sqrt{5} > \sqrt{11}$ must be true.

Question 6

The surface area of a curve $y = f(x)$ when rotated about the x-axis is given by

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \sqrt{4-x^2}, \quad x \in [-1,1], \quad a=-1,\ b=1$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}, \quad 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{x^2}{4-x^2} = \frac{4-x^2+x^2}{4-x^2} = \frac{4}{4-x^2} \quad \text{substituting}$$

$$S = 2\pi \int_{-1}^{1} \sqrt{4-x^2}\sqrt{\frac{4}{4-x^2}}\, dx$$

$$S = 2\pi \int_{-1}^{1} 2\, dx$$

$$S = 4\pi [x]_{-1}^{1} = 4\pi(1-(-1))$$

$$S = 8\pi \text{ units}^2$$

Question 7

The surface area of a curve $x = g(y)$ when rotated about the y-axis is given by

$$S = 2\pi \int_a^b x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy\,,$$

$$y = \sqrt[3]{x} = x^{\frac{1}{3}}, \quad x \in [0,8], \quad x=0,\ y=0, \quad x=8,\ y=2$$

$$y^3 = x, \quad \frac{dx}{dy} = 3y^2, \quad \text{substituting}$$

$$S = 2\pi \int_0^2 y^3\sqrt{1+9y^4}\, dy$$

let $u = 1+9y^4 \quad \dfrac{du}{dy} = 36y^3$

terminals, when $y=0,\ u=1$ and when $y=2,\ u=145$

$$S = 2\pi \int_1^{145} \frac{1}{36} u^{\frac{1}{2}}\, du$$

$$S = \frac{\pi}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{145}$$

$$S = \frac{\pi}{27}\left(145^{\frac{3}{2}} - 1\right)$$

$$S = \frac{\pi}{27}\left(145\sqrt{145} - 1\right) \text{ units}^2$$

Question 8

The surface area of the parametric curves $x = x(t)$ and $y = y(t)$

when rotated about the *y*-axis is given by $S = 2\pi \int_{t_1}^{t_2} x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

we can use θ instead of *t*.

$x = \sin^3(\theta)$ $\quad y = \cos^3(\theta)$ $\quad \theta \in \left[0, \frac{\pi}{2}\right], \quad \theta_1 = 0, \ \theta_2 = \frac{\pi}{2}$

$\frac{dx}{d\theta} = 3\sin^2(\theta)\cos(\theta)$, $\frac{dy}{d\theta} = -3\cos^2(\theta)\sin(\theta)$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9\sin^4(\theta)\cos^2(\theta) + 9\cos^4(\theta)\sin^2(\theta)$$

$$= 9\sin^2(\theta)\cos^2(\theta)\left(\sin^2(\theta) + \cos^2(\theta)\right)$$

$$= 9\sin^2(\theta)\cos^2(\theta)$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \left|3\sin(\theta)\cos(\theta)\right| = \left|\frac{3}{2}\sin(2\theta)\right|$$

$= \frac{3}{2}\sin(2\theta)$ the moduli are not needed since $\theta \in \left[0, \frac{\pi}{2}\right]$

substituting

$$S = 2\pi \int_0^{\frac{\pi}{2}} \sin^3(\theta) \times 3\sin(\theta)\cos(\theta)\, d\theta$$

$$S = 6\pi \int_0^{\frac{\pi}{2}} \sin^4(\theta)\cos(\theta)\, d\theta$$

let $u = \sin(\theta)$ $\quad \frac{du}{d\theta} = \cos(\theta)$

terminals, when $\theta = 0$, $u = \sin(0) = 0$ and when $\theta = \frac{\pi}{2}$, $u = \sin\left(\frac{\pi}{2}\right) = 1$

$$S = 6\pi \int_0^1 u^4\, du$$

$$S = 6\pi \left[\frac{u^5}{5}\right]_0^1 = \frac{6\pi}{5}(1 - 0)$$

$$S = \frac{6\pi}{5} \text{ units}^2$$

Question 9

The surface area of the parametric curves $x = x(t)$ and $y = y(t)$ when rotated about the x-axis is given by $S = 2\pi \int_{t_1}^{t_2} y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$x = \frac{4}{3}\sqrt{(t+1)^3}$ $\qquad y = \frac{1}{2}t^2$ $\qquad 0 \le t \le 1, \quad t_1 = 0, \quad t_2 = 1$

$\frac{dx}{dt} = \frac{4}{3} \times \frac{3}{2} \times (t+1)^{\frac{1}{2}}$ $\qquad \frac{dy}{dt} = t$

$= 2\sqrt{t+1}$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4(t+1) + t^2 = t^2 + 4t + 4 = (t+2)^2$$

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = |t+2| = t+2$ since $t > -2$ substituting

$$S = 2\pi \int_0^1 \frac{1}{2}t^2(t+2)\,dt = \pi \int_0^1 (t^3 + 2t^2)\,dt$$

$$S = \pi \left[\frac{t^4}{4} + \frac{2t^3}{3}\right]_0^1 = \pi\left(\frac{1}{4} + \frac{2}{3} - 0\right)$$

$$S = \frac{11\pi}{12} \text{ units}^2$$

Question 10

$$\frac{dP}{dt} = 2P\left(6 - \frac{P}{8000}\right), \quad P(0) = 4000$$

a. maximum number of bacteria $\frac{dP}{dt} = 0$,

$$6 - \frac{P}{8000} = 0$$

$$P = 6 \times 8000 = 48000$$

b. $\frac{dP}{dt} = 2\left(6P - \frac{P^2}{8000}\right)$

now the number of bacteria is growing at its fastest rate when

$$\frac{d^2P}{dt^2} = 2\left(6 - \frac{P}{4000}\right)\frac{dP}{dt} = 0$$

$$P = 6 \times 4000 = 24000$$

c. $\frac{dP}{dt} = 2P\left(\frac{48000-P}{8000}\right) = \frac{P(48000-P)}{4000}$

inverting both sides $\frac{dt}{dP} = \frac{4000}{P(48000-P)}$ now by partial fractions

$$\frac{4000}{P(48000-P)} = \frac{A}{P} + \frac{B}{48000-P} = \frac{A(48000-P)+BP}{P(48000-P)} = \frac{48000A+P(B-A)}{P(48000-P)}$$

equating $(1)\, B-A=0,\quad (2)\, 48000A=4000,\quad \Rightarrow A=B=\frac{4000}{48000}=\frac{1}{12}$

$$t = \frac{1}{12}\int\left(\frac{1}{P}+\frac{1}{48000-P}\right)dP$$

$12t = \log_e(|P|) - \log_e(|48000-P|)+c$ since $4000 \le P < 48000$, the moduli are not needed.

