2024
VCE
Specialist Mathematics
Year 12
Trial Examination 1

Detailed Answers

Kilbaha Education

Quality educational content

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Question 1

a. $\frac{dy}{dx} = 2x\sqrt{16-9y^2}$, $y(1) = \frac{2}{3}$ separating the variables

$$\int \frac{1}{\sqrt{16-9y^2}}\,dy = \int 2x\,dx$$

$$\frac{1}{3}\sin^{-1}\left(\frac{3y}{4}\right) = x^2 + c$$

to find c, when $x=1$, $y=\frac{2}{3}$, $\frac{1}{3}\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{18} = 1 + c$, $c = \frac{\pi}{18} - 1$ A1

$$\frac{1}{3}\sin^{-1}\left(\frac{3y}{4}\right) = x^2 + \frac{\pi}{18} - 1$$

$$\sin^{-1}\left(\frac{3y}{4}\right) = 3x^2 + \frac{\pi}{6} - 3$$ M1

$$\frac{3y}{4} = \sin\left(3x^2 + \frac{\pi}{6} - 3\right)$$

$$y = \frac{4}{3}\sin\left(3x^2 + \frac{\pi}{6} - 3\right)$$ A1

b. $y(1) = \frac{2}{3}$ when $x=1$, $y = \frac{2}{3}$, $\frac{dy}{dx} = 2\sqrt{16 - 9\left(\frac{4}{9}\right)} = 2\sqrt{12} = 4\sqrt{3}$

using implicit differentiation $\frac{dy}{dx} = 2x\sqrt{16-9y^2}$

$$\frac{d^2y}{dx^2} = 2\sqrt{16-9y^2} + \frac{-18y \times 2x}{2\sqrt{16-9y^2}}\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 2\sqrt{16-9y^2} - \frac{18xy}{\sqrt{16-9y^2}}\frac{dy}{dx}$$ M1

$$\left.\frac{d^2y}{dx^2}\right|_{x=1,\, y=\frac{2}{3}} = 4\sqrt{3} - \frac{18 \times \frac{2}{3} \times 4\sqrt{3}}{2\sqrt{3}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1,\, y=\frac{2}{3}} = 4\sqrt{3} - 24$$ A1

Question 2

$$f(x)=\frac{x^2-5x+4}{x}=\frac{(x-4)(x-1)}{x}=x-5+\frac{4}{x} \quad \text{domain } D=R\backslash\{0\}$$

i. a vertical asymptote at $x=0$ and an oblique asymptote at $y=x-5$. A1

ii. $f(x)=x-5+4x^{-1}$

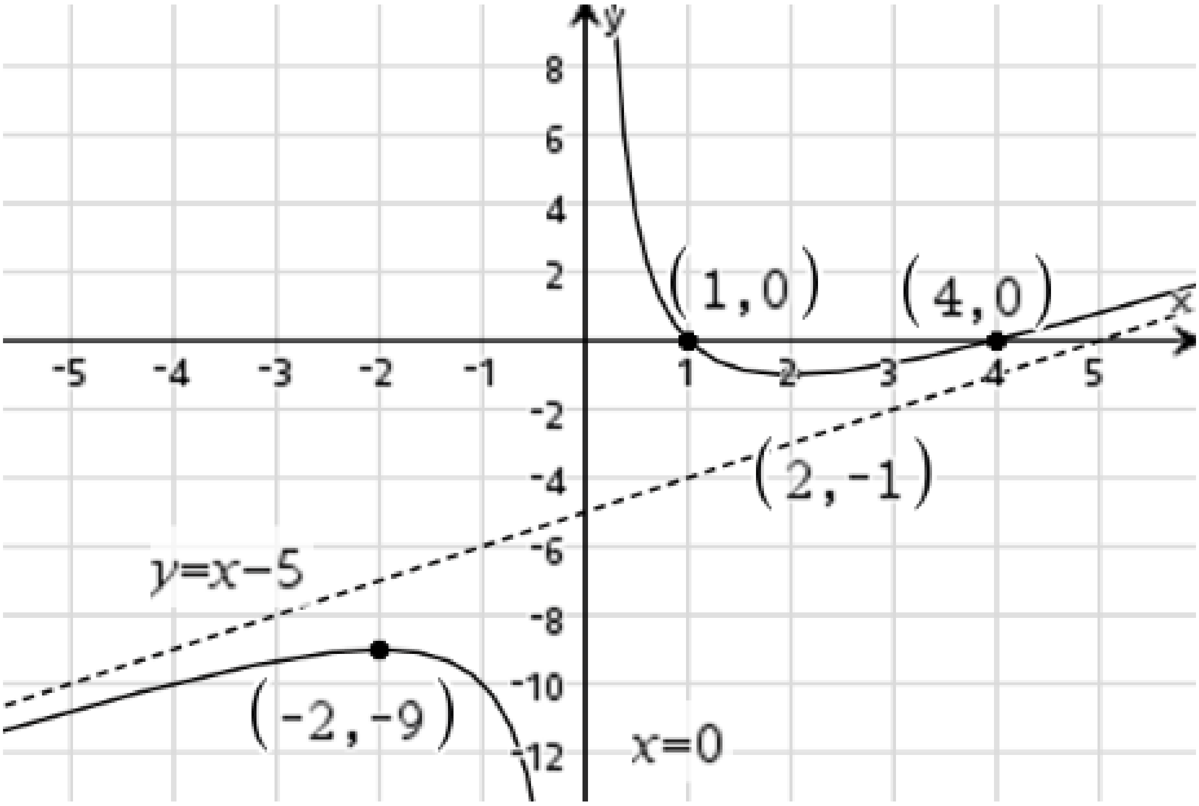
$f'(x)=1-4x^{-2}=1-\frac{4}{x^2}=\frac{x^2-4}{x^2}=0$ when $x=\pm 2$,

