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Specialist Mathematics 2

Written Examination 2

Question and Answer Book

2024 Insight Publications Year 12 Trial Examination

- **Reading time:** 15 minutes
- **Writing time:** 2 hours

Approved materials:

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied:

- Question and answer book of 27 pages
- Formula sheet
- Answer sheet for multiple choice answers

Instructions

- Unless otherwise indicated, the diagrams in this book are not drawn to scale.
- All responses must be in English.

Students are **not** permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Contents

Section A – Multiple choice

- Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
- Choose the response that is **correct** for the question.
- A correct answer scores 1; an incorrect answer scores 0.
- Marks will **not** be deducted for incorrect answers.
- No marks will be given if more than one answer is completed for any question.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1

Consider the following statement.

'For all integers n, if $5n + 1$ is odd, then n is even.'

Which one of the following is the **negation** of this statement?

A. There exists an integer n such that $5n + 1$ is even and n is odd.

B. There exists an integer n such that $5n + 1$ is even and n is even.

C. There exists an integer n such that $5n + 1$ is odd and n is odd.

D. For all integers n, if $5n + 1$ is even, then n is odd.

Question 2

Consider the function $f(x) = kx^2 + 2x - k,\ k \in \mathbb{R}$.

The graph of $y = \dfrac{1}{f(x)}$ will always have

A. at least three linear asymptotes.

B. at least two linear asymptotes.

C. at least one linear asymptote.

D. a stationary point.

Question 3

The *x*-coordinates of the points of intersection of the graphs of $\frac{1}{a+\sin(x)}$ and $\frac{1}{a+\cos(x)}$, where $a \in \mathbb{R}$, are given by

A. $\frac{3\pi}{4} + k\pi,\ k \in \mathbb{Z}$

B. $\frac{\pi}{4} + 2k\pi,\ k \in \mathbb{Z}$

C. $\frac{\pi}{4} + k\pi,\ k \in \mathbb{Z}$

D. $-\frac{5\pi}{4} + k\pi,\ k \in \mathbb{Z}$

Question 4

If $z = \frac{a+4i}{1+ai}$, $a \in \mathbb{R}$ is a real number, the possible values of a are

A. $a \geq 0$

B. $a \leq 0$

C. $a = \pm 4$

D. $a = \pm 2$

Question 5

If $z = 1 + \sqrt{3}i$ and $\frac{z}{w} = 2 + 2i$, then $\text{Arg}(w)$ is

A. $\frac{\pi}{12}$

B. $\frac{7\pi}{4}$

C. $\frac{\pi}{4}$

D. $\frac{\pi}{3}$

Use the following information to answer questions 6 and 7.

The algorithm below, described in pseudocode, uses Euler's method to estimate the coordinates of a solution to the differential equation $\frac{dy}{dx} = g(x, y)$.

```
Inputs:
    g(x, y), the right-hand side of the DE, as a function of
    x and y
    x0, the x-coordinate of the starting point
    y0, the y-coordinate of the starting point
    h, the step size
    n, the number of iterations to perform

define eulers(g(x, y), x0, y0, h, n)
    x ← x0
    y ← y0
for i from 1 to n
     [          ]
     x ← x + h
    print (x, y)
end for
return
```

Question 6

Which one of the following options would be the most appropriate to fill the empty box?

A. `y ← g(x, y)`

B. `y ← y + g(x, y)`

C. `y ← y + h × g(x, y)`

D. `y ← y + g(x + h, y)`

Question 7

Consider the algorithm implemented with the following inputs.

```
eulers(sin(x + y), 1, 4, 0.2, 10)
```

The value of the variable x after two iterations of the **for** loop would be closest to

A. 1.2

B. 1.4

C. 1.6

D. 3.6

Question 8

The particular solution to the differential equation $y\frac{dy}{dx} = 3x^2$ that passes through point $(0, 2)$ will also pass through point

A. $\left(2\sqrt[3]{2}, -6\right)$

B. $\left(-2\sqrt[3]{2}, 6\right)$

C. $\left(2\sqrt[3]{2}, 6\right)$

D. $\left(1, -\sqrt{6}\right)$

Question 9

A jug of water at a temperature of T °C is placed in a refrigerator for t minutes. The temperature inside the refrigerator is maintained at 4°C.

