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Specialist Mathematics

Written Examination 2

2024 Insight Publications Year 12 Trial Examination

Worked Solutions

This book presents:

- answers and worked solutions
- explanatory notes
- mark allocations
- tips.

Section A – Multiple choice questions

Question	Answer
1	C
2	B
3	C
4	D
5	A
6	C
7	B
8	C
9	D
10	D
11	B
12	D
13	A
14	D
15	B
16	D
17	D
18	D
19	B
20	A

Question 1

Answer: C

Worked solution

The negation of a statement, *S*, is the opposite statement; that is, it is equivalent to *S* being false. The statement given is a 'for all' statement. Its general form is 'For all x, if A is the case, then B is the case'. The negation of a 'for all' statement is 'There exists an x such that A is the case but B is not the case'. This matches only option C.

Question 2

Answer: B

Worked solution

When $k \neq 0$, the graph of $y = \frac{1}{f(x)}$ will have three linear asymptotes, a stationary point and no points of inflection. When $k = 0$, the graph of $y = \frac{1}{f(x)}$ becomes $y = \frac{1}{2x}$, which will have two linear asymptotes, no stationary points and no points of inflection. Therefore, option B is the correct answer.

Tip

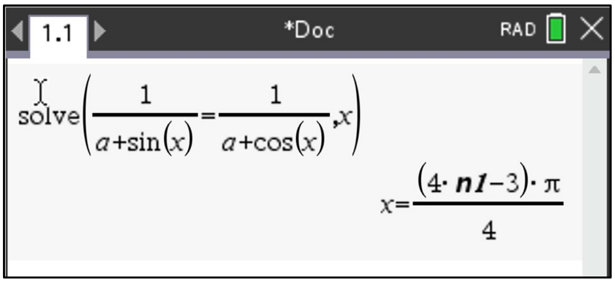
- *Using a slider for the pronumeral, in this case k, on a graph on a CAS calculator can be useful to analyse how the key features of the graph of a function change as the pronumeral changes.*

Question 3

Answer: C

Worked solution

Using the solve function on a calculator yields:



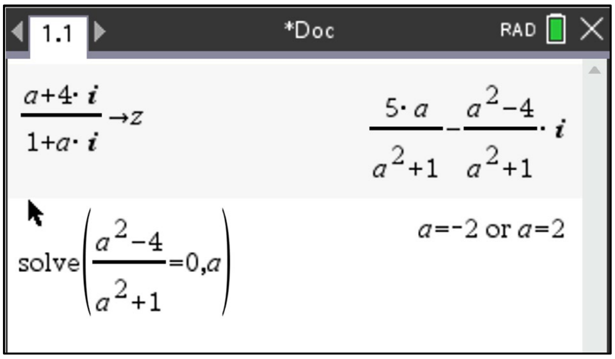
For $n \in \mathbb{Z}$, the general solution $x = \dfrac{(4n-3)\pi}{4}$ generates the same set of solutions as $x = \dfrac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$; therefore option C is the correct response.

Question 4

Answer: D

Worked solution

The denominator of z can be rationalised by multiplying it by $\dfrac{1-ai}{1-ai}$. This yields $z = \dfrac{5a}{a^2+1} - \dfrac{a^2-4}{a^2+1}i$, and if $z \in \mathbb{R}$ then $\operatorname{Im}(z) = 0$. Solving $\dfrac{a^2-4}{a^2+1} = 0$ for a gives $a = \pm 2$.



Tip

- *Defining a complex number on a CAS calculator will automatically give the number in rectangular form, which is very helpful when it is needed to analyse the real or imaginary part of the complex number.*

Question 5

Answer: A

Worked solution

1.1 *Doc RAD

$a+b\cdot i \to w \qquad a+b\cdot i$

$1+\sqrt{3}\cdot i \to z \qquad 1+\sqrt{3}\cdot i$

$\text{solve}\left(\frac{z}{w}=2+2\cdot i, a, b\right)$

$$a=\frac{\sqrt{3}+1}{4} \text{ and } b=\frac{\sqrt{3}-1}{4}$$

$$\left(\frac{\sqrt{3}+1}{4}+\frac{\sqrt{3}-1}{4}\cdot i\right) \blacktriangleright \text{Polar} \qquad e^{\frac{i\cdot\pi}{12}}\cdot\frac{\sqrt{2}}{2}$$

Alternatively, $\text{Arg}(w)$ can be found as shown below.

1.1 *Doc RAD

$$w=\frac{1+\sqrt{3}\cdot i}{2+2\cdot i} \qquad w=\frac{\sqrt{3}}{4}+\frac{1}{4}+\left(\frac{\sqrt{3}}{4}-\frac{1}{4}\right)\cdot i$$

$$\text{angle}\left(\frac{\sqrt{3}}{4}+\frac{1}{4}+\left(\frac{\sqrt{3}}{4}-\frac{1}{4}\right)\cdot i\right) \qquad \frac{\pi}{12}$$

Tip

- *It is helpful to be familiar with Euler's notation for complex numbers in polar form. Although not in the study design, a CAS calculator uses Euler's notation, so it is very useful to know that $re^{i\theta}$ and $r\text{cis}(\theta)$ both are equivalent to $r\cos(\theta)+r\sin(\theta)i$.*

Question 6

Answer: C

Worked solution

The function $g(x, y)$ is used to determine the gradient and calculate an approximation for the new y-value by moving along that gradient a horizontal distance of h. Option C gives the correct formula for determining the next successive y-coordinate using Euler's method.

Question 7

Answer: B

Worked solution

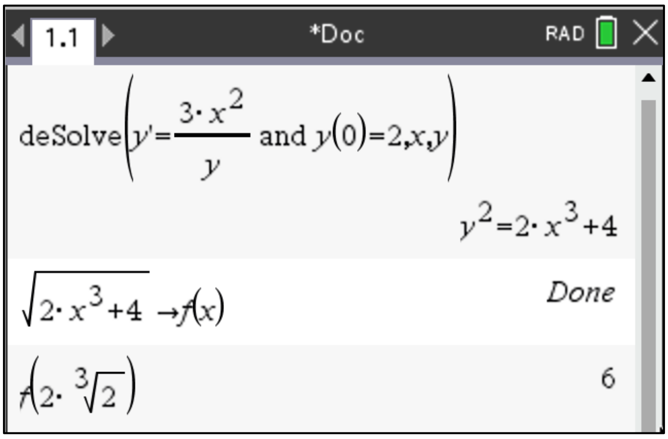
The x-value starts at 1 and increases by 0.2 for each loop, so its value after two loops will be 1.4.

