Billanook College

July Exam 2016

VCE Specialist Mathematics
Examination 2

Written Examination

Question and Answer Booklet

Reading time: 15 minutes
Writing time: 1½ hour

Student's Name: ______________________________

Teacher's Name : ______________________________

Structure of Booklet

Section	Number of Questions	Number of marks
1	22	22
2	5	60
total		82

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference book, one approved CAS calculator, and one scientific calculator. Calculator memory DOES NOT need to be cleared.
Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied:
Question and answer booklet
Multiple choice answer sheet

Instructions
Write your name and teacher's name in the space provided above.
Always show your working.
All written responses should be in English

*** Please note: questions are not in order ***

Students are NOT permitted to bring mobile phones and/or any other electronic communications equipment into the examination room.

SECTION 1 - Multiple Choice Questions

Instructions for Section 1
Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions. Choose the response that is **correct** for the question. A correct answer scores 1, an incorrect answer scores 0. Marks will **not** be deducted for incorrect answers. No marks will be given if more than one answer is completed for any question. Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1

The equations of the asymptotes of the hyperbola given by the parametric equations

$$x = 5 - 6\sec(2t)\,, y = 3 + 5\tan(2t)$$

are

A. $y = \frac{5}{6}x - \frac{7}{6}$ and $y = -\frac{5}{6}x + \frac{43}{6}$

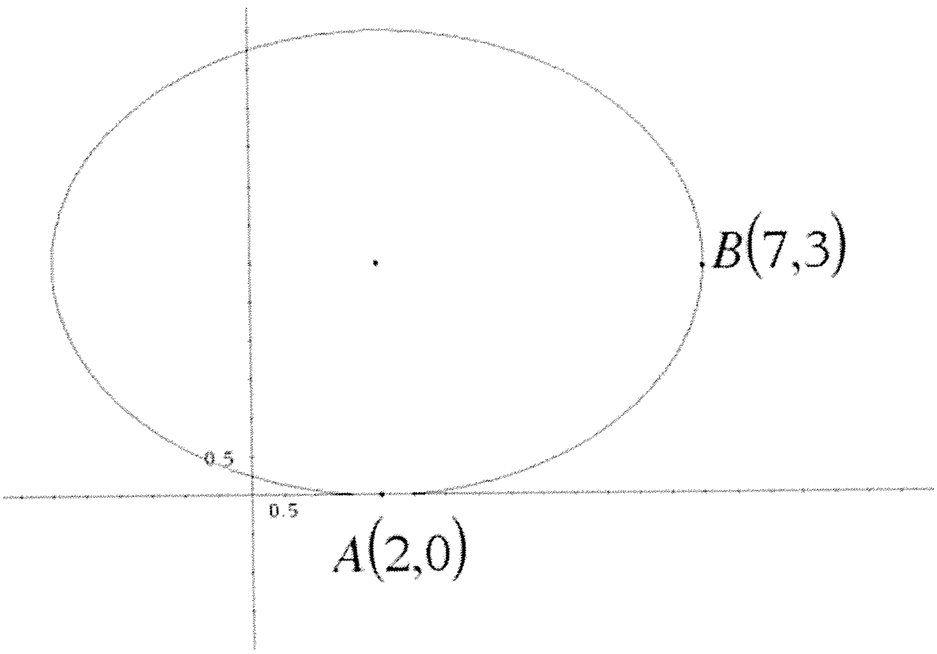
B. $y = \frac{5}{6}x$ and $y = -\frac{5}{6}x$

C. $y = \frac{5}{6}x - \frac{7}{6}$ only

D. $y = -\frac{5}{6}x - \frac{7}{6}$ and $y = \frac{5}{6}x + \frac{43}{6}$

E. $y - 3 = \pm\frac{6}{5}(x - 5)$

Question 2



SECTION 1 - continued

In the graph of an ellipse shown above, two vertices A(2, 0) and B(7, 3) are labelled. Then the equation of the ellipse is