The water has an initial temperature of 20°C and cools to 18°C after being in the refrigerator for 3 minutes.

The rate at which the water cools can be described by the differential equation $\frac{dT}{dt} = -k(T - S)$, where k is a positive constant and S is the constant temperature of the refrigerator, in °C.

The value of k is

A. 14

B. 2

C. $-\frac{1}{3}\ln\left(\frac{8}{7}\right)$

D. $\frac{1}{3}\ln\left(\frac{8}{7}\right)$

Question 10

The position of a particle moving in the Cartesian plane, at time t, is given by the parametric equations $x = a\cos(t)$ and $y = b\sin(t)$, where $t \geq 0$ and a and b are positive real constants.

Given that $a > b$, the maximum speed of the particle is given by

A. $a^2 - b^2$

B. $|a - b|$

C. $\sqrt{a^2 - b^2}$

D. a

Question 11

The total surface area of the volume generated by revolving part of the curve with equation $y = \sqrt[3]{3x}$, where $0 \leq y \leq 2$ around the y-axis, can be found by evaluating

A. $\frac{\pi}{6}\int_{1}^{17} \sqrt{u}\, du$

B. $\frac{\pi}{6}\int_{1}^{17} \sqrt{u}\, du + \frac{64\pi}{9}$

C. $\frac{\pi}{6}\int_{0}^{2} \sqrt{u}\, du + \frac{64\pi}{9}$

D. $\frac{\pi}{6}\int_{0}^{2} \sqrt{u}\, du$

Question 12

The acceleration, $a \text{ m s}^{-2}$, of a particle that starts with an initial velocity, $v_0 \text{ m s}^{-1}$, and moves in a straight line with velocity $v \text{ m s}^{-1}$ is described by $a = -g - \alpha v$, $\alpha \in \mathbb{R}$.

The velocity of the particle after moving for t seconds is given by

A. $v(t) = v_0 \cdot e^{-t} + \frac{g}{\alpha}\left(e^{-t} - 1\right)$

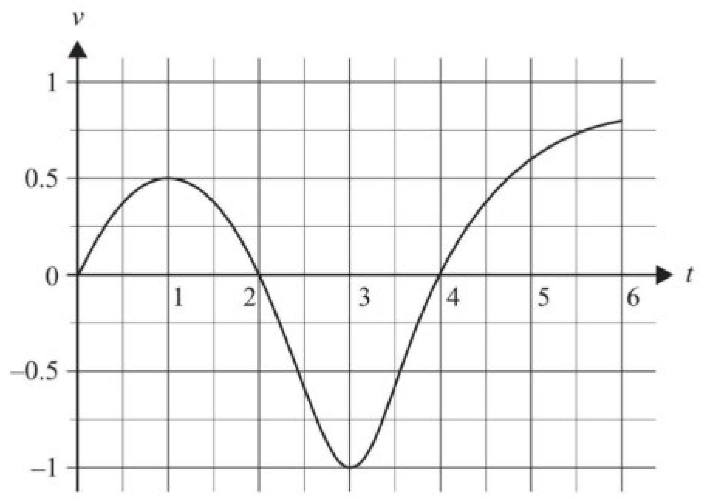
B. $v(t) = \frac{e^{-\alpha t} - g}{\alpha}$

C. $v(t) = e^{-\alpha t} + \frac{g}{\alpha}\left(e^{-\alpha t} - 1\right)$

D. $v(t) = v_0 \cdot e^{-\alpha t} + \frac{g}{\alpha}\left(e^{-\alpha t} - 1\right)$

Question 13

The velocity–time graph for a particle moving in a straight line is shown below. Velocity is measured in m s^{-1} and time in seconds.