$12t = \log_e\left(\frac{P}{48000-P}\right)+c$ using $t=0,\ P=4000$ to find c

$$0 = \log_e\left(\frac{4000}{48000-4000}\right)+c \quad \Rightarrow c = -\log_e\left(\frac{1}{11}\right) = \log_e(11)$$

$$12t = \log_e\left(\frac{P}{48000-P}\right)+\log_e(11) = \log_e\left(\frac{11P}{48000-P}\right)$$

$$e^{12t} = \frac{11P}{48000-P}, \qquad \frac{48000-P}{11P} = e^{-12t}$$

$$48000-P = 11Pe^{-12t}, \qquad 48000 = P + 11Pe^{-12t} = P\left(1+11e^{-12t}\right)$$

$$P = P(t) = \frac{48000}{1+11e^{-12t}}$$

Question 11

Let $C = \int x^2\cos(2x)\,dx$, using integration by parts

let $u = x^2 \qquad \frac{dv}{dx} = \cos(2x)$

$\frac{du}{dx} = 2x \qquad v = \int\cos(2x)\,dx = \frac{1}{2}\sin(2x)$

$C = \frac{x^2}{2}\sin(2x) - \int x\sin(2x)\,dx$ using integration by parts again on the last term

let $u = x \qquad \frac{dv}{dx} = \sin(2x)$

$\frac{du}{dx} = 1 \qquad v = \int\sin(2x) = -\frac{1}{2}\cos(2x)$

$$C = \frac{x^2}{2}\sin(2x) - \left[-\frac{x}{2}\cos(2x) + \frac{1}{2}\int\cos(2x)\,dx\right]$$

$$C = \frac{x^2}{2}\sin(2x) + \frac{x}{2}\cos(2x) - \frac{1}{4}\sin(2x) + c$$

Question 12

$\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k} \quad \underset{\sim}{b} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -3 & 1 \\ 4 & 2 & -3 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & -3 \\ 4 & 2 \end{vmatrix}$$

$\underset{\sim}{n} = 7\underset{\sim}{i} + 10\underset{\sim}{j} + 16\underset{\sim}{k}$

plane using the normal through the point $(3,2,1)$

$7(x-3) + 10(y-2) + 16(z-1) = 0$

$7x - 21 + 10y - 20 + 16z - 16 = 0$

$7x + 10y + 16z = 57$

Question 13

a. $P(3,3,6),\ Q(1,-1,2),\ R(5,2,0)$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -2\underset{\sim}{i} - 4\underset{\sim}{j} - 4\underset{\sim}{k}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = 2\underset{\sim}{i} - \underset{\sim}{j} - 6\underset{\sim}{k}$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & -4 & -4 \\ 2 & -1 & -6 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} -4 & -4 \\ -1 & -6 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -2 & -4 \\ 2 & -6 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -2 & -4 \\ 2 & -1 \end{vmatrix}$$

$\overrightarrow{PQ} \times \overrightarrow{PR} = 20\underset{\sim}{i} - 20\underset{\sim}{j} + 10\underset{\sim}{k} = 10\left(2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}\right)$

for the normal we can take $\underset{\sim}{n} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

plane using the normal through the point $P(3,3,6)$

$2(x-3) - 2(y-3) + 1(z-6) = 0$

$2x - 6 - 2y + 6 + z - 6 = 0$

$2x - 2y + z = 6$

b. line $\underset{\sim}{r} = 2\underset{\sim}{i} + 5\underset{\sim}{k} + t\left(2\underset{\sim}{i} - 4\underset{\sim}{j} - 3\underset{\sim}{k}\right)$ and a plane $2x - 2y + z = 6$

line has parametric equation $x = 2 + 2t,\ y = -4t,\ z = 5 - 3t$ substitute into the plane

$2(2+2t) - 2(-4t) + 5 - 3t = 6$

$4 + 4t + 8t + 5 - 3t = 6$

$9t = -3, \quad t = -\frac{1}{3}$ substitute this value of t back into the line

$x = 2 - \frac{2}{3} = \frac{4}{3},\ y = \frac{4}{3},\ z = 5 + 1$ the point of intersection is $\left(\frac{4}{3}, \frac{4}{3}, 6\right)$

Question 14

plane $2x+y+z=7$ and a line $\underset{\sim}{r}=11\underset{\sim}{i}+4\underset{\sim}{j}+3\underset{\sim}{k}+t\left(\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}\right)$

normal to the plane $\underset{\sim}{n}=2\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$, $\quad |\underset{\sim}{n}|=\sqrt{4+1+1}=\sqrt{6}$

direction of the line $\underset{\sim}{v}=\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}$, $\quad |\underset{\sim}{v}|=\sqrt{1+4+1}=\sqrt{6}$

$\underset{\sim}{n}.\underset{\sim}{v}=2+2-1=3$

now let α be the angle between the plane and the line, and let θ be the angle between the vectors $\underset{\sim}{n}$ and $\underset{\sim}{v}$, so that $\cos(\theta)=\dfrac{\underset{\sim}{n}.\underset{\sim}{v}}{|\underset{\sim}{n}||\underset{\sim}{v}|}$ now $\alpha+\theta=90^0$ so that $\alpha=90^0-\theta$ and

$\sin(\alpha)=\sin\left(90^0-\theta\right)=\cos(\theta)=\dfrac{3}{\sqrt{6}\sqrt{6}}=\dfrac{1}{2}$ so

$\alpha=30^0$

Question 15

a. $A(3,1,-1)$, $B(5,2,-6)$

the direction of the line $\underset{\sim}{v}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=2\underset{\sim}{i}+\underset{\sim}{j}-5\underset{\sim}{k}$, ,

the vector equation of the line through *A*, is

$\underset{\sim}{r}=3\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}+t\left(2\underset{\sim}{i}+\underset{\sim}{j}-5\underset{\sim}{k}\right)=(3+2t)\underset{\sim}{i}+(1+t)\underset{\sim}{j}+(-1-5t)\underset{\sim}{k}, \quad t\in R$

b. plane $x+2y-z=9$

normal to the plane $\underset{\sim}{n}=\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}$, $\quad |\underset{\sim}{n}|=\sqrt{1+4+1}=\sqrt{6}$

direction of the line $\underset{\sim}{v}=2\underset{\sim}{i}+\underset{\sim}{j}-5\underset{\sim}{k}$, $\quad |\underset{\sim}{v}|=\sqrt{4+1+25}=\sqrt{30}$