Question 8

Answer: C

Worked solution

The differential equation can be easily solved, including the initial condition, using a calculator's differential equation solver. Note that when rearranging the solution in the form $y = f(x)$, the positive square root is used to satisfy the initial condition.



Tip

- *The differential equation solver on a CAS calculator can quickly solve first- and second-order differential equations with and without initial conditions. It is important to be familiar with the calculator syntax to input a differential equation and its initial conditions.*

Question 9

Answer: D

Worked solution

The differential equation can be solved using the differential equation solver on a calculator, with the substitutions of $T \Rightarrow y$ and $t \Rightarrow x$.

$\text{deSolve}(y'=-k\cdot(y-4) \text{ and } y(0)=20,x,y)$	$y=16\cdot e^{-k\cdot x}+4$
$\text{solve}(18=16\cdot e^{-k\cdot 3}+4,k)$	$k=\dfrac{-\ln\left(\frac{7}{8}\right)}{3}$

Alternatively, the solution can be obtained through integration and substitution.

$$\frac{dT}{dt} \propto -(T-4)$$

$$\frac{dT}{dt} = -k(T-4),\ k \in \mathbb{R}$$

$$\frac{dt}{dT} = -\frac{1}{k}\cdot\frac{1}{T-4}$$

$$t = -\frac{1}{k}\ln|T-4| + C,\ C \in \mathbb{R}$$

$$0 = -\frac{1}{k}\ln(16) + C$$

$$C = \frac{1}{k}\ln(16)$$

$$t = -\frac{1}{k}\ln(T-4) + \frac{1}{k}\ln(16)$$

$$t = \frac{1}{k}\ln\left(\frac{16}{T-4}\right)$$

$$e^{kt} = \frac{16}{T-4}$$

$$T(t) = 16e^{-kt} + 4$$

$$18 = 16e^{-3k} + 4$$

$$k = \frac{1}{3}\ln\left(\frac{8}{7}\right)$$

Question 10

Answer: D

Worked solution

The parametric equations that describe the position of the particle can be differentiated once with respect to time to obtain the following parametric equations for the x and y components of the velocity of the particle:

$v_x(t) = -a\sin(t)$ and $v_y(t) = b\cos(t)$, where $t \geq 0$ and a and b are positive real constants.

The speed of the particle is given by

$$s(t) = \sqrt{(v_x(t))^2 + (v_y(t))^2} = \sqrt{(-a\sin(t))^2 + (b\cos(t))^2} = \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}$$

Since $a > b$, $s(t) = \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}$ will be a maximum when $t = \frac{\pi}{2}$ and have a value of

$$s\left(\frac{\pi}{2}\right) = \sqrt{a^2\sin^2\left(\frac{\pi}{2}\right) + b^2\cos^2\left(\frac{\pi}{2}\right)} = \sqrt{a^2} = a.$$

Question 11

Answer: B

Worked solution

The total surface area of the solid of revolution will be the curved surface area plus the circular area of the 'top' of the solid.

The curved surface area can be found by evaluating the integral.

$$I = 2\pi\int_0^2 \left(\frac{1}{3}y^3\right)\sqrt{1+y^4}\,dy$$

$$= \frac{2\pi}{3}\int_0^2 y^3\sqrt{1+y^4}\,dy$$

This integral can be evaluated by making a change of the variables and terminals.

$$u = y^4 + 1$$

$$\frac{du}{dx} = 4y^3$$

$$y = 0 \Rightarrow u = 1$$

$$y = 2 \Rightarrow u = 17$$

$$I = \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}\,du = \frac{\pi}{6}\int_1^{17} \sqrt{u}\,du$$

When $y = 2$, $x = \frac{1}{3}(2)^3 = \frac{8}{3}$. Therefore the area of the circular top of the solid is $\pi\left(\frac{8}{3}\right)^2 = \frac{64\pi}{9}$.

The total surface area of the solid is $\frac{\pi}{6}\int_1^{17} \sqrt{u}\,du + \frac{64\pi}{9}$.

Tip

- *For solids of revolution, the integral formula supplied on the VCAA formula sheets gives only the curved surface area of the solid. If asked for the total surface area, the end of the solid must also be taken into consideration.*

Question 12

Answer: D

Worked solution

The differential equation solver on a CAS calculator can be used, giving

deSolve$(v'=-g-k\cdot v$ and $v(0)=b,t,v)$

$$v=\left(b+\frac{g}{k}\right)\cdot \mathbf{e}^{-k\cdot t}-\frac{g}{k}$$

Alternatively, by hand, the differential equation can be solved as follows.

$$\frac{dv}{dt}=-g-\alpha v$$

$$\frac{dt}{dv}=\frac{-1}{g+\alpha v}$$

$$t=\int\frac{-1}{g+\alpha v}\,dv$$

$$-t=\frac{1}{\alpha}\ln\left|g+\alpha v\right|+C$$

$$0=\frac{1}{\alpha}\ln\left(g+\alpha v_0\right)+C$$

$$C=-\frac{1}{\alpha}\ln\left(g+\alpha v_0\right)$$

$$-\alpha t=\ln\left(g+\alpha v\right)-\ln\left(g+\alpha v_0\right)$$

$$-\alpha t=\ln\left(\frac{g+\alpha v}{g+\alpha v_0}\right)$$

$$e^{-\alpha t}=\frac{g+\alpha v}{g+\alpha v_0}$$

$$e^{-\alpha t}(g+\alpha v_0)=g+\alpha v$$

$$v=\frac{e^{-\alpha t}(g+\alpha v_0)-g}{\alpha}$$

$$v=v_0\cdot e^{-\alpha t}+\frac{g}{\alpha}\left(e^{-\alpha t}-1\right)$$

Question 13

Answer: A

Worked solution

The average velocity of the particle is the total displacement divided by the total time. For the interval $0 \le t \le 4$, this will be the area under the curve from $0 \le t \le 2$ less the area under the curve from $2 \le t \le 4$, divided by 4.