The average velocity of the particle over the interval $0 \le t \le 4$ is closest to

A. $-\frac{3}{32}$

B. $-\frac{1}{2}$

C. $\frac{3}{32}$

D. $\frac{7}{16}$

Question 14

Given that $(\underset{\sim}{a} \cdot \underset{\sim}{b})^2 = 1$, it follows that the unit vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ must be

A. collinear.

B. in the same direction.

C. in opposite directions.

D. linearly dependent.

Question 15

Given θ is the angle between two vectors, $\underset{\sim}{c}$ and $\underset{\sim}{d}$, and $\cos(\theta) = -\frac{3}{7}$, then the expression $\frac{\left|\underset{\sim}{c} \times \underset{\sim}{d}\right|}{\underset{\sim}{c} \cdot \underset{\sim}{d}}$ has a value of

A. $\frac{2\sqrt{10}}{3}$

B. $-\frac{2\sqrt{10}}{3}$

C. $-\frac{3\sqrt{10}}{20}$

D. $\frac{3\sqrt{10}}{20}$

Question 16

The position vectors of two ships, A and B, relative to a fixed origin, O, at t seconds after they begin to move are given by $\underset{\sim}{r}_a(t) = \left(4\cos(t)+2\right)\underset{\sim}{i} + \left(4\sin(t)+2\right)\underset{\sim}{j}$ and $\underset{\sim}{r}_b(t) = \left(2\cos(t)+1\right)\underset{\sim}{i} + \left(4\sin(t)+1\right)\underset{\sim}{j}$, respectively.

The coordinates of the points of intersection of the paths of the two ships, correct to two decimal places, are

A. $(0.00, 5.46)$ and $(0.00, 4.46)$

B. $(0.00, -1.46)$ and $(0.00, -2.46)$

C. $(0.76, 0.89)$ and $(2.33, 1.99)$

D. $(-0.76, -0.89)$ and $(2.33, -1.99)$

Question 17

Consider the vectors $\underset{\sim}{a} = m\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + n\underset{\sim}{j} + 2\underset{\sim}{k}$, where $m, n \in \mathbb{R}$.

If the unit vector in the direction of $\underset{\sim}{a} \times \underset{\sim}{b}$ is $\sqrt{6}\left(\frac{1}{15}\underset{\sim}{i} - \frac{1}{6}\underset{\sim}{j} - \frac{11}{30}\underset{\sim}{k}\right)$, then it follows that the values of m and n, respectively, are

A. 1 and -3

B. 2 and 4

C. -2 and 4

D. 2 and -4

Question 18

Consider two parallel planes given by the equations $\Pi_1:\ 2x + 3y + 6z = 14$ and $\Pi_2:\ 4x + 6y + 12z = -42$.

The equation of a third plane, Π_3, parallel to both Π_1 and Π_2, and at the same distance from Π_1 as Π_2, is

A. $2x + 3y + 6z = -56$

B. $4x + 6y + 12z = -14$

C. $4x + 6y + 12z = -5$

D. $6x + 9y + 18z = 147$

Question 19

Consider two independent normal random variables, X and Y, both with a mean of zero and a standard deviation of $\sqrt{2}$.

Given that $W = X + Y$ and $Z = X - Y$, determine the probability of W being larger than Z.

A. $\frac{3}{4}$

B. $\frac{1}{2}$

C. $\frac{1}{4}$

D. $\frac{1}{8}$

Question 20

A confidence interval is to be used to estimate the population mean, μ, based on a sample mean, $\bar{x}$.

If the sample size is multiplied by 4, the width of the confidence interval will

A. decrease by a factor of 2.

B. decrease by a factor of 4.

C. increase by a factor of 2.

D. increase by a factor of 4.

End of Section A

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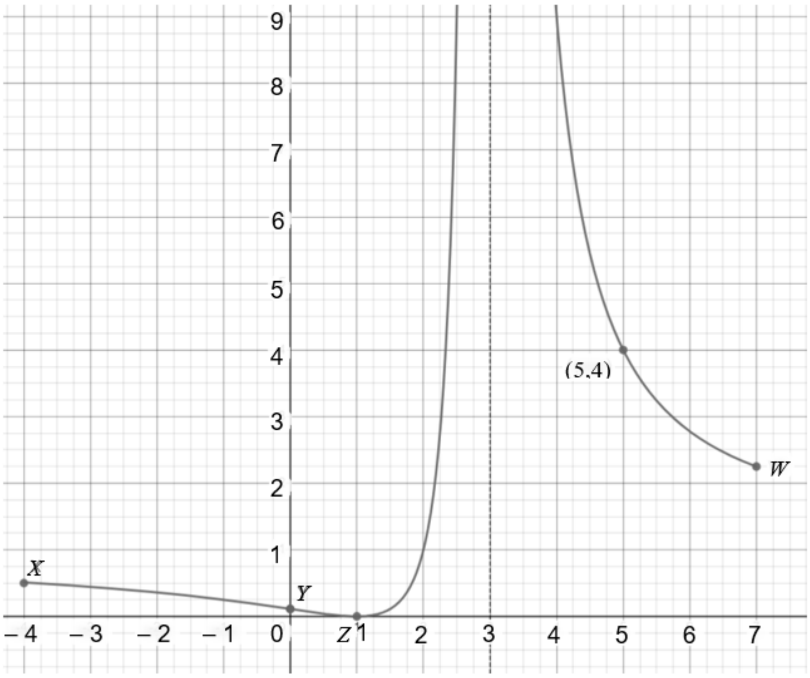
Section B – Short answer