$\underset{\sim}{n}.\underset{\sim}{v}=2+2+5=9$

now let α be the angle between the plane and the line, and let θ be the angle between the vectors $\underset{\sim}{n}$ and $\underset{\sim}{v}$, so that $\cos(\theta)=\dfrac{\underset{\sim}{n}.\underset{\sim}{v}}{|\underset{\sim}{n}||\underset{\sim}{v}|}$ now $\alpha+\theta=90^0$ so that

$\alpha=90^0-\theta$ and $\sin(\alpha)=\sin\left(90^0-\theta\right)=\cos(\theta)=\dfrac{9}{\sqrt{30}\sqrt{6}}=\dfrac{9}{\sqrt{6\times5\times6}}=\dfrac{9}{6\sqrt{5}}$

so $\sin(\alpha)=\dfrac{3\sqrt{5}}{10}$

Question 16

$\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + 5t\underset{\sim}{j} + (t^2 - 16t)\underset{\sim}{k}$

$\dot{\underset{\sim}{r}}(t) = 2t\underset{\sim}{i} + 5\underset{\sim}{j} + (2t - 16)\underset{\sim}{k}$

$\left|\dot{\underset{\sim}{r}}(t)\right| = \sqrt{(2t)^2 + 5^2 + (2t-16)^2} = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256}$

$v(t) = \left(8t^2 - 64t + 281\right)^{\frac{1}{2}}$

$\frac{dv}{dt} = \frac{16t - 64}{2\sqrt{8t^2 - 64t + 281}} = 0 \quad \Rightarrow \quad t = 4$

$\dot{\underset{\sim}{r}}(4) = 8\underset{\sim}{i} + 5\underset{\sim}{j} - 8\underset{\sim}{k}$

$\left|\dot{\underset{\sim}{r}}(4)\right| = \sqrt{64 + 25 + 64} = \sqrt{153} = 3\sqrt{17}$

Question 17

plane (1) $x + y - z = 3$

normal to the plane $\underset{\sim}{n}_1 = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, $\quad |\underset{\sim}{n}_1| = \sqrt{1+1+1} = \sqrt{3}$

plane (2) $2x - y - 2z = 4$

normal to the plane $\underset{\sim}{n}_2 = 2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$, $\quad |\underset{\sim}{n}_2| = \sqrt{4+1+4} = \sqrt{9} = 3$

$\underset{\sim}{n}_1 . \underset{\sim}{n}_2 = 2 - 1 + 2 = 3$

$\cos(\theta) = \frac{\underset{\sim}{n}_1 . \underset{\sim}{n}_2}{|\underset{\sim}{n}_1||\underset{\sim}{n}_2|} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\sec(\theta) = \frac{1}{\cos(\theta)} = \sqrt{3}$

Question 18

$\underset{\sim}{a} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k} \qquad \underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & -4 & 2 \\ 1 & -2 & 3 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} -4 & 2 \\ -2 & 3 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = -8\underset{\sim}{i} - 4\underset{\sim}{j}$$

The area of the triangle is $\frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|$

$\frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}| = \frac{1}{2}\sqrt{64+16} = \frac{1}{2}\sqrt{80} = \frac{1}{2}\sqrt{16 \times 5}$

$= 2\sqrt{5}$

Question 19

$O(0,0,0)$, $A(1,2,-1)$, $C(3,m,1)$, $m \in R$

$\overrightarrow{OA} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, $\overrightarrow{OC} = 3\underset{\sim}{i} + m\underset{\sim}{j} + \underset{\sim}{k}$

$$\overrightarrow{OA} \times \overrightarrow{OC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & -1 \\ 3 & m & 1 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 2 & -1 \\ m & 1 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} 1 & 2 \\ 3 & m \end{vmatrix} = (m+2)\underset{\sim}{i} - 4\underset{\sim}{j} + (m-6)\underset{\sim}{k}$$

Now the area of the parallelogram is $\left|\overrightarrow{OA} \times \overrightarrow{OC}\right| = 4\sqrt{5}$

$\sqrt{(m+2)^2 + 16 + (m-6)^2} = 4\sqrt{5}$

$\sqrt{m^2 + 4m + 4 + 16 + m^2 - 12m + 36} = 4\sqrt{5}$

$\sqrt{2m^2 - 8m + 56} = 4\sqrt{5}$

$2\left(m^2 - 4m + 28\right) = 16 \times 5 = 80$

$m^2 - 4m + 28 = 40$

$m^2 - 4m - 12 = 0$

$(m-6)(m+2) = 0$

$m = 6, -2$ both answers are acceptable

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SECTION A-Multiple-choice

Question 1 **Answer D.**

$\forall n \in Z$, let p: "n^2 is even" and q: "n is even", the given statement is $p \to q$

The contrapositive is $\neg q \to \neg p$ that is "if n is odd then n^2 is odd.

Question 2 **Answer C.**

	f	t_1	t_2	n
	0	2	3	3
once	8	2	8	
twice	18	2	18	
third time	38	2	38	

Question 3 **Answer E.**

line (1) $\underset{\sim}{r}(t)=2\underset{\sim}{i}+3\underset{\sim}{j}+t\left(\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}\right)$ has direction $\underset{\sim}{v}_1=\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}$

line (2) $\underset{\sim}{r}(t)=3\underset{\sim}{i}+\underset{\sim}{j}-2\underset{\sim}{k}+t\left(2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}\right)$ has direction $\underset{\sim}{v}_2=2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k}$

$$\underset{\sim}{v}_1\times\underset{\sim}{v}_2=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 1 & 2 & -1\\ 2 & 1 & -1\end{vmatrix}$$

Question 4 **Answer A.**

$A(5,-6,4),\ B(-3,-1,-10)$

the direction of the line $\underset{\sim}{v}=\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}=8\underset{\sim}{i}-5\underset{\sim}{j}+14\underset{\sim}{k}$,

the equation of the line through *B*, is $\underset{\sim}{r}=-3\underset{\sim}{i}-\underset{\sim}{j}-10\underset{\sim}{k}+t\left(8\underset{\sim}{i}-5\underset{\sim}{j}+14\underset{\sim}{k}\right)$

Question 5 **Answer B.**

normal to the plane $\underset{\sim}{n}=\underset{\sim}{i}-\underset{\sim}{j}+3\underset{\sim}{k}$, through the point $(3,2,-4)$