A. $\frac{(x+2)^2}{25} - \frac{(y-3)^2}{9} = 1$

B. $\frac{(x-2)^2}{25} + \frac{(y+3)^2}{9} = 1$

C. $\frac{(x-2)^2}{25} - \frac{(y+3)^2}{9} = 1$

D. $\frac{(x+2)^2}{25} + \frac{(y-3)^2}{9} = 1$

E. $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{9} = 1$

Question 3

The graph of the function $f(x) = \frac{2x^2+3x+7}{x^2+2x+2}$ has

A. a horizontal asymptote at $y = 2$ and a minimal point at $x = -1$.

B. a horizontal asymptote at $y = 2$ and a maximal point at $x = -1$.

C. a vertical asymptote at $x = -1$ and a maximal point at $x = -1$.

D. a y-intercept at $(0, \frac{7}{3})$ and a x-intercept at $x = 3$.

E. no intersection with the horizontal asymptote $y = 2$.

Question 4

The range of the function $f(x) = 4\arctan(2 - 3x) + 5$ is

A. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

B. $(-\infty, \infty)$

C. $\left(2 - \frac{3\pi}{2}, 2 + \frac{3\pi}{2}\right)$

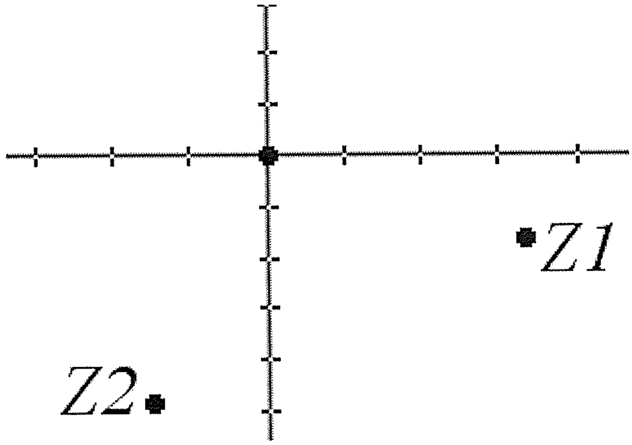
D. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

E. $(5 - 2\pi, 5 + 2\pi)$

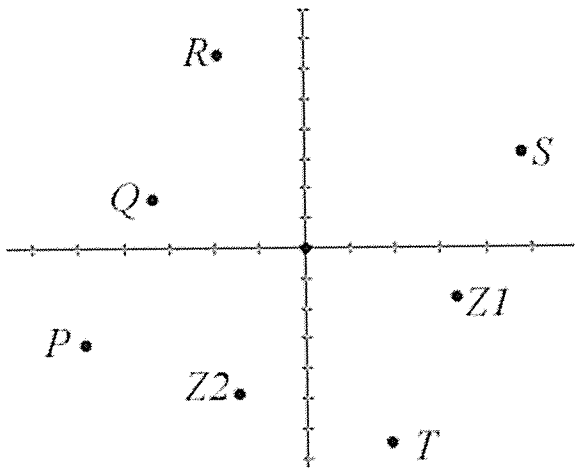
SECTION 1 - continued
TURN OVER

Question 5

Two complex numbers z_1, z_2 are shown in the axes below.



Then the point representing the correct position of $z = z_1 - z_2$ in the diagram shown below is



A. P

B. Q

C. R

D. S

E. T

SECTION 1 - continued

Question 6

The shape of the graph of the locus represented by $|z - z_1| = |z - z_2|$ is

A. an ellipse

B. a hyperbola

C. a parabola

D. a circle

E. a straight line

Question 7

Let $z = -5\sqrt{3} - 5i$.