- Answer **all** questions in the spaces provided.
- Unless otherwise specified, an **exact** answer is required to a question.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Take the **acceleration due to gravity** to have magnitude $g \text{ m s}^{-2}$, where $g = 9.8$

Question 1 (10 marks)

The curve described by the rational function $f(x) = \dfrac{(x-a)^2}{(x-b)^2}, c \le x \le 7$ and $a, b, c \in \mathbb{R}$ is shown on the graph below. The curve has a stationary point at Z, a vertical asymptote at $x = 3$, a point of inflection at Y and passes through $(5, 4)$.



a. Show that the coordinates of the stationary point, Z, are $(1, 0)$.

1 mark

b. State the value of c.

1 mark

c. Find the coordinates of the point of inflection, Y.

1 mark

d. Find the length of the curve described by $f(x)$ from X to Y, correct to two decimal places.

2 marks

e. Find the volume of the solid of revolution formed when the section of the curve from Y to Z is rotated about the x-axis, correct to two decimal places.

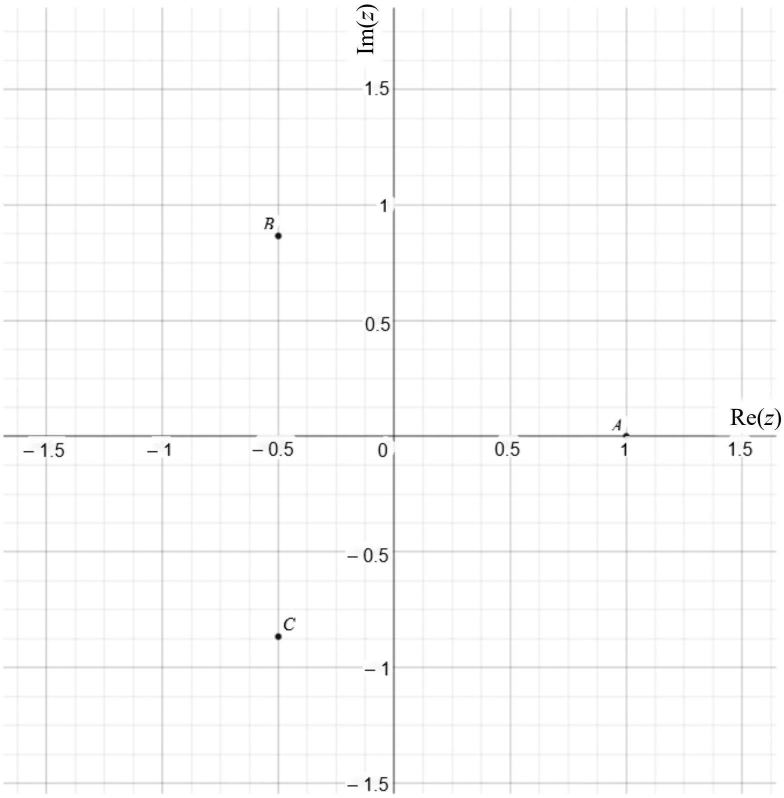
2 marks

f. A second curve is described by the function $g(x) = x + k,\ k \in \mathbb{R}$.
Find the values of k, correct to three decimal places, such that $f(x)$ and $g(x)$ intersect once.