$1(x-3)-1(y-2)+3(z+4)=0$

$x-3-y+2+3z+12=0$

$-x+y-3z=11$

Question 6 **Answer D.**

The distance of a plane $ax+by+cz=d$ from the origin is $\dfrac{d}{\sqrt{a^2+b^2+c^2}}$

plane (1) $5x-4y-12z=10$ distance from the origin $\dfrac{10}{\sqrt{25+16+144}}=\dfrac{10}{\sqrt{185}}$

plane (2) $-15x+12y+36z=20$ distance from the origin $\dfrac{20}{\sqrt{225+144+1296}}=\dfrac{20}{3\sqrt{185}}$

the planes are parallel, the shortest distance is the distance of each to the origin,
but their normals are pointing in the opposite directions, so the planes lie on opposite sides of the origin, so the distance between them is $\dfrac{10}{\sqrt{185}}+\dfrac{20}{3\sqrt{185}}=\dfrac{50}{3\sqrt{185}}$

Question 7 **Answer A.**

$X \sim N(20,4)$

$H_0:\ \mu=20$

$H_1:\ \mu<20$ one sided

$\Pr\left(X\le C^*\right)=0.05,\quad n=16,\quad C^*=19.2$

$\beta=\Pr\left(\bar{X}>C^*\mid\mu=18.5\right)=0.08075\quad\approx\ 8\%$

SECTION B

Question 1

a. $|z| = \left|z - 2\text{cis}\left(\frac{\pi}{4}\right)\right| = \left|z - \left(\sqrt{2} + \sqrt{2}i\right)\right|$ let $z = x + yi$

$|x + yi| = \left|\left(x - \sqrt{2}\right) + \left(y - \sqrt{2}\right)i\right|$

$\sqrt{x^2 + y^2} = \sqrt{\left(x - \sqrt{2}\right)^2 + \left(y - \sqrt{2}\right)^2}$

$x^2 + y^2 = x^2 - 2\sqrt{2}x + 2 + y^2 - 2\sqrt{2}y + 2$

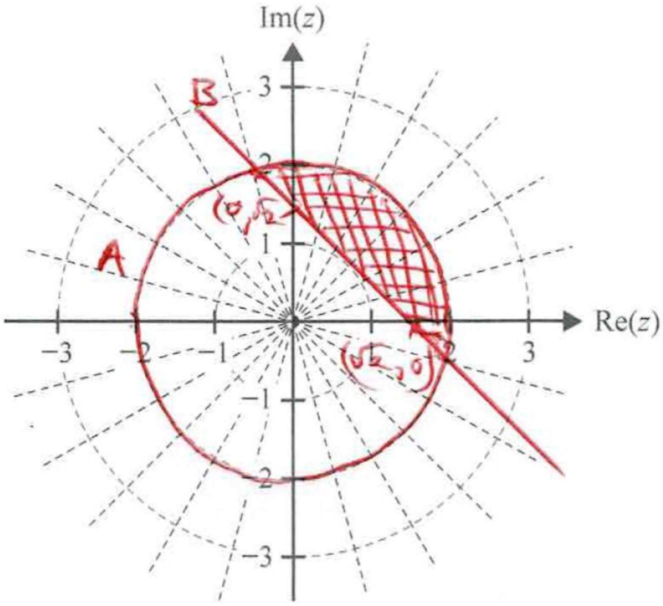
$2\sqrt{2}y = 4 - 2\sqrt{2}x$

$y = \sqrt{2} - x$

b. $z\,\overline{z} = (x + yi)(x - yi) = x^2 - y^2 i^2 = x^2 + y^2 = 4$

circle centre at the origin and radius 2.

c.



d. The required region is the area of a sector, with radius $r = 2$ and angle $\theta = \frac{\pi}{2} + 2\times\frac{\pi}{12} = \frac{2\pi}{3}$, area is $A = \frac{r^2}{2}\left(\theta - \sin(\theta)\right)$

$A = \frac{2^2}{2}\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right) = \frac{4\pi - 3\sqrt{3}}{3}$

e. $w = z^3$

$z_1 = 2\text{cis}\left(-\frac{\pi}{12}\right)$, $z_2 = 2\text{cis}\left(\frac{7\pi}{12}\right)$, $z_3 = 2\text{cis}\left(-\frac{3\pi}{4}\right)$ as these three roots are equally spaced around the circle (they do not occur in conjugate pairs)

$w = z_1^{\,3} = \left(2\text{cis}\left(-\frac{\pi}{12}\right)\right)^3 = 4\sqrt{2} - 4\sqrt{2}i$

$z:=x+y\cdot i$	$x+y\cdot i$
$u:=2\cdot e^{\frac{\pi\cdot i}{4}}$	$\sqrt{2}+\sqrt{2}\cdot i$
$\lvert z\rvert=\lvert z-u\rvert$	$\sqrt{x^2+y^2}=\sqrt{x^2-2\cdot\sqrt{2}\cdot x+y^2-2\cdot\sqrt{2}\cdot y+4}$
$\left(\sqrt{x^2+y^2}=\sqrt{x^2-2\cdot\sqrt{2}\cdot x+y^2-2\cdot\sqrt{2}\cdot y+4}\right)^2$	$x^2+y^2=x^2-2\cdot\sqrt{2}\cdot x+y^2-2\cdot\sqrt{2}\cdot y+4$
$\text{solve}\left(x^2+y^2=x^2-2\cdot\sqrt{2}\cdot x+y^2-2\cdot\sqrt{2}\cdot y+4,y\right)$	$y=\sqrt{2}-x$
$z\cdot\text{conj}(z)=4$	$x^2+y^2=4$
$\text{solve}\left(x^2+y^2=4 \text{ and } y=\sqrt{2}-x,\{x,y\}\right)$	$x=\frac{-(\sqrt{3}-1)\cdot\sqrt{2}}{2} \text{ and } y=\frac{(\sqrt{3}+1)\cdot\sqrt{2}}{2} \text{ or } x=\frac{(\sqrt{3}+1)\cdot\sqrt{2}}{2} \text{ and } y$▸
$2\cdot e^{\frac{-\pi\cdot i}{12}}$	$\frac{(\sqrt{3}+1)\cdot\sqrt{2}}{2}-\frac{(\sqrt{3}-1)\cdot\sqrt{2}}{2}\cdot i$
$\left(2\cdot e^{\frac{-\pi\cdot i}{12}}\right)^3$	$4\cdot\sqrt{2}-4\cdot\sqrt{2}\cdot i$
$x1:=\frac{-(\sqrt{3}-1)\cdot\sqrt{2}}{2}$	$\frac{-(\sqrt{3}-1)\cdot\sqrt{2}}{2}$
$y1:=\frac{(\sqrt{3}+1)\cdot\sqrt{2}}{2}$	$\frac{(\sqrt{3}+1)\cdot\sqrt{2}}{2}$
$x2:=\frac{(\sqrt{3}+1)\cdot\sqrt{2}}{2}$	$\frac{(\sqrt{3}+1)\cdot\sqrt{2}}{2}$
$y2:=\frac{-(\sqrt{3}-1)\cdot\sqrt{2}}{2}$	$\frac{-(\sqrt{3}-1)\cdot\sqrt{2}}{2}$
$(x1+y1\cdot i)^3$	$4\cdot\sqrt{2}-4\cdot\sqrt{2}\cdot i$
$(x2+y2\cdot i)^3$	$4\cdot\sqrt{2}-4\cdot\sqrt{2}\cdot i$