One square of the graph represents a displacement of $\frac{1}{8}$ metres. An estimate for the average velocity is

Average velocity $= \dfrac{\frac{1}{8} \times (5\frac{1}{2} - 8\frac{1}{2})}{4} = \dfrac{-\frac{3}{8}}{4} = -\dfrac{3}{32}$ m s^{-1}

Tip

- *There is a difference between average speed and average velocity. Average speed is a scalar and is calculated by taking the total distance travelled divided by the total time. Remember: on a velocity–time graph the area under the curve gives the signed displacement of the object measured from the starting position.*

Question 14

Answer: D

Worked solution

If $\left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right)^2 = 1$, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = \pm 1$. This implies that either $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are a pair of parallel or anti-parallel unit vectors. This can be seen by considering the formula for the scalar product: $\underset{\sim}{a} \cdot \underset{\sim}{b} = \left|\underset{\sim}{a}\right|\left|\underset{\sim}{b}\right|\cos(\theta)$. Since $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{b}\right| = 1$, $\cos(\theta) = \pm 1$, which gives $\theta = 0°$ or $\theta = 180°$.

Option A is incorrect because vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ must pass through at least one common point to be collinear and there is not enough information to conclude that this must be the case.

Options B and C are both incorrect because the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ may be in the same or opposite directions.

Option D is correct because either $\underset{\sim}{a} = 1 \cdot \underset{\sim}{b}$ or $\underset{\sim}{a} = -1 \cdot \underset{\sim}{b}$, both of which satisfy the conditions for two vectors being linearly dependent.

Question 15

Answer: B

Worked solution

Using the geometrical definitions of the dot and cross products gives

$\dfrac{\left|\underset{\sim}{\text{c}} \times \underset{\sim}{\text{d}}\right|}{\underset{\sim}{\text{c}} \cdot \underset{\sim}{\text{d}}} = \dfrac{\left|\underset{\sim}{\text{c}}\right|\left|\underset{\sim}{\text{d}}\right| \sin(\theta)}{\left|\underset{\sim}{\text{c}}\right|\left|\underset{\sim}{\text{d}}\right| \cos(\theta)} = \tan(\theta)$, where θ is the angle between $\underset{\sim}{\text{c}}$ and $\underset{\sim}{\text{d}}$.

If $\cos(\theta) = -\dfrac{3}{7}$, then it follows that $\theta \in \left(\dfrac{\pi}{2}, \pi\right)$ and, using a right-angled triangle and Pythagoras, $\tan(\theta) = -\dfrac{\sqrt{40}}{3} = -\dfrac{2\sqrt{10}}{3}$.

Question 16

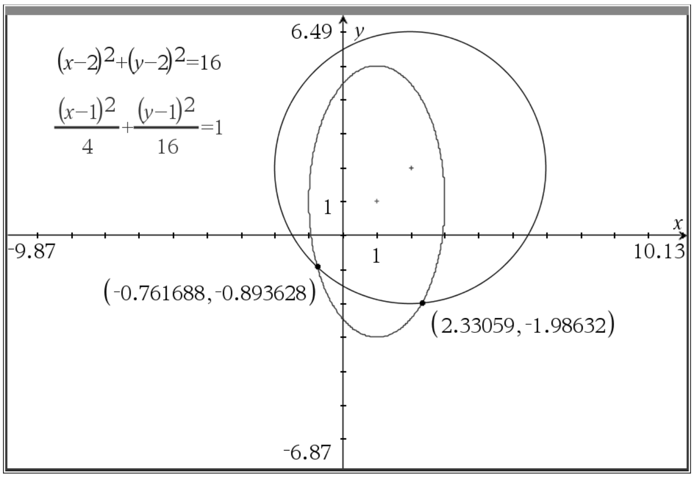
Answer: D

Worked solution

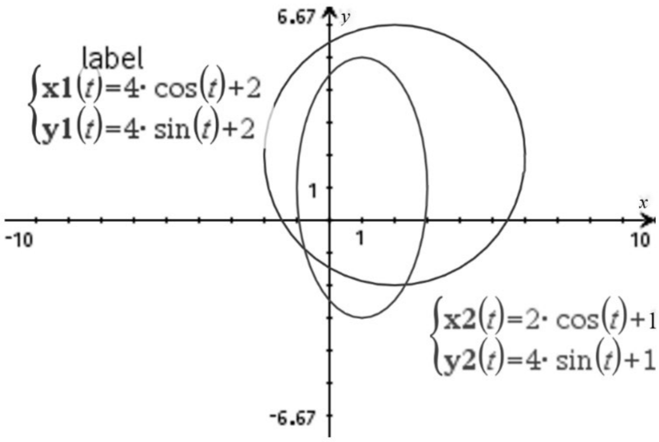
The Cartesian relation describing the path of ship A is a circle with the equation $(x-2)^2+(y-2)^2=16$.

The Cartesian relation describing the path of ship B is an ellipse with the equation $\frac{(x-1)^2}{4}+\frac{(y-1)^2}{16}=1$.

The points of intersection of the two paths can be found by graphing both Cartesian relations and analysing the graphs for the points of intersection.



Alternatively, the parametric equations can be graphed on a CAS calculator explicitly, as shown below.



Tip

- *The number of decimal places displayed on a graph page can generally be changed by going into the document settings on the graph page and changing the float value.*

Question 17

Answer: D

Worked solution

A vector equation can be set up and solved on a calculator, using the fact that the unit vector will be the vector scaled by the reciprocal of its magnitude.

$\begin{bmatrix} m & 3 & ^-1 \end{bmatrix} \rightarrow a$	$\begin{bmatrix} m & 3 & ^-1 \end{bmatrix}$
$\begin{bmatrix} 1 & n & 2 \end{bmatrix} \rightarrow b$	$\begin{bmatrix} 1 & n & 2 \end{bmatrix}$
$\text{solve}\left(\dfrac{\text{crossP}(a,b)}{\text{norm}(\text{crossP}(a,b))}=\sqrt{6}\cdot\begin{bmatrix} \frac{1}{15} & \frac{^-1}{6} & \frac{^-11}{30} \end{bmatrix},m,n\right)$	$m=2$ and $n={}^-4$

Tip

- *Many CAS calculators have a powerful set of predefined operations for vectors, including dot product and cross product. Learning how to input a vector as a matrix with one row (that is, a row vector) on a CAS and being familiar with what vector operations are available on the CAS will save you a lot of time during the exam.*

Question 18

Answer: D

Worked solution

The signed distance between Π_1 and the origin is $\dfrac{14}{\sqrt{2^2+3^2+6^2}}=2.$

The signed distance between Π_2 and the origin is $\dfrac{-21}{\sqrt{2^2+3^2+6^2}}=-3.$

Therefore, the distance between Π_1 and Π_2 is 5.