Then the magnitude and principal argument of z^{10} are respectively

A. 10^{10} and $\left(-\frac{5\pi}{6}\right)^{10}$

B. 10^{10} and $-\frac{\pi}{3}$

C. 10^{10} and $-\frac{25\pi}{3}$

D. 100 and $-\frac{\pi}{3}$

E. 5^{10} and $-\frac{\pi}{3}$

Question 8

The complex number, $z = 4cis(72°) \cdot z_1$, can be obtained from z_1 by

A. a rotation of 72° around the origin in clockwise.

B. a rotation of 72° around the origin in anti-clockwise.

C. a rotation of 72° around the origin in anti-clockwise, followed by a dilation of factor 4 from the origin.

D. a rotation of 72° around the origin in clockwise, followed by a dilation of factor 4 from the origin.

E. a reflection about the origin.

SECTION 1 - continued
TURN OVER

Question 9

u and v are two complex numbers with $u + v = \frac{3i}{5}$, $u \times v = \frac{7}{10}$. Then u and v must be the two roots of the equation

A. $10z^2 - 6iz + 7 = 0$

B. $\left(z - \frac{3i}{5}\right)\left(z - \frac{7}{10}\right) = 0$

C. $z^2 + \frac{3i}{5}z + \frac{7}{10} = 0$

D. $10z^2 + 6iz + 7 = 0$

E. $10z^2 - 6iz - 7 = 0$

Question 10

By an appropriate substitution, the definite integral

$$\int_{\log_e\left(\frac{\pi}{6}\right)}^{\log_e\left(\frac{\pi}{2}\right)} \frac{e^x}{1 + e^{2x}}\,dx$$

is equivalent to

A. $\int_{\log_e\left(\frac{\pi}{6}\right)}^{\log_e\left(\frac{\pi}{2}\right)} \frac{u}{1+u^2}\,du$

B. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{1+u^2}\,du$

C. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{u}{1+u^2}\,du$

D. $\int_{\log_e\left(\frac{\pi}{6}\right)}^{\log_e\left(\frac{\pi}{2}\right)} \frac{1}{1+u^2}\,du$

E. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{u}{1+u}\,du$

Question 11

$G(x)$ is a differentiable function over real numbers. $g(x)$ is the derivative function of $G(x)$ and $\int_1^5 g(x)\,dx = 8$. $(1,\ 2)$ is a point on the graph of $G(x)$. Then the value of $G(5)$ is

A. 5

B. 2

C. 8

D. 1

E. 10

Question 12

A type of chemical solution with concentration 12g/L flows into a cylindrical tank at a rate of 2L per minute. This tank initially has 85 litres of the same type of solution. At the same time the solution flows out of the tank at a rate of 1.5 L per minute. Let x g be the amount of chemical in the tank after t minutes. Then a correct differential equation regarding x is

A. $\frac{dx}{dt} = 24 - \frac{3x}{170-t}$

B. $\frac{dx}{dt} = 24 - \frac{1.5x}{85}$

C. $\frac{dx}{dt} = 24 + \frac{1.5x}{85+0.5t}$

D. $\frac{dx}{dt} = 24 - \frac{3x}{170+4t}$

E. $\frac{dx}{dt} = 24 - \frac{3x}{170+t}$

Question 13

The graph of $y = \dfrac{-x^2+1}{2x}$ has

A. no straight line asymptotes.

B. $y = 2x$ as its only straight line asymptote.

C. $x = 0$ as its only straight line asymptote.

D. $y = 0 \quad and \quad y = -\frac{1}{2}x$ as its only straight line asymptotes.

E. $x = 0 \quad and \quad y = -\frac{1}{2}x$ as its only straight line asymptotes.

SECTION 1 - continued
TURN OVER

Question 14

An antiderivative of $\dfrac{2}{\sqrt{4-x^2}}$ could be:

A. $\cos^{-1}\left(\dfrac{x}{2}\right)$

B. $2\cos^{-1}\left(\dfrac{x}{2}\right)$

C. $\sin^{-1}\left(\dfrac{x}{2}\right)$

D. $1-2\cos^{-1}\left(\dfrac{x}{2}\right)$

E. $\dfrac{1}{2}\sin^{-1}\left(\dfrac{x}{2}\right)$

Question 15

Three vectors given by $\underset{\sim}{u} = \underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{v} = 2\underset{\sim}{i} + a\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{w} = 5\underset{\sim}{i} + 5\underset{\sim}{j} + a\underset{\sim}{k}$ are linearly dependent. Then the possible values of a are

A. 1, 4

B. 2, 5

C. 3, 6

D. 0, 3

E. −1, 2

SECTION 1 - continued

Question 16

If θ is the angle between $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + \underset{\sim}{j} - \sqrt{11}\,\underset{\sim}{k}$ then $\tan(2\theta)$ is

A. $\frac{5}{13}$

B. $\frac{9}{40}$

C. $-\frac{120}{119}$

D. $\frac{40}{169}$

E. $\frac{120}{169}$

Question 17

The velocity of a particle is $\underset{\sim}{v} = \frac{e^t+e^{-t}}{2}\underset{\sim}{i} + \frac{e^t-e^{-t}}{2}\underset{\sim}{j} + 2t\underset{\sim}{k}$. The initial position of the particle is $\underset{\sim}{r}(0) = 2\underset{\sim}{j} + 2\underset{\sim}{k}$. Then the position of the particle at time t is

A. $\frac{e^t-e^{-t}}{2}\underset{\sim}{i} + \frac{\left(e^{\frac{t}{2}}+e^{-\frac{t}{2}}\right)^2}{2}\underset{\sim}{j} + (t^2+2)\underset{\sim}{k}$

B. $\frac{e^t-e^{-t}}{2}\underset{\sim}{i} + \frac{e^t+e^{-t}}{2}\underset{\sim}{j} + t^2\underset{\sim}{k}$

C. $\frac{e^t-e^{-t}}{2}\underset{\sim}{i} + \frac{e^t+e^{-t}}{2}\underset{\sim}{j} + 2\underset{\sim}{k}$

D. $\frac{e^t-e^{-t}}{2}\underset{\sim}{i} + \frac{e^t+e^{-t}}{2}\underset{\sim}{j} + (t^2+2)\underset{\sim}{k}$

E. $\frac{e^t+e^{-t}}{2}\underset{\sim}{i} + \frac{e^t-e^{-t}}{2}\underset{\sim}{j}$

SECTION 1 - continued
TURN OVER

Question 18

If $z = 3 + 2i$ then $\frac{\bar{z}}{z}$ is equal to:

A. 13

B. $\frac{5-12i}{5}$

C. $\frac{5-12i}{13}$

D. $\frac{13-12i}{5}$

E. $\frac{13-12i}{13}$

Question 19

The position vector, in metres of a particle is given by

$$\underset{\sim}{r}(t) = (8\sin(3t) + 2)\underset{\sim}{i} + (5 - 15\cos(3t))\underset{\sim}{j},\ t \geq 0.$$

Then the minimum speed of the particle is

A. 289

B. 51

C. 24

D. 66

E. 29

SECTION 1 - continued

Question 20

The implied domain and range of $\sin^{-1}\left(\frac{x-1}{2}\right)$ respectively are:

A. $\left[-\frac{1}{2},\ \frac{1}{2}\right]$ and $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$

B. $[-2,\ 2]$ and $\left[-\frac{\pi}{4},\ \frac{\pi}{4}\right]$

C. $[-2,\ 2]$ and $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$

D. $\left[\frac{1}{2},\ \frac{3}{2}\right]$ and $\left[-\frac{\pi}{4},\ \frac{\pi}{4}\right]$

E. $[-1,\ 3]$ and $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$

Question 21

The acceleration, in m/s^2, of a particle is given by $a = ve^{\frac{x^2}{100}}$, where v and x are the velocity and displacement t seconds after $t = 0$. $v = 3\ m/s$ when $x = 1$. Then the velocity when $x = 5$ is approximately