$f(2)=2-5+\frac{4}{2}=-1, \quad f(-2)=-2-5-\frac{4}{2}=-9$

so there are turning points at $(2,-1)$, $(-2,-9)$ A1

iii. $f''(x)=8x^{-3}=\frac{8}{x^3}\neq 0$ so there are no points of inflexion A1

iv. The graph crosses the *x*-axis at $(1,0),(4,0)$ G1



Question 3

a. $a = 9.8 - 0.2v^2 = \frac{98}{10} - \frac{2v^2}{10} = \frac{1}{5}\left(49 - v^2\right)$

use $a = v\frac{dv}{dx} = \frac{1}{5}\left(49 - v^2\right)$

$\frac{dv}{dx} = \frac{49 - v^2}{5v}$ inverting

$\frac{dx}{dv} = \frac{5v}{49 - v^2}$

$x = -\frac{5}{2}\int \frac{-2v}{49 - v^2}\,dv$

$x = -\frac{5}{2}\log_e\left(\left|49 - v^2\right|\right) + c$ A1

$x = 0,\ v = 0,\ \Rightarrow\ 0 = -\frac{5}{2}\log_e(49) + c,\quad c = \frac{5}{2}\log_e(49)$

$x = \frac{5}{2}\log_e(49) - \frac{5}{2}\log_e\left(\left|49 - v^2\right|\right)$

$x = \frac{5}{2}\log_e\left(\frac{49}{49 - v^2}\right)$

since $0 \le v < 7$ the modulus is not needed. A1

$\frac{2x}{5} = \log_e\left(\frac{49}{49 - v^2}\right)$

$\frac{49}{49 - v^2} = e^{\frac{2x}{5}}$

$\frac{49 - v^2}{49} = e^{-\frac{2x}{5}}$ M1

$49 - v^2 = 49e^{-\frac{2x}{5}}$

$v^2 = 49 - 49e^{-\frac{2x}{5}}$

$v^2 = 49\left(1 - e^{-\frac{2x}{5}}\right)$

b. use $a=\frac{dv}{dt}=\frac{1}{5}\left(49-v^2\right)$ inverting

$$\frac{dt}{dv}=\frac{5}{49-v^2}$$

$t=5\int\frac{1}{49-v^2}\,dv$ now by partial fractions

$$\frac{1}{49-v^2}=\frac{A}{7+v}+\frac{B}{7-v}=\frac{A(7-v)+B(7+v)}{(7+v)(7-v)}=\frac{v(B-A)+7(A+B)}{49-v^2}$$

M1

$$(1)\ B-A=0 \quad (2)\ 7(A+B)=1 \quad A=B=\frac{1}{14}$$

$$t=\frac{5}{14}\int\left(\frac{1}{7+v}+\frac{1}{7-v}\right)dv$$

$$t=\frac{5}{14}\left[\log_e\left(|7+v|\right)-\log_e\left(|7-v|\right)+c\right]$$