Question 2 $\frac{dP}{dt} = rP\left(1-\frac{P}{30000}\right), \quad P(0)=500$

a. $\frac{dP}{dt} = rP\left(\frac{30000-P}{30000}\right) = \frac{rP(300000-P)}{30000}$

inverting both sides $\frac{dt}{dP} = \frac{30000}{rP(30000-P)}$ now by partial fractions

$$\frac{30000}{P(30000-P)} = \frac{A}{P} + \frac{B}{30000-P} = \frac{A(30000-P)+BP}{P(30000-P)} = \frac{30000A + P(B-A)}{P(30000-P)}$$

equating $(1)\, B-A=0, \quad (2)\, 30000A = 30000, \quad \Rightarrow A=B=1$

$$t = \frac{1}{r}\int\left(\frac{1}{P}+\frac{1}{30000-P}\right)dP$$

$$rt = \log_e(|P|) - \log_e(|30000-P|) + c$$

but since $500 \le P < 30000$ the moduli are not needed.

$rt = \log_e\left(\frac{P}{30000-P}\right) + c$ using $t=0,\ P=500$ to find c

$$0 = \log_e\left(\frac{500}{30000-500}\right) + c \quad \Rightarrow c = -\log_e\left(\frac{1}{59}\right) = \log_e(59)$$

$$rt = \log_e\left(\frac{P}{30000-P}\right) + \log_e(59) = \log_e\left(\frac{59P}{30000-P}\right)$$

$$e^{rt} = \frac{59P}{30000-P}, \qquad \frac{30000-P}{59P} = e^{-rt}$$

$$30000 - P = 59Pe^{-rt}$$

$$30000 = P + 59Pe^{-rt} = P\left(1+59e^{-rt}\right)$$

$$P = P(t) = \frac{30000}{1+59e^{-rt}}$$

b. $P(10) = 1930, \quad r = \frac{1}{10}\log_e\left(\frac{11387}{2807}\right) \approx 0.14$

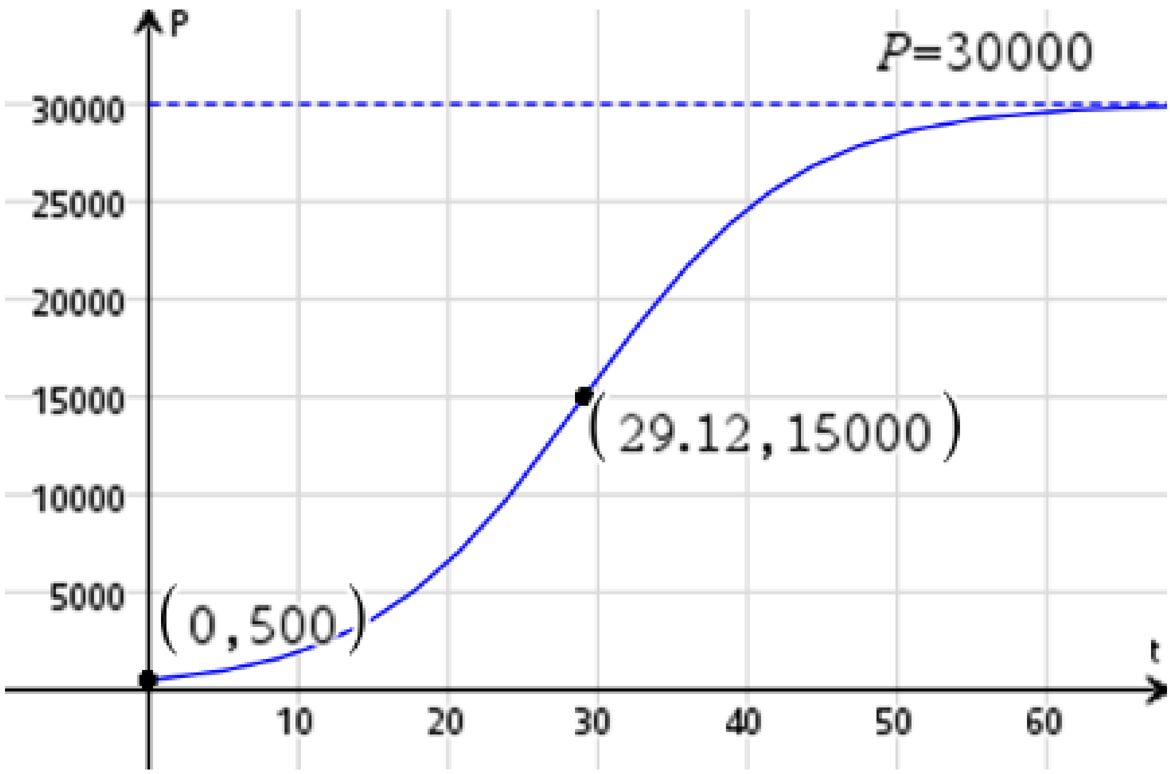
c. when $t=0,\ P=500,\ \frac{dP}{dt} = 0.14\times 500\left(1-\frac{500}{30000}\right) = 68.9$

d. when $P=10000, \quad t=?$ solving $10000 = \frac{30000}{1+59e^{-0.14t}} \quad \Rightarrow t = 24.2$

e. $\frac{dP}{dt} = r\left(\frac{30000P - P^2}{30000}\right)$ now the faster growth rate at the point of inflexion which

occurs when $\frac{d^2P}{dt^2} = r\left(\frac{30000-2P}{30000}\right)\frac{dP}{dt} = 0, \quad P = 15000$