A. 5.45 m/s

B. 7.45 m/s

C. 4.45 m/s

D. 4.12 m/s

E. 14.70 m/s

Question 22

The velocity of a particle travelling on a straight line is given by

$$v(t) = t^3 - 4t^2 - 4t + 16, t \geq 0.$$

Then the displacement of the particle from the initial position when $t = 6$ is

A. $\frac{220}{3}\ m$

B. 76 m

C. 64 m

D. 60 m

E. 48 m

END OF SECTION 1

TURN OVER

SECTION 2- Extended Response questions

Instructions for Section 2

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m/s^2, where $g = 9.8$.

Question 1 (12 marks)

A curve is defined by the parametric equations

$$x = 2 + 3\tan(t), \qquad y = 4\sec(t), \qquad t \in [0, 2\pi]$$

a. Find the Cartesian equation of the curve.

2 marks

b. State the equations of the asymptotes.

2 marks

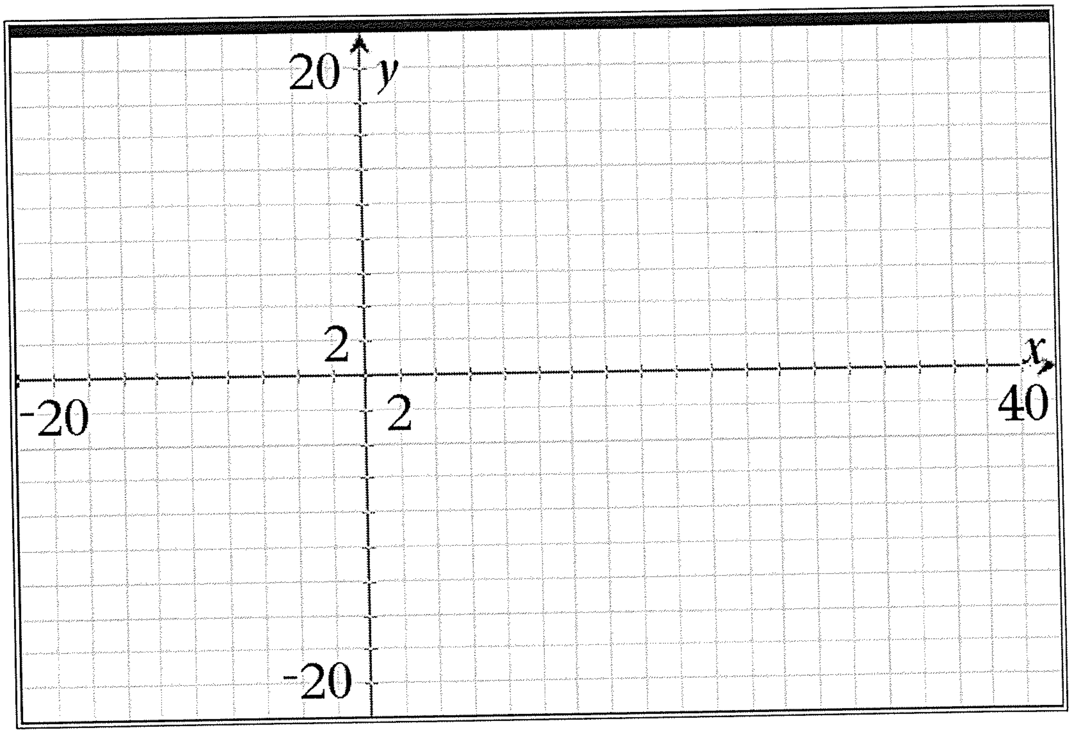
c. Find the equations of the tangent at $(6, \frac{20}{3})$

2 marks

SECTION 2- Question 1- continued

d. Sketch the graphs of the curve, showing the asymptotes, the stationary points and the tangents found in Part c.