A1

$$t=\frac{5}{14}\left[\log_e\left(\frac{|7+v|}{|7-v|}\right)+c\right]$$

$$t=0,\quad v=0,\quad c=0$$

since $0\le v<7$ the modulus is not needed.

$$t=\frac{5}{14}\log_e\left(\frac{7+v}{7-v}\right)$$

$$\frac{14t}{5}=\log_e\left(\frac{7+v}{7-v}\right)$$

$$e^{\frac{14t}{5}}=\frac{7+v}{7-v}$$

$$e^{-\frac{14t}{5}}=\frac{7-v}{7+v}$$

$$7-v=(7+v)e^{-\frac{14t}{5}}=7e^{-\frac{14t}{5}}+ve^{-\frac{14t}{5}}$$

$$7-7e^{-\frac{14t}{5}}=v+ve^{-\frac{14t}{5}}=v\left(1+e^{-\frac{14t}{5}}\right)$$ M1

$$v=\frac{7\left(1-e^{-\frac{14t}{5}}\right)}{1+e^{-\frac{14t}{5}}}$$

Question 4

a. $P(1,2,3)$, $Q(-1,2,-1)$, $R(-1,-2,5)$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -2\underset{\sim}{i} - 4\underset{\sim}{k}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = -2\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -2 & 0 & -4 \\ -2 & -4 & 2 \end{vmatrix} = \underset{\sim}{i}\begin{vmatrix} 0 & -4 \\ -4 & 2 \end{vmatrix} - \underset{\sim}{j}\begin{vmatrix} -2 & -4 \\ -2 & 2 \end{vmatrix} + \underset{\sim}{k}\begin{vmatrix} -2 & 0 \\ -2 & -4 \end{vmatrix}$ M1

$\overrightarrow{PQ} \times \overrightarrow{PR} = -16\underset{\sim}{i} + 12\underset{\sim}{j} + 8\underset{\sim}{k} = 4\left(-4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

Let $\underset{\sim}{n} = -4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ be the normal to the plane A1

The equation of the plane using the normal through the point $P(1,2,3)$

$-4(x-1) + 3(y-2) + 2(z-3) = 0$

$-4x + 4 + 3y - 6 + 2z - 6 = 0$

$-4x + 3y + 2z = 8$

b. The plane $-4x + 3y + 2z = 8$ has a normal $\underset{\sim}{n} = -4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$ and $|\underset{\sim}{n}| = \sqrt{16+9+4} = \sqrt{29}$

the line $2 - x = \dfrac{y-4}{2} = \dfrac{z-2}{c}$, $\dfrac{x-2}{-1} = \dfrac{y-4}{2} = \dfrac{z-2}{c}$

has a direction $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j} + c\underset{\sim}{k}$ and $|\underset{\sim}{v}| = \sqrt{1+4+c^2} = \sqrt{5+c^2}$

$\underset{\sim}{n}.\underset{\sim}{v} = 4 + 6 + 2c = 10 + 2c = 2(5+c)$

let α be the angle between the line and the plane, let θ be the angle between the line and the normal to the plane, then $\alpha = \cos^{-1}\left(\dfrac{5\sqrt{29}}{29}\right) = \cos^{-1}\left(\dfrac{5}{\sqrt{29}}\right) = \sin^{-1}\left(\dfrac{2}{\sqrt{29}}\right)$

$\theta = \dfrac{\pi}{2} - \alpha = \cos^{-1}\left(\dfrac{2}{\sqrt{29}}\right)$, $\cos(\theta) = \left(\dfrac{2}{\sqrt{29}}\right)$ A1

$\cos(\theta) = \dfrac{2}{\sqrt{29}} = \dfrac{\underset{\sim}{n}.\underset{\sim}{v}}{|\underset{\sim}{n}||\underset{\sim}{v}|}$