$P = 15000, \quad t=?$ solving $15000 = \frac{30000}{1+59e^{-0.14t}} \quad \Rightarrow t = 29.12$



deSolve$\left(p'=r\cdot p\cdot\left(1-\frac{p}{30000}\right) \text{ and } p(0)=500,t,p\right)$

$$p=\frac{30000\cdot e^{r\cdot t}}{e^{r\cdot t}+59}$$

Define $f1(x)=\dfrac{30000}{1+59\cdot e^{-r\cdot x}}$ *Done*

solve$(f1(10)=1930,r)$

$$r=\frac{\ln(11387)}{10}-\frac{\ln(2807)}{10}$$

$r:=\dfrac{\ln(11387)}{10}-\dfrac{\ln(2807)}{10}$ 0.140

Define $f1(x)=\dfrac{30000}{1+59\cdot e^{-r\cdot x}}$ *Done*

$\dfrac{d}{dx}(f1(x))|x=0$ 68.851

solve$(f1(x)=15000,x)$ $x=29.118$

Question 3

a. plane (1) $x=1+2s+3t,\quad y=-2-s-2t,\quad z=2-s+t$

vector equation (1) $\underset{\sim}{r}(s,t)=\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}+s\left(2\underset{\sim}{i}-\underset{\sim}{j}-\underset{\sim}{k}\right)+t\left(3\underset{\sim}{i}-2\underset{\sim}{j}+\underset{\sim}{k}\right)$

b. $\underset{\sim}{n}=\begin{vmatrix}\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k}\\ 2 & -1 & -1\\ 3 & -2 & 1\end{vmatrix}=-3\underset{\sim}{i}-5\underset{\sim}{j}-\underset{\sim}{k}$

[2 -1 -1]→a	[2 -1 -1]
b:=[3 -2 1]	[3 -2 1]
crossP(a,b)	[-3 -5 -1]

the equation of plane through $P_0(1,-2,2)$ is

$3(x-1)+5(y+2)+1(z-2)=0$

$3x-3+5y+10+z-2=0$

$3x+5y+z=-5$

c. the second plane contains the point $P(1,0,3)$, its equation is

$3(x-1)+5(y-0)+1(z-3)=0$

$3x-3+5y+z-3=0$

$3x+5y+z=6$

d.i. The distance of a plane $ax+by+cz=d$ from the origin is $\dfrac{d}{\sqrt{a^2+b^2+c^2}}$

plane (1) $3x+5y+z=-5$ distance from the origin $\dfrac{5}{\sqrt{9+25+1}}=\dfrac{5}{\sqrt{35}}$

plane (2) $3x+5y+z=6$ distance from the origin $\dfrac{6}{\sqrt{35}}$

the planes are parallel, the shortest distance is the distance of each to the origin, but their normals are pointing in the opposite directions, so the planes lie on opposite sides of the origin, so the distance between them is $\dfrac{5}{\sqrt{35}}+\dfrac{6}{\sqrt{35}}=\dfrac{11}{\sqrt{35}}=\dfrac{11\sqrt{35}}{35}$

ii. the point $P(1,0,3)$ is on plane (2)

line QP: $x=1+3t,\ y=5t,\ z=3+t$ this intersects plane (1) $3x+5y+z=-5$

$3(1+3t)+25t+3+t=-5$

$3+9t+25t+3+t=-5,\qquad 35t=-11,\quad t=-\dfrac{11}{35}$

substitute into line QP: $x=1-\dfrac{33}{35},\ y=-\dfrac{55}{35},\ z=3-\dfrac{11}{35},\quad M\left(\dfrac{2}{35},-\dfrac{11}{7},\dfrac{94}{35}\right)$

Now M is the midpoint of QP, so coordinates of Q are

$\dfrac{x_Q+1}{2}=\dfrac{2}{35},\ \dfrac{y_Q+0}{2}=-\dfrac{11}{7},\ \dfrac{z_Q+3}{2}=\dfrac{94}{35}$

$x_Q=-\dfrac{31}{35},\ y_Q=-\dfrac{22}{7},\ z_Q=\dfrac{83}{35}\qquad Q\left(-\dfrac{31}{35},-\dfrac{22}{7},\dfrac{83}{35}\right)$

Question 4

a. line (1) $\underset{\sim}{r}(t)=4\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}+t\left(-\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}\right)$ has direction $\underset{\sim}{v}_1=-\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}$

line (2) $\underset{\sim}{r}(s)=5\underset{\sim}{i}+4\underset{\sim}{j}-2\underset{\sim}{k}+s\left(-\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}\right)$ has direction $\underset{\sim}{v}_2=\underset{\sim}{b}=-\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}$

consider the point $P_T(4,2,1)$ on line (1)

and the point $P_S(5,4,-2)$ on line (2)

let $\underset{\sim}{a}=\overrightarrow{P_TP_s}=\overrightarrow{OP_S}-\overrightarrow{OP_T}=\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}$. The shortest distance between these parallel lines is the magnitude of the vector resolute of the vector $\underset{\sim}{a}$ perpendicular to the vector $\underset{\sim}{b}$.

$[1\ \ 2\ \ -3]\to a$	$[1\ \ 2\ \ -3]$
$[-1\ \ 1\ \ 3]\to b$	$[-1\ \ 1\ \ 3]$
a−dotP(a,unitV(b))· unitV(b)	$\left[\frac{3}{11}\ \ \frac{30}{11}\ \ \frac{-9}{11}\right]$
$\text{norm}\left(\left[\frac{3}{11}\ \ \frac{30}{11}\ \ \frac{-9}{11}\right]\right)$	$\frac{3\cdot\sqrt{110}}{11}$

$$\left|\underset{\sim}{a}-\left(\underset{\sim}{a}.\hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}\right| = \left|\left(\underset{\sim}{i}+2\underset{\sim}{j}-3\underset{\sim}{k}\right)-\left(\frac{8}{11}\underset{\sim}{i}-\frac{8}{11}\underset{\sim}{j}-\frac{24}{11}\underset{\sim}{k}\right)\right|=\left|\frac{3}{11}\underset{\sim}{i}+\frac{30}{11}\underset{\sim}{j}-\frac{9}{11}\underset{\sim}{k}\right|=\frac{3\sqrt{110}}{11}$$

b. line (1) $\underset{\sim}{r}(t)=\underset{\sim}{i}-3\underset{\sim}{j}+6\underset{\sim}{k}+t\left(3\underset{\sim}{i}+5\underset{\sim}{j}-a\underset{\sim}{k}\right)$, line (2) $\underset{\sim}{r}(s)=-6\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}+s\left(4\underset{\sim}{i}-10\underset{\sim}{j}+6\underset{\sim}{k}\right)$

in parametric form line (1) $x=1+3t,\ y=-3+5t,\ z=6-at$

and line (2) $x=-6+4s,\ y=2-10s,\ z=1+6s$ equating

$x:\ 1+3t=-6+4s \qquad y:\ -3+5t=2-10s \qquad z:=6-at=1+6s$

$x:\ (3)\ 4s-3t=7 \qquad\qquad 10s+5t=5$ or $2s+t=1\ \ \times 2$

$(4)\ 4s+2t=2$ subtracting $(4)-(3)$ gives

$5t=-5,\quad t=-1,\ \ s=1$ into $z:\ \ 6+a=7,\quad a=1$

The point of intersection is found by substituting $s=1$ or $t=-1$ into the parametric forms for the lines and gives $(-2,-8,7)$.

c. line $\underset{\sim}{r}(t)=\underset{\sim}{i}+\underset{\sim}{j}-5\underset{\sim}{k}+t\left(4\underset{\sim}{i}+b\underset{\sim}{j}+2\underset{\sim}{k}\right)$ is parallel to the plane $2x-3y-z=2$

The direction of the line is $\underset{\sim}{v}=4\underset{\sim}{i}+b\underset{\sim}{j}+2\underset{\sim}{k}$ and normal to the plane is $\underset{\sim}{n}=2\underset{\sim}{i}-3\underset{\sim}{j}-\underset{\sim}{k}$.