4 marks



A solid revolution is formed by rotating the region bounded by the x-axis and the curve for $x \in [0, 4]$ about the x-axis.

e. Write down a definite integral in terms of x that gives the volume of the solid of revolution.

1 mark

____

____

f. Find the volume of this solid, correct your answer to two decimal places.

1 mark

____

____

SECTION 2- continued
TURN OVER

Question 2 (10 marks)

a. Show that $z = 2\sqrt{2} + 2\sqrt{2}i$ is a solution of the equation $z^4 = -256$.

2 marks

b. Find the other solutions of $z^4 = -256$ in the form of $z = rcis(\theta)$.

2 marks

c. Find the Cartesian equation of the locus $|z - 2\sqrt{2} - 2\sqrt{2}i| = |z + 2\sqrt{2} + 2\sqrt{2}i|$

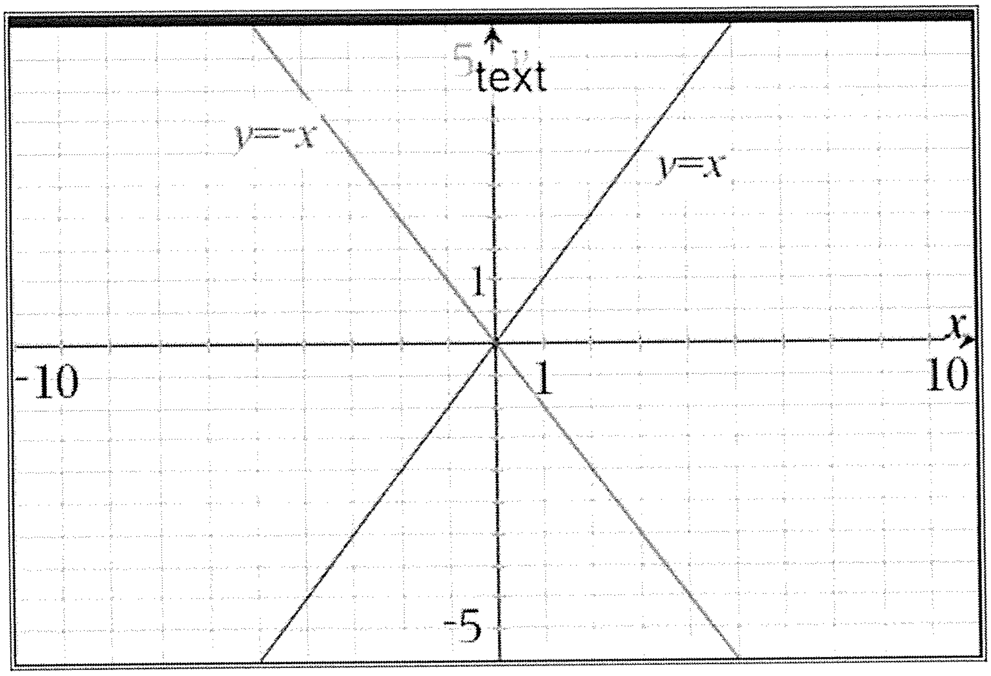
2 marks

SECTION 2- Question 2- continued

d. Sketch the graphs of the sets of points in the set of axes below, where two axes of symmetry are shown.

$$S = \{z \in C, |z| = |2\sqrt{2} + 2\sqrt{2}i|\}$$

1 mark



e.

i. Show that $z = 2\sqrt{2} + 2\sqrt{2}i$ is a point on the ellipse represented by the equation

$$\left|z - \frac{8\sqrt{42}}{7}\right| + \left|z + \frac{8\sqrt{42}}{7}\right| = 16$$

SECTION 2- Question 2- continued
TURN OVER

ii. Hence find the Cartesian form of the equation of the locus in part i.

$1 + 2 = 3$ marks

SECTION 2- continued

Question 3 (12 marks)

Let $\overrightarrow{OA} = \underset{\sim}{i}, \overrightarrow{OB} = \underset{\sim}{j}$. M is the middle point between O and A.

a. Find $|\overrightarrow{MB}|$.