To be the same distance from Π_1 as Π_2, Π_3 must be 7 units from the origin.

As Π_3 is parallel to Π_1, its equation should be of the form $2x+3y+6z=49$.

If we multiply this equation by a factor of 3, we obtain $6x+9y+18z=147$.

Therefore, option D is correct.

Question 19

Answer: B

Worked solution

Since both X and Y, (and hence W and Z) are normal variables with a mean of zero and a standard deviation of $\sqrt{2}$, the difference $W-Z$ is a standard normal random variable. Therefore

$\Pr(W > Z) = \Pr(W - Z > 0) = \Pr((X+Y)-(X-Y)) > 0 = \Pr(2Y) > 0$

$\mathrm{E}(2Y) = 2\mathrm{E}(Y) = 0$

$\mathrm{Var}(2Y) = 2^2\,\mathrm{Var}(Y) = 4\times 2 = 8$

$\mathrm{sd}(2Y) = \sqrt{8} = 2\sqrt{2}$

Using a CAS calculator will give the required answer:

0.5

$\mathrm{normCdf}(0,\infty,0,2\cdot\sqrt{2})$ 0.5

Question 20

Answer: A

Worked solution

From the VCAA formula sheet, the width of the confidence interval is $2z\dfrac{s}{\sqrt{n}}$. Therefore, increasing *n* by a factor of 4 will decrease the confidence interval by a factor of $\sqrt{4}$.

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Section B – Short answer

Question 1a.

Worked solution

$f'(x) = \dfrac{4-4x}{(x-3)^3}$

The stationary point of $f(x)$ will occur when $f'(x) = 0$.

$$\frac{4-4x}{(x-3)^3} = 0$$

$$4 = 4x$$

$$x = 1$$

$$f(1) = \frac{(1-1)^2}{(1-3)^2} = 0$$

Therefore, the stationary point has coordinates $(1, 0)$.

Mark allocation: 1 mark

- 1 mark for the correct use of the first derivative to show a stationary point at $(1, 0)$

Question 1b.

Worked solution

c is the left-hand end point of the domain of $f(x)$. By inspection of the graph, $c = -4$.

Mark allocation: 1 mark

- 1 mark for the correct value of c

Question 1c.

Worked solution

The point of inflection can be found where $f''(x) = 0$.

$$f''(x) = \frac{8x}{(x-3)^4}$$

$$\frac{8x}{(x-3)^4} = 0$$

$$8x = 0$$

$$x = 0$$

$$f(0) = \frac{1}{9}$$

Therefore, the coordinates of the point of inflection are $\left(0, \dfrac{1}{9}\right)$.

Mark allocation: 1 mark

- 1 mark for the correct coordinates for the inflection point

Question 1d.

Worked solution

$$L = \int_{-4}^{0} \sqrt{1 + \left(\frac{4 - 4x}{(x-3)^3} \right)^2} \, dx$$
$$= 4.02$$

Mark allocation: 2 marks

- 1 mark for the correct definite integral that can be evaluated to find the arc length
- 1 mark for the correct rounded value of 4.02

Question 1e.

Worked solution

$$V = \pi \int_{0}^{1} \left(\frac{(x-1)^2}{(x-3)^2} \right)^2 dx$$
$$= 0.01$$

Mark allocation: 2 marks

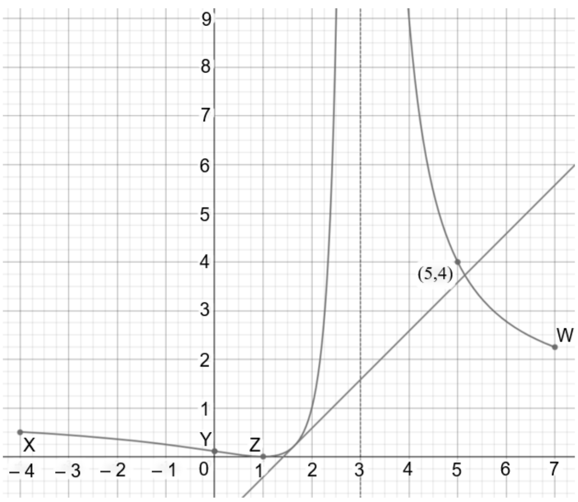
- 1 mark for the correct definite integral that can be evaluated to find the volume of revolution
- 1 mark for the correct rounded value of 0.01

Question 1f.

Worked solution

$g(x)$ is a linear function with a gradient of 1 and has a y-intercept of k.

By inspecting the graph, we see that the only linear function with a gradient of 1 that intersects $f(x)$ twice is one that is tangent to $f(x)$, as shown below.



Using calculus to find the point on $f(x)$ where the gradient of the tangent line is 1 gives

$$f'(x) = \frac{4-4x}{(x-3)^3} = 1$$

$$x \approx 1.63534$$

$$f(1.63534) \approx 0.21675657195129$$

The approximate coordinates of the point are $(1.63534,\ 0.216756)$.

The point slope form of a line can then be used to find the y-intercept of this tangent line.

$$y - f(1.63534) = 1(x - 1.63534)$$

$$y \approx x - 1.419$$

$$\therefore k \approx -1.419$$

The values of k, such that $f(x)$ and $g(x)$ intersect once, must be $k < -2.458$.

The smallest value of k such that $f(x)$ and $g(x)$ intersect once will be the value where $g(x)$ passes through point W.

From the graph, W has coordinates $\left(7, f(7)\right) = \left(7, \frac{9}{4}\right)$.

Using point slope form again gives

$$y - \frac{9}{4} = 1\left(x - 7\right)$$

$$y = x - \frac{19}{4}$$

This corresponds to a k value of $-\frac{19}{4}$ or -4.750.

Therefore, the range of the values of k, correct to three decimal places, such that $f(x)$ and $g(x)$ intersect once is $k \in [-4.750, -1.419)$.

Mark allocation: 3 marks

- 1 mark for finding the maximum k value of -1.419
- 1 mark for finding the minimum k value of -4.750
- 1 mark for the correct range of $k \in [-4.750, -1.419)$

Tip

- *Using a slider on a graph on a CAS calculator is a powerful way to visualise the solutions to graphical problems.*

Question 2a.