Now these are perpendicular, so that $\underset{\sim}{v}.\underset{\sim}{n}=8-3b-2=0 \quad 3b=6, \quad b=2$

Now the line is parallel to the plane, so the shortest distance between the line and the plane is the perpendicular distance from any point on the line to the plane.

Let $A(1,1,-5)$ be the point on the line when $t=0$.

Let $P_0(1,0,0)$ be a point on the plane, $\overrightarrow{AP_0}=\overrightarrow{OP_0}-\overrightarrow{OA}=-\underset{\sim}{j}+5\underset{\sim}{k}$, now

$\left|\underset{\sim}{n}\right|=\sqrt{4+9+1}=\sqrt{14}$ so a unit vector normal to the plane is $\hat{\underset{\sim}{n}}=\frac{1}{\sqrt{14}}\left(2\underset{\sim}{i}-3\underset{\sim}{j}-\underset{\sim}{k}\right)$.

as the dot product could be negative so we need to take the magnitude,

The required distance is $d=\left|\hat{\underset{\sim}{n}}.\overrightarrow{AP_0}\right|=\left|\frac{1}{\sqrt{14}}\left(2\underset{\sim}{i}-3\underset{\sim}{j}-\underset{\sim}{k}\right).\left(-\underset{\sim}{j}+5\underset{\sim}{k}\right)\right|=\left|\frac{3-5}{\sqrt{14}}\right|=\frac{2}{\sqrt{14}}=\frac{\sqrt{14}}{7}$

Question 5

a.i. $A(1,0,2),\ B(2,3,0),\ C(1,2,1)$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}, \quad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{AO} = 2\underset{\sim}{j} - \underset{\sim}{k}$

$$\underset{\sim}{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 3 & -2 \\ 0 & 2 & -1 \end{vmatrix} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$$

ii. the plane through any of the points, use A is

$1(x-1)+1(y-0)+2(z-2)=0$

$x-1+y+2z-4=0$

$\Pi_1: \quad x+y+2z=5$

b.i. plane $\Pi_2: \ x-y-z=0$, the line L is the intersection of the two planes.

Where they cross the y-z plane, when $x=0$, so $(1)\ \ y+2z=5 \quad (2)\ -y-z=0$

solving gives $y=-5,\ z=5,\ P(0,-5,5)$

ii. normal to first plane $\underset{\sim}{n}_1 = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ normal to second plane $\underset{\sim}{n}_2 = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$, $|\underset{\sim}{n}_2| = \sqrt{3}$

direction of the line is perpendicular to both normals

$$\underset{\sim}{v} = \underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 1 & 2 \\ 1 & -1 & -1 \end{vmatrix} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$$ so line through $P(0,-5,5)$ is

$\underset{\sim}{r}(t) = \left(-5\underset{\sim}{j} + 5\underset{\sim}{k}\right) + t\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right), \ t \in R$

iii. distance of the point $A(1,0,2)$ to the plane $\Pi_2: \ x-y-z=0$,

let $P_0(1,0,1)$ be a point on the plane is, so $\overrightarrow{AP_0} = \overrightarrow{OP_0} - \overrightarrow{OA} = -\underset{\sim}{k}$

as the dot product could be negative so we need to take the magnitude,

to find the distance $d = \left|\hat{\underset{\sim}{n}}_2 . \overrightarrow{AP_0}\right| = \left|\frac{1}{\sqrt{3}}\left(\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}\right).\left(-\underset{\sim}{k}\right).\right| = \left|\frac{1}{\sqrt{3}}\right| = \frac{\sqrt{3}}{3}$

iv. the distance of the point $A(1,0,2)$ to the line L now a point on L is $P(0,-5,5)$

and the direction of the line $\underset{\sim}{v} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$, $\quad |\underset{\sim}{v}| = \sqrt{14}, \quad \overrightarrow{PA} = \underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}$

$$D = \left|\overrightarrow{PA} \times \hat{\underset{\sim}{v}}\right| = \left|\left(\underset{\sim}{i} + 5\underset{\sim}{j} - 3\underset{\sim}{k}\right) \times \frac{1}{\sqrt{14}}\left(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)\right|$$

$$= \frac{1}{\sqrt{14}}\left|-\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right| = \sqrt{\frac{6}{14}} = \sqrt{\frac{3}{7}}$$

$$= \frac{\sqrt{21}}{7}$$

$\begin{bmatrix}1 & 0 & 2\end{bmatrix} \to a$	$\begin{bmatrix}1 & 0 & 2\end{bmatrix}$
$\begin{bmatrix}2 & 3 & 0\end{bmatrix} \to b$	$\begin{bmatrix}2 & 3 & 0\end{bmatrix}$
$\begin{bmatrix}1 & 2 & 1\end{bmatrix} \to c$	$\begin{bmatrix}1 & 2 & 1\end{bmatrix}$
$n1{:=}\text{crossP}(b-a,c-a)$	$\begin{bmatrix}1 & 1 & 2\end{bmatrix}$
$\begin{bmatrix}x & y & z\end{bmatrix} \to r$	$\begin{bmatrix}x & y & z\end{bmatrix}$
$\text{dotP}(r-a,n1)$	$x+y+2\cdot z-5$
$n2{:=}\begin{bmatrix}1 & -1 & -1\end{bmatrix}$	$\begin{bmatrix}1 & -1 & -1\end{bmatrix}$
$\text{crossP}(n1,n2)$	$\begin{bmatrix}1 & 3 & -2\end{bmatrix}$

$p{:=}\begin{bmatrix}1 & 0 & 1\end{bmatrix}$	$\begin{bmatrix}1 & 0 & 1\end{bmatrix}$
$\text{dotP}(a-p,\text{unitV}(n2))$	$\dfrac{-\sqrt{3}}{3}$