1 mark

b. $\overrightarrow{ON}$ is in the opposite direction to $\overrightarrow{OA}$ with $|\overrightarrow{MN}| = |\overrightarrow{MB}|$. Find an expression for $\overrightarrow{ON}$ in the form $\overrightarrow{ON} = a\underset{\sim}{i}$.

2 marks

c. Find the magnitude of $\overrightarrow{BN}$.

1 mark

SECTION 2- Question 3- continued
TURN OVER

Let $\overrightarrow{OC} = -\frac{\sqrt{10+2\sqrt{5}}}{4}\underset{\sim}{i} + \frac{\sqrt{5}-1}{4}\underset{\sim}{j}$.

d. Show that the angle between $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is $\frac{2\pi}{5}$, given that $\cos\left(\frac{\pi}{5}\right) = \frac{\sqrt{5}+1}{4}$

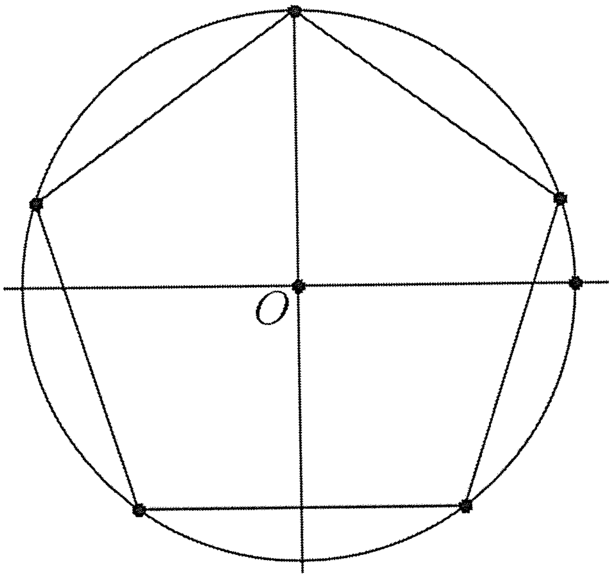
3 marks

e. Show that $|\overrightarrow{BC}| = |\overrightarrow{BN}|$.

3 marks

SECTION 2- Question 3- continued

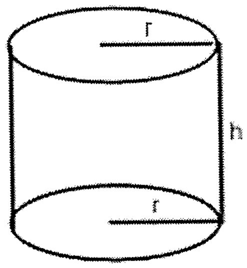
f. A regular pentagon is inscribed in a unit circle, shown below. Find the length of the sides of the pentagon.



2 marks

SECTION 2- continued
TURN OVER

Question 4 (9 marks)



In a factory a cylindrical container is made by expanding a cylindrical model. The radius r of the base is increasing at a rate of $\frac{r+2}{r+1}$ cm/s and the height h is increasing at a rate of $\frac{h}{1+h}$ cm/s.

a. Find the rate of change of the height h relative to the radius r, in terms of h and r.

2 marks

b. Find the rate which the volume of the container is increasing at per second when $h = 8$ cm and $r = 5$cm.

3 marks

SECTION 2- Question 4- continued

When the container is completed, it is filled with 400L of fresh water. A salt solution with concentration of 40g/L flows into this container at a rate of 8L per minute and the solution in the container is stirred constantly. The mixed solution flows out from a hole on the bottom at the same rate. Let x be the quantity in grams of the salt in the container after t minutes.

c. Write down a differential equation representing the change of the quantity of the salt in the container.

2 marks

d. Find the time when there is 200g of salt in the container, rounding your answer to 2 decimal places.