Worked solution

$\text{cSolve}\left(z^3=1,z\right)$	$z=\frac{^-1}{2}+\frac{\sqrt{3}}{2}\cdot \boldsymbol{i}$ or $z=\frac{^-1}{2}-\frac{\sqrt{3}}{2}\cdot \boldsymbol{i}$ or $z=1$
$\left(z=\frac{^-1}{2}+\frac{\sqrt{3}}{2}\cdot \boldsymbol{i} \text{ or } z=\frac{^-1}{2}-\frac{\sqrt{3}}{2}\cdot \boldsymbol{i} \text{ or } z=1\right)$▸Polar	$z=\boldsymbol{e}^{\frac{2\cdot \boldsymbol{i}\cdot \pi}{3}}$ or $z=\boldsymbol{e}^{\frac{^-2\cdot \boldsymbol{i}\cdot \pi}{3}}$ or $z=1$

$z^3-1=0$ has three unique solutions, given by $A=\text{cis}(0)$, $B=\text{cis}\left(\frac{2\pi}{3}\right)$, $C=\text{cis}\left(-\frac{2\pi}{3}\right)$.

Mark allocation: 1 mark

- 1 mark for finding the three correct roots, in polar form

Tip

- *Remember that equations of the form $z^n=a,\ a\in\mathbb{C}$ will always have n solutions that lie on the circumference of a circle on the Argand plane, with an angular separation of $\frac{2\pi}{n}$ radians.*

Question 2bi.

Worked solution

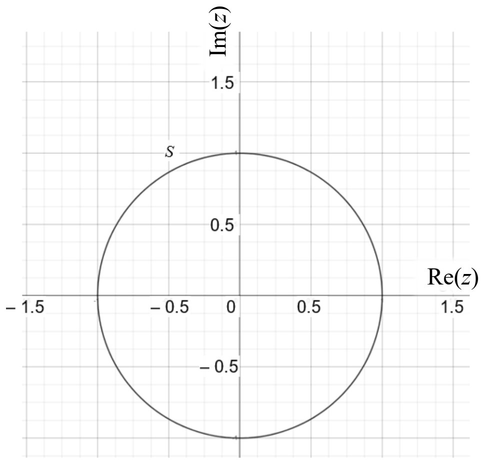
The three roots of $z^3-1=0$, in polar form, all have a radius of 1. Therefore, since S describes a circle centre $(0, 0)$, $r=1$.

Mark allocation: 1 mark

- 1 mark for stating the correct value of $r=1$

Question 2bii.

Worked solution



Mark allocation: 1 mark

- 1 mark for sketching S as a circle with radius 1 and centre $(0, 0)$ on the Argand plane

Question 2c.

Worked solution

The coordinates of the three roots of $z^3 - 1 = 0$ on the Argand plane are

$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), (1, 0).$

The distance between any two roots can be calculated using the distance formula.

$$d = \sqrt{\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2} - -\frac{1}{2}\right)^2} = \sqrt{\left(-\frac{\sqrt{3}}{2} - 0\right)^2 + \left(-\frac{1}{2} - 1\right)^2} = \sqrt{\left(\frac{\sqrt{3}}{2} - 0\right)^2 + \left(-\frac{1}{2} - 1\right)^2} = \sqrt{3}$$

Mark allocation: 2 marks

- 1 mark for showing that the distance between any two roots is the same
- 1 mark for the correct value of $d = \sqrt{3}$

Question 2di.

Worked solution

The centre of Q is the point z_0, and is also the mid-point of $\overline{AB}$, which can be found using the mid-point formula.

$$z_0 = \left(\frac{1 + -\frac{1}{2}}{2}, \frac{0 + \frac{\sqrt{3}}{2}}{2} \right) = \left(\frac{1}{4}, \frac{\sqrt{3}}{4} \right)$$

The radius of Q, r_0, will be half of the length of $\overline{AB}$, which is known from part **2c.**

$$r_0 = \frac{\sqrt{3}}{2}$$

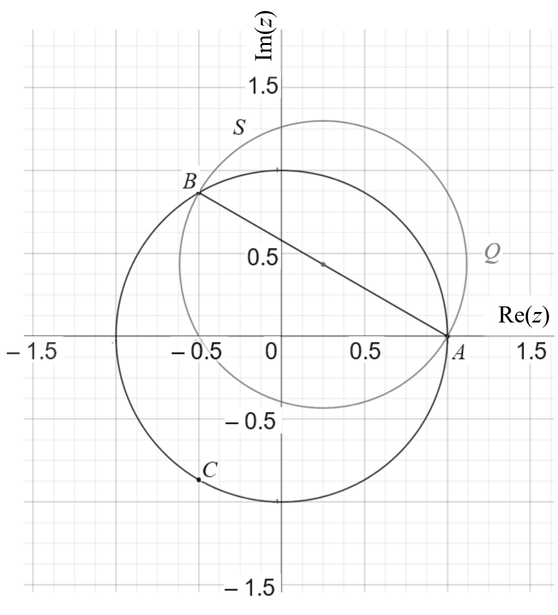
Therefore, $Q = \left\{ z : \left| z - \left(\frac{1}{4} + \frac{\sqrt{3}}{4} i \right) \right| = \frac{\sqrt{3}}{2} \right\}$

Mark allocation: 2 marks

- 1 mark for finding the correct centre and radius of Q
- 1 mark for the correct form of the relation for Q: $\left| z - \left(\frac{1}{4} + \frac{\sqrt{3}}{4} i \right) \right| = \frac{\sqrt{3}}{2}$

Question 2dii.

Worked solution



Mark allocation: 1 mark

- 1 mark for sketching the correct circle for Q, with centre $\left(\frac{1}{4}, \frac{\sqrt{3}}{4} \right)$ and radius $\frac{\sqrt{3}}{2}$

Question 2e.

Worked solution

The area of the region will be half the area of Q plus the area of the segment bound by the chord $\overline{AB}$ and the minor arc of S.

$$
\begin{aligned}
A &= \frac{1}{2}\times\pi\times\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1^2}{2}\left(\frac{2\pi}{3}-\sin\left(\frac{2\pi}{3}\right)\right)\\
&= \frac{3\pi}{8}+\frac{1}{2}\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\\
&= \frac{3\pi}{8}+\frac{\pi}{3}-\frac{\sqrt{3}}{4}\\
&= \frac{9\pi+8\pi-6\sqrt{3}}{24}\\
&= \frac{17\pi-6\sqrt{3}}{24}
\end{aligned}
$$

Mark allocation: 2 marks

- 1 mark for writing the correct expressions for the areas of the semicircle and segment
- 1 mark for the correct total area of $\frac{17\pi-6\sqrt{3}}{24}$ or equivalent

Tip

- *The formula to calculate the area of a segment is provided on the VCAA formula sheet. Remember that you must use radians for the value of the angle when using the formula.*

Question 3a.