$\dfrac{2}{\sqrt{29}} = \dfrac{2(c+5)}{\sqrt{29}\sqrt{5+c^2}}$

$\sqrt{5+c^2} = c + 5$ M1

$5 + c^2 = c^2 + 10c + 25$

$10c = -20$

$c = -2$ A1

Question 5

a. $z^5+1=0$

$z^5=-1$

$z^5=\text{cis}(\pi+2k\pi)=\text{cis}(\pi(2k+1))$

$z=\text{cis}\left(\frac{\pi(2k+1)}{5}\right)$ let $k=-2,-1,0,1,2$ M1

$k=-2 \quad z=\text{cis}\left(-\frac{3\pi}{5}\right)$

$k=-1 \quad z=\text{cis}\left(-\frac{\pi}{5}\right)$

$k=0 \quad z=\text{cis}\left(\frac{\pi}{5}\right)$ A1

$k=1 \quad z=\text{cis}\left(\frac{3\pi}{5}\right)$

$k=2 \quad z=\text{cis}(\pi)=-1$

b. $(z+1)Q(z)=0$

$(z+1)(z^4-z^3+z^2-z+1)$ expanding

$=z(z^4-z^3+z^2-z+1)+(z^4-z^3+z^2-z+1)$

$=z^5-z^4+z^3-z^2+z+z^4-z^3+z^2-z+1$ M1

$=z^5+1$

c. $u=z+\frac{1}{z}, \quad u^2=z^2+2+\frac{1}{z^2}$

u^2-u-1

$=\left(z^2+2+\frac{1}{z^2}\right)-\left(z+\frac{1}{z}\right)-1$

$=\frac{(z^4+2z^2+1)}{z^2}-\left(\frac{z^3+z}{z^2}\right)-1$ M1

$=\frac{1}{z^2}\left(z^4+2z^2+1-(z^3+z)-z^2\right)$

$=\frac{z^4-z^3+z^2-z+1}{z^2}=\frac{Q(z)}{z^2}$

$Q(z)=0 \iff u^2-u-1=0$

d. $\cos\left(\frac{3\pi}{5}\right)$ is the real part of one of the solutions to $Q(z)=0$

so that $z=\text{cis}\left(\frac{3\pi}{5}\right)=\cos\left(\frac{3\pi}{5}\right)+i\sin\left(\frac{3\pi}{5}\right)$

and $\frac{1}{z}=\text{cis}\left(-\frac{3\pi}{5}\right)=\cos\left(-\frac{3\pi}{5}\right)+i\sin\left(-\frac{3\pi}{5}\right)=\cos\left(\frac{3\pi}{5}\right)-i\sin\left(\frac{3\pi}{5}\right)$

$Q(z)=0, \quad \Rightarrow \quad u^2-u-1=0, \quad u=\frac{1\pm\sqrt{5}}{2}$

Now $\frac{3\pi}{5}=108^0$ and $\cos\left(\frac{3\pi}{5}\right)$ is in the second quadrant so $\cos\left(\frac{3\pi}{5}\right)<0$

$u=z+\frac{1}{z}=2\cos\left(\frac{3\pi}{5}\right)=\frac{1-\sqrt{5}}{2}$

Hence $\cos\left(\frac{3\pi}{5}\right)=\frac{1-\sqrt{5}}{4}$ A1

Question 6

a. $C\overset{d}{=}N\left(70,5^2\right), \quad T=C_1+C_2+C_3+C_4$ be the total of four independent carrots

$E(T)=4E(C)=4\times 70=280$

$\text{Var}(T)=4\text{Var}(C)=4\times 5^2=100 \quad \sigma_T=\sqrt{100}=10$

$T\overset{d}{=}N\left(280,10^2\right)$

$\Pr(T>292)=\Pr\left(Z>\frac{292-280}{10}\right)$ M1

$=\Pr(Z>1.2)=\frac{1}{2}(1-0.770)$

$=0.115$ A1

b. $E(\bar{C})=70, \quad \text{Var}(\bar{C})=\frac{5^2}{4}=\frac{25}{4}=\left(\frac{5}{2}\right)^2$

$\bar{C}\overset{d}{=}N\left(70,\left(\frac{5}{2}\right)^2\right)$

$\Pr(67<\bar{C}<73)=\Pr\left(\frac{67-70}{\frac{5}{2}}<Z<\frac{73-70}{\frac{5}{2}}\right)$

$=\Pr(-1.2<Z<1.2)$ A1

$=0.770$

Question 7

To prove $47^n + 53 \times 47^{n-1}$ is divisible by 100 for all $n \in N$.