2 marks

Question 4 5

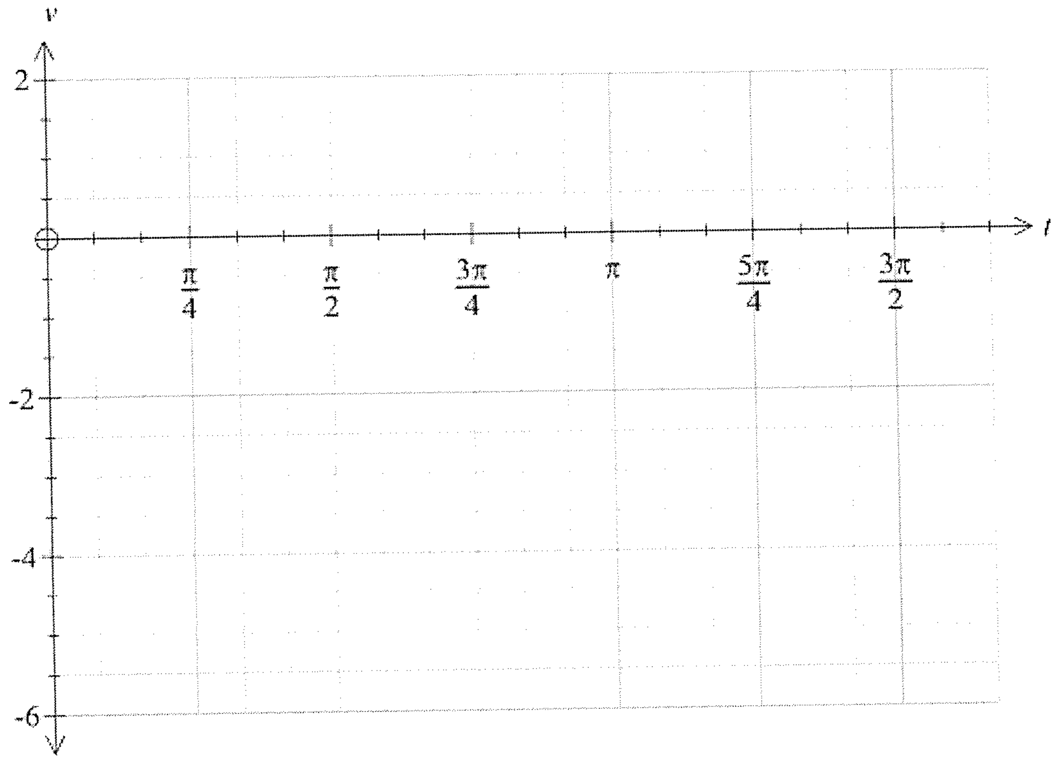
A particle is moving in a straight line with velocity v ms^{-1}. At any time, t seconds the velocity of the particle is given by the differential equation $2\frac{dv}{dt}+v^2+1=0$, where $0 \le t < a$.

a. If $v=1$ when $t=0$, use calculus methods to show that $v=\tan\left(\frac{\pi}{4}-\frac{t}{2}\right)$.

3 marks

SECTION 2 – continued

b. If $t = a$ is an asymptote, where $\pi \le a \le 2\pi$, find a.
Hence sketch a graph of v against t on the axes provided for $0 \le t < a$. Clearly show the coordinates of any intercepts, and the equation of the asymptote $t = a$.



3 marks

c. Find, correct to two decimal places, the distance travelled by the particle in the first $\frac{5\pi}{4}$ seconds.

1 mark

SECTION 2 – continued
TURN OVER

d. Find the time taken, correct to two decimal places, for the particle to travel a distance of 0.75 metres.

2 marks

e. Given that $x = 0$ when $t = 0$, use calculus methods to find an expression for x, the displacement of the particle, in terms of t.

4 marks

SECTION 2 – continued

f. Find, correct to two decimal places, the position of the particle after $\frac{5\pi}{4}$ seconds.

1 mark

g. Show that the position, x in terms of v is $x = \log_e\left(\frac{2}{v^2+1}\right)$.

3 marks
Total 17 marks

END OF QUESTION AND ANSWER BOOK