Worked solution

$\left[t^2+1 \quad 4\cdot t+1\right] \rightarrow ra(t)$	*Done*
$\left[3\cdot t \quad t+2\right] \rightarrow rb(t)$	*Done*
$\mathrm{norm}(rb(0)-ra(0))$	$\sqrt{2}$

Mark allocation: 1 mark

- 1 mark for the correct linear distance of $\sqrt{2}$ metres

Tip

- *Often the best first step in approaching a multi-part vector calculus question is to define the position vectors on a CAS calculator. They can be defined as vector functions in terms of t (with each being a row vector).*

Question 3bi.

Worked solution

$x = t^2 + 1$

When $y = 4t + 1,\ t = \dfrac{y-1}{4}$.

Substituting gives

$$x = \left(\frac{y-1}{4}\right)^2 + 1$$

$$\sqrt{x-1} = \frac{y-1}{4}$$

$$y = 4\sqrt{x-1} + 1$$

Mark allocation: 1 mark

- 1 mark for the correct Cartesian equation of $y = 4\sqrt{x-1} + 1$

Question 3bii.

Worked solution

$x = 3t \Rightarrow t = \dfrac{x}{3}$

$y = t + 2$

Substituting gives

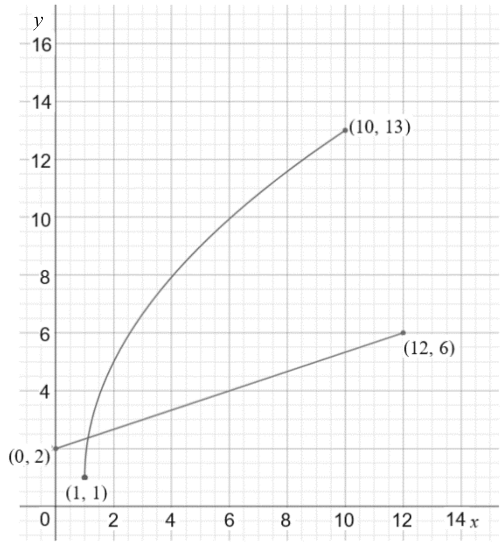
$y = \dfrac{x}{3} + 2$

Mark allocation: 1 mark

- 1 mark for the correct Cartesian equation of $y = \dfrac{x}{3} + 2$

Question 3c.

Worked solution



Mark allocation: 2 marks

- 1 mark for drawing each correct curve shape
- 1 mark for correctly labelling the endpoint coordinates for both curves

Question 3d.

Worked solution

Let L_A be the length of the path of Adam's ball and L_C be the length of the path of Cynthia's ball.

$$L_{\text{A}} = \int_0^3 \sqrt{\left(\frac{d}{dt}\left(t^2+1\right)\right)^2 + \left(\frac{d}{dt}\left(4t+1\right)\right)^2}\, dt = \int_0^3 \sqrt{4t^2+16}\, dt \approx 15.5957$$

$$L_{\text{C}} = \int_0^4 \left(\frac{d}{dt}\left(3t\right)\right)^2 + \left(\frac{d}{dt}\left(t+2\right)\right)^2 dt = \int_0^4 \sqrt{9+1}\, dt \approx 12.6491$$

$$L_{\text{A}} > L_{\text{C}}$$

$$L_{\text{A}} - L_{\text{C}} \approx 15.5957 - 12.6491 \approx 2.95$$

Mark allocation: 1 mark

- 1 mark for stating that Adam's golf ball travels further by 2.95 metres

Question 3e.

Worked solution

The average speed is the total distance travelled divided by the total time taken.

Let $\overline{v}_a$ be the average speed of Adam's ball and $\overline{v}_c$ be the average speed of Cynthia's ball.

$$\overline{v}_a \approx \frac{15.5957}{3} \approx 5.20 \text{ m s}^{-1}$$
$$\overline{v}_c = \frac{12.6491}{4} \approx 3.16 \text{ m s}^{-1}$$
$$\therefore \overline{v}_a > \overline{v}_c$$

Mark allocation: 1 mark

- 1 mark for stating that Adam's ball had a greater average speed

Question 3f.

Worked solution

$\begin{bmatrix} t^2+1 & 4\cdot t+1 \end{bmatrix} \rightarrow ra(t)$	*Done*
$\begin{bmatrix} 3\cdot t & t+2 \end{bmatrix} \rightarrow rb(t)$	*Done*
$ra(3)$	$\begin{bmatrix} 10 & 13 \end{bmatrix}$
$rb(4)$	$\begin{bmatrix} 12 & 6 \end{bmatrix}$
$\begin{bmatrix} 11 & 9 \end{bmatrix} \rightarrow h$	$\begin{bmatrix} 11 & 9 \end{bmatrix}$
$\mathrm{norm}(ra(3)-h)$	$\sqrt{17}$
$\mathrm{norm}(rb(4)-h)$	$\sqrt{10}$

Therefore, Cynthia's ball was closer by $\sqrt{17}-\sqrt{10}$ metres.

Mark allocation: 2 marks

- 1 mark for finding the correct distances from the hole: $\sqrt{17}$ and $\sqrt{10}$ metres
- 1 mark for stating that Cynthia's ball was closer to the hole by $\sqrt{17}-\sqrt{10}$ metres

Question 4a.

Worked solution

Two equations can be written from the information provided.

$$7500 = r \cdot 3000\left(1 - \frac{3000}{K}\right) \qquad [1]$$

$$7500 \cdot \frac{8}{5} = r \cdot 6000\left(1 - \frac{6000}{K}\right) \qquad [2]$$

Solving the simultaneous equations [1] and [2] gives values of $r = 3$ and $K = 18\,000$.

Mark allocation: 2 marks

- 1 mark for correct working showing value of $r = 3$
- 1 mark for correct working showing value of $K = 18\,000$

Tip

- *The maximum rate of growth for a population modelled by the logistic differential equation will always occur at the point of inflection of the population versus the time graph.*

Question 4b.

Worked solution

Solving the differential equation using the DE solver on the calculator gives

$$\text{deSolve}\left(p' = 3 \cdot p \cdot \left(1 - \frac{p}{18000}\right) \text{ and } p(0) = 3000, t, p\right)$$

$$p = \frac{18000 \cdot \mathbf{e}^{3 \cdot t}}{\mathbf{e}^{3 \cdot t} + 5}$$

Therefore, the particular solution is $P(t) = \dfrac{18000e^{3t}}{e^{3t} + 5}$.