3 marks

Question 2 (10 marks)

The solutions of $z^3 = 1, z \in \mathbb{C}$ are shown by points A, B and C on the Argand diagram below.



a. List the roots of $z^3 - 1 = 0$, in polar form.

1 mark

b. The roots of $z^3 - 1 = 0$ all belong to the set of points S, such that $\{z \in S: z\bar{z} = r\}$.

i. State the value of r.

1 mark

ii. Sketch and label the set S on the Argand diagram above.

1 mark

c. Show that the distance, d, between any two roots of $z^3 - 1 = 0$ is the same, and determine the value of d.

2 marks

d. **i.** The line segment $\overline{AB}$ forms the diameter of a second circle, Q, on the Argand plane.
Find the equation of Q in the form $|z - z_0| = r_0$, where $z_0 \in \mathbb{C}$ and $r_0 \in \mathbb{R}$.

2 marks

ii. Sketch and label Q on the Argand diagram on the previous page.

1 mark

e. Find the area of the region enclosed by the intersection of S and Q.

2 marks

Question 3 (9 marks)

Adam and Cynthia are playing a round of mini golf. On the last hole their scores are tied. They decide to hit their balls at the same time and the person whose ball is closest to the hole after one stroke is the winner.

The position of their golf balls, relative to a fixed origin, O, after t seconds, are given by $\underset{\sim}{r}_a(t) = (t^2+1)\underset{\sim}{i} + (4t+1)\underset{\sim}{j}$ for $0 \le t \le 3$ and $\underset{\sim}{r}_c(t) = 3t\,\underset{\sim}{i} + (t+2)\underset{\sim}{j}$ for $0 \le t \le 4$, respectively, where the displacement components are measured in metres.

a. Calculate the initial distance between Adam and Cynthia's golf balls.

1 mark

b. **i.** Determine the Cartesian equation of Adam's ball in the form $y = f(x)$.

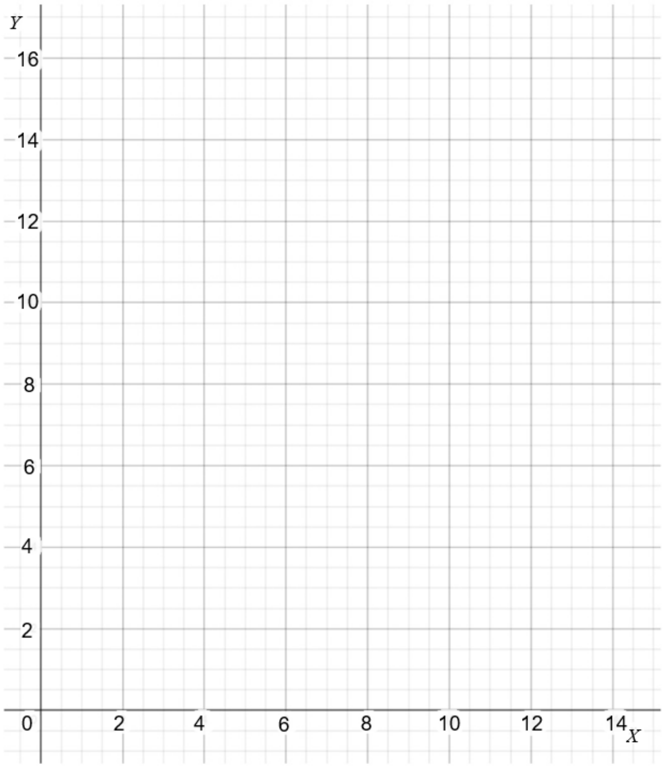
1 mark

ii. Determine the Cartesian equation of Cynthia's golf ball in the form $y = g(x)$.

1 mark

c. On the axes below, sketch the paths of Adam's and Cynthia's golf balls, including the coordinates of any end points.

2 marks



d. Whose golf ball, Adam's or Cynthia's, travelled further during its motion and by how many metres? Give your answer correct to two decimal places.

1 mark

e. Whose golf ball, Adam's or Cynthia's, had a greater average speed over the course of its motion?

1 mark

f. If the hole has a position vector of $11\underset{\sim}{i}+9\underset{\sim}{j}$ relative to the origin, O, whose golf ball, Adam's or Cynthia's, had a final position closest to the hole and by how many metres?