$\begin{bmatrix}0 & -5 & 5\end{bmatrix} \to p$	$\begin{bmatrix}0 & -5 & 5\end{bmatrix}$
$a-p$	$\begin{bmatrix}1 & 5 & -3\end{bmatrix}$
$\begin{bmatrix}1 & 3 & -2\end{bmatrix} \to v$	$\begin{bmatrix}1 & 3 & -2\end{bmatrix}$
$\text{crossP}(a-p,v)$	$\begin{bmatrix}-1 & -1 & -2\end{bmatrix}$
$\text{norm}(\begin{bmatrix}-1 & -1 & -2\end{bmatrix})$	$\sqrt{6}$
$\text{norm}(v)$	$\sqrt{14}$
$\dfrac{\sqrt{6}}{\sqrt{14}}$	$\dfrac{\sqrt{21}}{7}$

Question 6

$$\underset{\sim}{r}_S(t) = 23t\underset{\sim}{i} + 5t\underset{\sim}{j} + \left(4\sqrt{2}\sin\left(\frac{\pi t}{2}\right) + 4\sqrt{2}\right)\underset{\sim}{k}, \quad t \ge 0$$

a. when it **first** lands $\underset{\sim}{r}_S(t).\underset{\sim}{k} = 0 \quad 4\sqrt{2}\sin\left(\frac{\pi t}{2}\right) + 4\sqrt{2} = 0$

$\sin\left(\frac{\pi t}{2}\right) = -1, \quad \frac{\pi t}{2} = \frac{3\pi}{2}, \quad t = 3$

b. the distance of the sparrow from O when it lands

$\underset{\sim}{r}_S(3) = 69\underset{\sim}{i} + 15\underset{\sim}{j}$

$\left|\underset{\sim}{r}_S(3)\right| = \sqrt{69^2 + 15^2} = 70.6$ cm

c. $\underset{\sim}{v}_S(t) = 23\underset{\sim}{i} + 5\underset{\sim}{j} + \left(2\pi\sqrt{2}\cos\left(\frac{\pi t}{2}\right)\right)\underset{\sim}{k}$

$\left|\underset{\sim}{v}_S(t)\right| = \sqrt{23^2 + 5^2 + \left(8\pi^2\cos^2\left(\frac{\pi t}{2}\right)\right)}$

the maximum speed occurs when coming down, since $0 < t < 3$

or at $\cos^2\left(\frac{\pi t}{2}\right) = 1$, first time when $t = 2$

$\underset{\sim}{v}_S(2) = 23\underset{\sim}{i} + 5\underset{\sim}{j} - 2\pi\sqrt{2}\underset{\sim}{k}$

$\left|\underset{\sim}{v}_S(2)\right|_{\max} = \sqrt{23^2 + 5^2 + 8\pi^2} = 25.2\,\text{cm/s}$

d. $\underset{\sim}{v}_M(t) = 6\underset{\sim}{i} + \underset{\sim}{j} + \left(\frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{k}, \quad t \ge 0$

$\underset{\sim}{r}_M(t) = \int\left(6\underset{\sim}{i} + \underset{\sim}{j} + \left(\frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{k}\right)dt = 6t\underset{\sim}{i} + t\underset{\sim}{j} + \sin\left(\frac{\pi t}{6}\right)\underset{\sim}{k} + \underset{\sim}{c}$

$\underset{\sim}{r}_M(0) = 10\underset{\sim}{i} + 4\underset{\sim}{j} + 4\sqrt{2}\underset{\sim}{k} = \underset{\sim}{c}$

$\underset{\sim}{r}_M(t) = (6t + 10)\underset{\sim}{i} + (t + 4)\underset{\sim}{j} + \left(4\sqrt{2} + \sin\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{k}$

e. $\underset{\sim}{r}_S(t) = \underset{\sim}{r}_M(s)$

$\underset{\sim}{i}: (1) \quad 23t = 6s + 10$

$\underset{\sim}{j}: (2) \quad 5t = s + 4$

$\underset{\sim}{k}: (3) \quad 4\sqrt{2}\sin\left(\frac{\pi t}{2}\right) + 4\sqrt{2} = \sin\left(\frac{\pi s}{6}\right) + 4\sqrt{2}$

solving all (1) and (2) and verifying it checks in (3) yes it does, gives $s = 6, \; t = 2$

so the sparrow and the miner are at the same position but at different times,

so $\underset{\sim}{r}_S(2) = \underset{\sim}{r}_M(6) = 46\underset{\sim}{i} + 10\underset{\sim}{j} + 4\sqrt{2}\underset{\sim}{k}$ the point is $\left(46, 10, 4\sqrt{2}\right)$

$\left[23\cdot t \quad 5\cdot t \quad 4\cdot\sqrt{2}\cdot\sin\left(\frac{\pi\cdot t}{2}\right)+4\cdot\sqrt{2}\right]\rightarrow rs(t)$	*Done*
$k:=\left[0 \quad 0 \quad 1\right]$	$\left[0 \quad 0 \quad 1\right]$
$\text{solve}\left(\text{dotP}\left(rs(t),k\right)=0,t\right)\mid 0<t<5$	$t=3$
$rs(3)$	$\left[69 \quad 15 \quad 0\right]$
$\text{norm}\left(rs(3)\right)$	70.6116
$\text{Define } vs(t)=\frac{d}{dt}\left(rs(t)\right)$	*Done*
$\text{solve}\left(\left(\cos\left(\frac{\pi\cdot t}{2}\right)\right)^2=1,t\right)\mid 0<t<3$	$t=2$
$vs(2)$	$\left[23 \quad 5 \quad -2\cdot\pi\cdot\sqrt{2}\right]$
$\text{norm}\left(vs(2)\right)$	25.1586
$\text{Define } vs(t)=\frac{d}{dt}\left(rs(t)\right)$	*Done*
$\text{Define } rm(t)=\int_0^t vm(u)\,du+\left[10 \quad 4 \quad 4\cdot\sqrt{2}\right]$	*Done*
$rm(t)$	$\left[6\cdot t+10 \quad t+4 \quad \sin\left(\frac{\pi\cdot t}{6}\right)+4\cdot\sqrt{2}\right]$
$\text{solve}\left(rs(t)=rm(s),\{s,t\}\right)$	$s=6 \text{ and } t=2$
$rs(2)$	$\left[46 \quad 10 \quad 4\cdot\sqrt{2}\right]$
$rm(6)$	$\left[46 \quad 10 \quad 4\cdot\sqrt{2}\right]$

End of detailed answers for the
2023 Kilbaha VCE Specialist Mathematics Sample questions Solutions

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