Dividing the particular solution by e^{3t} gives the required form $P(t) = \dfrac{18000}{1 + 5e^{-3t}}$.

$a = 18\,000$

$b = 3$

$c = 5$

Mark allocation: 3 marks

- 1 mark for correct value of $a = 18\,000$
- 1 mark for correct value of $b = 3$
- 1 mark for correct value of $c = 5$

Question 4c.

Worked solution

The maximum rate of growth occurs at a time when

$$\frac{d^2P}{dt^2}=0$$

$$\frac{-180\,000\cdot e^{3t}(e^{3t}-5)}{(e^{3t}+5)^3}=0$$

$$t=\frac{\log_e(5)}{3}$$

$$P\left(\frac{\log_e(5)}{3}\right)=9000$$

Mark allocation: 2 marks

- 1 mark for the correct time value of $t=\frac{\log_e(5)}{3}$
- 1 mark for the correct population value of $P=9000$

Question 4d.

Worked solution

The population and rate of growth will both be increasing from $t=0$ up to the point of inflection of the curve described by $P(t)$, which occurs at $\frac{\log_e(5)}{3}$ (known from part **c.**).

Therefore, the correct interval is $t\in\left[0,\frac{\log_e(5)}{3}\right]$.

Mark allocation: 1 mark

- 1 mark for the correct time interval of $t\in\left[0,\frac{\log_e(5)}{3}\right]$

Question 4e.

Worked solution

The third minute is the time interval between $t=2$ and $t=3$.

The amount of new bacteria produced during this time interval can be found by evaluating $P(3)-P(2)=209.257\approx 209$.

Mark allocation: 1 mark

- 1 mark for the correct integer population of 209 bacteria

Question 5a.

Worked solution

By inspection of the parametric equations

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

Mark allocation: 1 mark

- 1 mark for the correct vector equation of Π_1

Question 5b.

Worked solution

A vector normal to Π_1 can be found using the vector cross product of two vectors in the plane.

$$\underset{\sim}{\mathrm{n}}_1 = \underset{\sim}{\mathrm{b}} \times \underset{\sim}{\mathrm{c}} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -1 \end{pmatrix}$$

A vector equation of Π_1 can now be determined, which can be used to write a Cartesian expression.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \underset{\sim}{\mathrm{n}}_1 = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \cdot \underset{\sim}{\mathrm{n}}_1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$$

$$3x - 5y - z = -27$$

Mark allocation: 2 marks

- 1 mark for finding the correct normal vector to Π_1
- 1 mark for the correct Cartesian equation for Π_1 of $3x - 5y - z = -27$

Question 5c.

Worked solution

Π_2 contains both $\underset{\sim}{n}_1$ and L. Therefore, a vector normal to Π_2 can be found by evaluating the vector cross product of $\underset{\sim}{n}_1$ and a vector in the direction of L.

$$\underset{\sim}{n}_2 = \underset{\sim}{n}_1 \times \underset{\sim}{c} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} \times \begin{pmatrix} -5 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} -25 \\ -17 \\ 10 \end{pmatrix}$$

As in the previous part of the question, we can use $\underset{\sim}{n}_2$ to determine a vector and hence a Cartesian expression for Π_2.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -25 \\ -17 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 13 \end{pmatrix} \cdot \begin{pmatrix} -25 \\ -17 \\ 10 \end{pmatrix}$$

$$-25x - 17y + 10z = 38$$

Mark allocation: 2 marks

- 1 mark for the correct normal vector to Π_2
- 1 mark for the correct algebra resulting in the Cartesian equation for Π_2: $-25x - 17y + 10z = 38$

Question 5d.

Worked solution

First, find a vector from $P(5,4,14)$ to any point on Π_2.

$$\underset{\sim}{u} = \begin{pmatrix} 3 \\ 1 \\ 13 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 14 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$$

Next, find the vector projection of $\underset{\sim}{u}$ on $\underset{\sim}{n}_2$, which will be a vector from $P(5,4,14)$ to Π_2 that is perpendicular to Π_2.

$$\underset{\sim}{v} = \frac{\underset{\sim}{u} \times \underset{\sim}{n}_2}{\underset{\sim}{n}_2 \times \underset{\sim}{n}_2}\underset{\sim}{n}_2 = \begin{pmatrix} \frac{-175}{78} \\ \frac{-119}{78} \\ \frac{35}{39} \end{pmatrix}$$

Finally, the position vector of the reflection of $P(5,4,14)$ across the plane Π_2 can be found by evaluating $\underset{\sim}{P} + 2\underset{\sim}{v}$, where $\underset{\sim}{P}$ is the position vector of $P(5,4,14)$.

$$\begin{pmatrix} 5 \\ 4 \\ 14 \end{pmatrix} + 2\begin{pmatrix} \frac{-175}{78} \\ \frac{-119}{78} \\ \frac{35}{39} \end{pmatrix} = \begin{pmatrix} \frac{20}{39} \\ \frac{37}{39} \\ \frac{616}{39} \end{pmatrix}$$

This corresponds to coordinates $\left(\frac{20}{39}, \frac{37}{39}, \frac{616}{39}\right)$.

Mark allocation: 3 marks

- 1 mark for the correct expression for a vector, $\underset{\sim}{v}$, from $P(5,4,14)$ to Π_2 that is perpendicular to Π_2
- 1 mark for the correct expression using $\underset{\sim}{v}$ that can be evaluated to find the position vector of the reflected point
- 1 mark for the correct coordinates of reflected point of $\left(\frac{20}{39}, \frac{37}{39}, \frac{616}{39}\right)$

Question 5ei.

Worked solution

Since line L lies in plane Π_3, any point on L will also satisfy the equation for Π_3.

Using the equation for L, $\underset{\sim}{r}_2(t) = 3\underset{\sim}{i} + \underset{\sim}{j} + 13\underset{\sim}{k} + t(-5\underset{\sim}{i} + 5\underset{\sim}{j} - 4\underset{\sim}{k})$, the position vectors, and hence the coordinates of two points on Π_3, can be determined.