Base case $n = 1$ $LHS = 47 + 53 = 100$. This is divisible by 100. A1

The statement is true when $n = 1$

Assume it is true when $n = k$ so that $47^k + 53 \times 47^{k-1}$ is divisible by 100, so that we can write $47^k + 53 \times 47^{k-1} = 100a$ where $a \in N$, so $53 \times 47^{k-1} = 100a - 47^k$ A1

Now consider

$47^{k+1} + 53 \times 47^k$

$= 47 \times 47^k + 53 \times 47 \times 47^{k-1}$

$= 47 \times 47^k + 47\left(100a - 47^k\right)$ M1

$= 47 \times 100a$

$= 100(47a)$

$= 100b$ where $b = 47a$, $b \in N$ so it is divisible by 100.

Since it is true when $n = 1$ and assuming it is true when $n = k$ then it is true when $n = k + 1$, so by the principle of mathematical induction, it is true for $n \in N$. A1

Question 8

a. $x = \log_e\left(\sec(2t) + \tan(2t)\right) - \sin(2t)$

$$\frac{dx}{dt} = \frac{\frac{d}{dt}\left(\sec(2t) + \tan(2t)\right)}{\sec(2t) + \tan(2t)} - 2\cos(2t)$$

$$\frac{dx}{dt} = \frac{2\tan(2t)\sec(2t) + 2\sec^2(2t)}{\sec(2t) + \tan(2t)} - 2\cos(2t)$$

$$\frac{dx}{dt} = \frac{2\sec(2t)\left(\sec(2t) + \tan(2t)\right)}{\sec(2t) + \tan(2t)} - 2\cos(2t)$$ M1

$$\frac{dx}{dt} = 2\sec(2t) - 2\cos(2t)$$

$$\frac{dx}{dt} = \frac{2}{\cos(2t)} - 2\cos(2t)$$

$$\frac{dx}{dt} = \frac{2\left(1 - \cos^2(2t)\right)}{\cos(2t)}$$ M1

$$\frac{dx}{dt} = \frac{2\sin^2(2t)}{\cos(2t)} = \frac{2\sin(2t)\sin(2t)}{\cos(2t)}$$

$$\frac{dx}{dt} = 2\tan(2t)\sin(2t)$$

b. $x = \log_e\left(\sec(2t) + \tan(2t)\right) - \sin(2t), \quad y = \cos(2t)$

$\frac{dx}{dt} = 2\tan(2t)\sin(2t) \qquad \frac{dy}{dt} = -2\sin(2t)$ A1

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4\tan^2(2t)\sin^2(2t) + 4\sin^2(2t)$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4\sin^2(2t)\left(1 + \tan^2(2t)\right)$

$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 4\sin^2(2t)\sec^2(2t) = 4\tan^2(2t)$

$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 2\left|\tan(2t)\right|$ A1

$0 \le t \le \frac{\pi}{4}, \quad 0 \le 2t \le \frac{\pi}{2}$ we are in the first quadrant

so therefore the modulus is not needed $\left|\tan(2t)\right| = \tan(2t)$

$S = 2\pi \int_{t_1}^{t_2} y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$S = 2\pi \int_0^{\frac{\pi}{4}} \cos(2t)\, 2\tan(2t)\, dt$

$S = 4\pi \int_0^{\frac{\pi}{4}} \sin(2t)\, dt$

$S = 2\pi\left[-\cos(2t)\right]_0^{\frac{\pi}{4}}$

$S = 2\pi\left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right)$

$S = 2\pi$ A1

Question 9

$I = \int t^3 \sin\left(t^2\right) dt$

$I = \int t.t^2 \sin\left(t^2\right) dt$

let $x = t^2 \quad \frac{dx}{dt} = 2t$

$I = \frac{1}{2}\int x \sin(x)\, dx$ A1

integration by parts

let $u = x \quad \frac{dv}{dx} = \sin(x)$

$\frac{du}{dx} = 1 \quad v = \int \sin(x)\, dx = -\cos(x)$

M1

$I = \frac{1}{2}\left[-x\cos(x) + \int \cos(x)\, dx\right]$

$I = \frac{1}{2}\left[-x\cos(x) + \sin(x)\right]$

$I = \frac{1}{2}\left(\sin\left(t^2\right) - t^2 \cos\left(t^2\right)\right) + c$ A1

End of detailed answers for the
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