2 marks

Question 4 (9 marks)

The population, P, of a sample of bacteria t minutes after the bacteria were introduced to a petri dish can be modelled by the logistic differential equation $\frac{dP}{dt}=rP\left(1-\frac{P}{K}\right)$.

a. 3000 bacteria were initially introduced to the petri dish and the initial rate of growth of the population was 7500 bacteria per minute. When the population of bacteria had doubled, the rate of growth had increased by a factor of $\frac{8}{5}$.

Show that $r=3$ and $K=18\,000$.

2 marks

b. The particular solution of the differential equation can be written in the form

$P(t)=\frac{a}{ce^{-bt}+1},\ a,b,c\in R$.

Find the values of a, b and c.

3 marks

c. After how many minutes was the population growing at the maximum rate and what was the population at that time?

2 marks

d. State the values of t for when both the population and the rate of growth are increasing. Give your answer in interval notation.

1 mark

e. How many new bacteria were produced in the third minute after the bacteria were introduced? Give your answer to the nearest integer.

1 mark

Question 5 (12 marks)

A plane, Π_1, is described by the parametric equations

$x = 2 + s + 3t$

$y = 5 + s + 2t$ where $s, t \in R$.

$z = 8 - 2s - t$

a. Find a vector equation of the plane Π_1 in the form $\underset{\sim}{r} = \underset{\sim}{a} + s\underset{\sim}{b} + t\underset{\sim}{c}$.

1 mark

b. Hence, find the Cartesian equation of the plane Π_1.

2 marks

A second plane, Π_2, is perpendicular to Π_1 and intersects it along line L, which is given by $\underset{\sim}{r}_2 = 3\underset{\sim}{i} + \underset{\sim}{j} + 13\underset{\sim}{k} + t(-5\underset{\sim}{i} + 5\underset{\sim}{j} - 4\underset{\sim}{k})$.

c. Show that the Cartesian equation of plane Π_2 is given by $-25x - 17y + 10z = 38$.

2 marks

d. Find the reflection of point $P(5, 4, 14)$ across plane Π_2.

3 marks

e. A third plane, Π_3, intersects Π_1 and Π_2 at L and has an equation of the form $13x + 29y + cz = d$.

i. Find the values of c and d.

2 marks

ii. Find the acute angle, in degrees, between Π_1 and Π_3, correct to one decimal place.

2 marks

Question 6 (10 marks)

Daniel owns an industrial design factory and is considering purchasing a new machine that produces metal rectangular frames. The machine produces rectangles with length L and width W. The lengths, L and W, measured in cm, are independent and each vary according to the following probability density function.

$$f(x) = \begin{cases} -\frac{1}{2}x + 1 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the mean and the standard deviation of the perimeter of these rectangles.

2 marks

A contractor wishes to purchase rectangles to stack one on top of another inside a box that has a square bottom with dimensions $1\ \text{cm}$ x $1\ \text{cm}$.

b. Show that the probability of randomly selecting a rectangle made from Daniel's machine that will fit inside the box is equal to $\frac{9}{16}$.

1 mark

c. Find the probability that two randomly selected rectangles will both fit inside the box.

1 mark

d. Find the expected value and the standard deviation of the average sample length of a sample of 100 rectangles produced by the machine.

2 marks

e. Find the 90% confidence interval for the sample mean length from a random sample of 100 rectangles, correct to three decimal places.

1 mark

f. Find the confidence level required such that the confidence interval for the sample mean length has a width of less than 0.1 cm. Give your answer correct to three decimal places.

1 mark

Rather than use the new machine, Daniel decides to use his old machine, which produces rectangles with areas that follow a normal distribution with $\sigma = 2 \text{ cm}^2$. The mean area, μ, can be adjusted on the machine using a dial.

However, the potential buyer is cautious and will conduct a statistical test at the 5% level of significance, using a sample of 55 rectangles. If the buyer concludes that the mean area of the rectangles is less than 100 cm^2, they will not make the purchase.

g. What is the minimum setting that Daniel can configure for the mean area μ such that he still has an 80% probability or higher of making the sale? Give your answer correct to three decimal places.

2 marks

End of Question and Answer Book