$\underset{\sim}{r}_2(0) = 3\underset{\sim}{i} + \underset{\sim}{j} + 13\underset{\sim}{k}$

$\underset{\sim}{r}_2(1) = -2\underset{\sim}{i} + 6\underset{\sim}{j} + 9\underset{\sim}{k}$

So the coordinates of two points on plane Π_3 are $(3,1,13)$ and $(-2,6,9)$.

The above coordinates can now be substituted into the equation for Π_3 to find the values of c and d.

$13(3) + 29(1) + c(13) = d$ [1]
$13(-2) + 29(6) + c(9) = d$ [2]

Solving the simultaneous equations [1] and [2] gives $c = 20$ and $d = 328$.

Mark allocation: 2 marks

- 1 mark for finding the coordinates of any two points on Π_3
- 1 mark for the correct values of $c = 20$ and $d = 328$

Question 5eii.

Worked solution

First, find the angle, θ, between the normal vectors of Π_1 and Π_3, using the dot product.

$$\cos(\theta) = \frac{\begin{pmatrix}13\\29\\20\end{pmatrix} \cdot \begin{pmatrix}3\\-5\\-1\end{pmatrix}}{\left|\begin{pmatrix}13\\29\\20\end{pmatrix}\right|\left|\begin{pmatrix}3\\-5\\-1\end{pmatrix}\right|}$$

$$\theta = 124.6°$$

The acute angle between the planes is therefore $180 - 124.55 = 55.4°$.

Mark allocation: 2 marks

- 1 mark for finding the angle between the normal vectors of Π_1 and Π_2
- 1 mark for the correct acute angle between the planes of $55.4°$

Question 6a.

Worked solution

$$\mathrm{E}(W) = \mathrm{E}(L) = \int_0^2 x \cdot f(x)\,dx = \frac{2}{3}$$

$$\mathrm{Var}(W) = \mathrm{Var}(L) = \int_0^2 x^2 \cdot f(x)\,dx - \left(\frac{2}{3}\right)^2 = \frac{2}{9}$$

$$P = 2W + 2L$$

$$\mathrm{E}(P) = 2\times\mathrm{E}(W) + 2\times\mathrm{E}(L) = 2\times\frac{2}{3} + 2\times\frac{2}{3} = \frac{8}{3}$$

$$\mathrm{sd}(P) = \sqrt{2^2\,\mathrm{Var}(W) + 2^2\,\mathrm{Var}(L)} = \sqrt{4\times\frac{2}{9} + 4\times\frac{2}{9}} = \sqrt{\frac{16}{9}} = \frac{4}{3}$$

Mark allocation: 2 marks

- 1 mark for the expected value of the perimeter: $\frac{8}{3}$
- 1 mark for the standard deviation of the perimeter: $\frac{4}{3}$

Question 6b.

Worked solution

$$\Pr(W < 1) = \Pr(L < 1) = \int_0^1 -\frac{1}{2}x + 1\,dx = \left[-\frac{1}{4}x^2 + x\right]_0^1 = \frac{3}{4}$$

$$\Pr(W < 1 \cap L < 1) = \frac{3}{4}\times\frac{3}{4} = \frac{9}{16}$$

Mark allocation: 1 mark

- 1 mark for showing any correct sequential working leading to the answer of $\frac{9}{16}$

Question 6c.

Worked solution

Pr(2 rectangles will fit) $= \frac{9}{16}\times\frac{9}{16} = \frac{81}{256}$

Mark allocation: 1 mark

- 1 mark for the correct answer of $\frac{81}{256}$

Question 6d.

Worked solution

$\mathrm{E}\left(\overline{L}\right) = \mathrm{E}(L) = \frac{2}{3}$

$\mathrm{sd}\left(\overline{L}\right) = \frac{\mathrm{sd}(L)}{\sqrt{100}} = \frac{\sqrt{2}}{30}$

Mark allocation: 2 marks

- 1 mark for $\mathrm{E}\left(\overline{L}\right) = \frac{2}{3}$
- 1 mark for $\mathrm{sd}\left(\overline{L}\right) = \frac{\sqrt{2}}{30}$

Question 6e.

Worked solution

$90\%\ CI = (0.589, 0.744)$

Mark allocation: 1 mark

- 1 mark for the correct answer of $(0.589, 0.744)$

Question 6f.

Worked solution

$$2z \times \frac{\frac{\sqrt{2}}{3}}{\sqrt{100}} = \frac{1}{10}$$

$$z = \frac{3\sqrt{2}}{4}$$

$$\Pr\left(z > \frac{3\sqrt{2}}{4}\right) = 0.1444$$

$$1 - 2 \times 0.1444 \approx 0.711$$

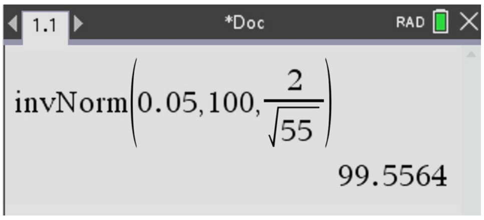
Mark allocation: 1 mark

- 1 mark for the correct answer of 0.711

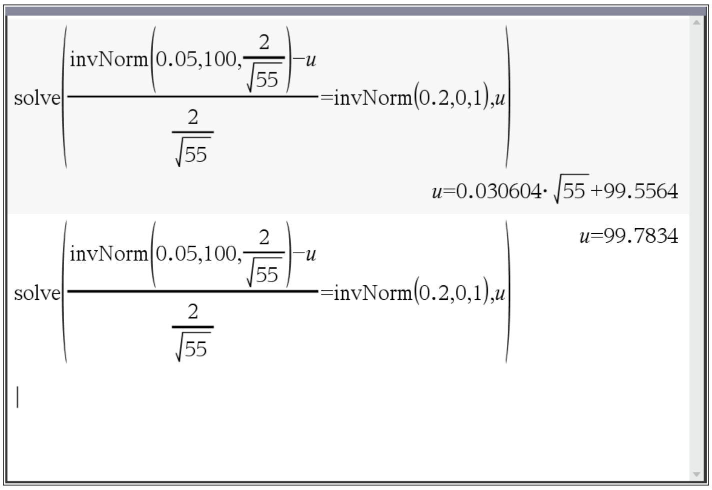
Question 6g.

Worked solution

The critical value for a statistical test at the 5% level can be found using the inverse normal function on a calculator.



This critical value can then be used to find the value of μ that gives an 80% or higher probability of making the sale.



Mark allocation: 2 marks

- 1 mark for the correct expression or value for the critical value at the 5% level
- 1 mark for the correct mean area of 99.783

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