Billanook College

September Exam 2016

VCE Specialist Mathematics
Examination 1

Written Examination

Question and Answer Booklet

Reading time: 15 minutes
Writing time: 1 hour

Student's Name: ________________________________

Structure of Booklet

Section	Number of Questions	Number of marks
Exam 1	10	40

Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
No calculator is permitted in this examination.

Materials supplied:
Question and answer booklet
Formula sheet.

Instructions
Write your name and teacher's name in the space provided above.
Always show your working.
All written responses should be in English

Students are NOT permitted to bring mobile phones and/or any other electronic communications equipment into the examination room.

Question 1 (4 marks)

Let $z_1 = 2i$.

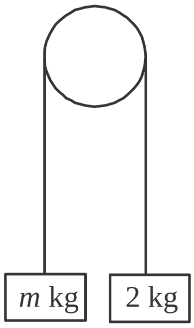
a. Simplify $\dfrac{\bar{z}_1}{1+i}$. 1 mark

b. Show that z_1 is a solution to the equation $z^3 - 3z^2 + 4z - 12 = 0$. 1 mark

c. Find all the solutions to this equation. 2 marks

Question 2 (4 marks)

A light inextensible string passes over a smooth pulley with particles of mass m kg and 2 kg attached to the ends of the string as shown.



When released, the acceleration of the 2 kg particle downwards is 1 ms^{-2}.

a. Find the value of m, in terms of g. 2 marks

The system is reset and the m kg particle is replaced by a 3 kg particle. The acceleration of the 2 kg particle is now a ms^{-2} upwards.

b. Find the value of a, in terms of g. 2 marks

Question 3 (3 marks)

Given that $f(x) = \arcsin(3x)$, find $f''\left(\frac{1}{6}\right)$.

Question 4 (3 marks)

Consider the relation $xy^2 + y\log_e(x) - 2y - 3 = 0, \quad x > 0$. Evaluate $\frac{dy}{dx}$ at the point (1, 3).

Question 5 (6 marks)

A company operates a telephone helpline for its clients. The waiting time, in minutes, for clients using the helpline is normally distributed with mean μ.
A random sample is taken of 100 of the company's clients who use the helpline and an approximate 95% confidence interval for μ is $15.02 < \mu < 16.98$.

a. Find the mean $\bar{x}$ and the standard deviation s for this sample. 2 marks

The company upgrades its telephone helpline by outsourcing it to a call centre business.
The waiting times, in minutes, for callers to this call centre are normally distributed with an advertised mean waiting time of 5.2 minutes and a standard deviation of 1.5 minutes.
After the upgrade, the company is suspicious that the waiting times on average are longer than those suggested by the advertising. It randomly samples 25 of its clients and finds that the mean waiting time is 5.8 minutes. Assume that the population standard deviation remains at 1.5 minutes.

b. **i.** Write down appropriate null and alternative hypotheses to test whether the mean waiting time is longer than that advertised by the call centre business. 1 mark

ii. Find the p value for this test. 2 marks

iii. Hence explain whether or not the null hypothesis should be rejected or not rejected at the 5% level of significance. 1 mark

Question 6 (3 marks)

Evaluate $\int_0^{\frac{\pi}{4}} \cos^2(x)\sin(2x)dx$.

Question 7 (3 marks)

The region in the first quadrant enclosed by the graph with equation $y = \frac{3}{\sqrt{x^2+4}}$, the coordinate axes and the line with equation $x = 2$, is rotated about the x–axis to form a solid of revolution.

a. Write down a definite integral that gives the volume of this solid of revolution. 1 mark

b. Find the volume of the solid of revolution. 2 marks

Question 8 (5 marks)

Consider the three vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} + m\,\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\underset{\sim}{c} = -\underset{\sim}{i} + 3\,\underset{\sim}{j}$ where $m \in R$.

a. Find the value of *m* such that $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$. 1 mark

b. Consider the vector $\underset{\sim}{d}$ which has a magnitude of 6, is parallel to $\underset{\sim}{a}$, and runs in the same direction as $\underset{\sim}{a}$.

Find $\underset{\sim}{d}$. 2 marks

c. Find a value of *m* such that $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are **linearly dependent**. 2 marks

Question 9 (5 marks)

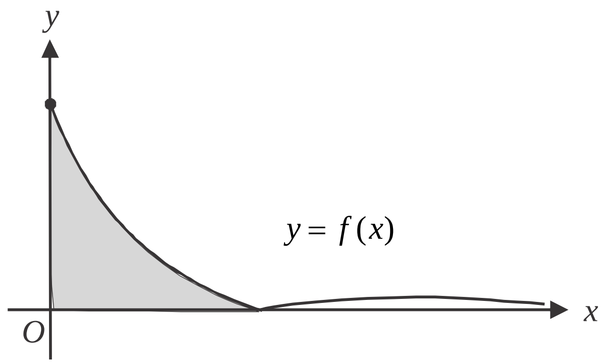
The solution to the differential equation $\frac{dy}{dx} = \frac{x\sqrt{x^2 - 1}}{e^{2y}}$, where $y = 0$ when $x = 1$, is given by

$y = \log_e \sqrt{\frac{a}{b}(x^2 - c)^{\frac{b}{a}} + c}$ where *a*, *b* and *c* are integers. Find the values of *a*, *b* and *c*.

Question 10 (4 marks)

Let $f : [0, \infty) \to R,\ f(x) = \left| \frac{x - 1}{(x + 1)(x^2 + 1)} \right|$.

The graph of *f* is shown below.



The region enclosed by the graph of *f* and the *x* and *y* axes is shaded. Find the area of this shaded region.

Specialist Mathematics Formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder	$2\pi rh$
volume of a cylinder	$\pi r^2 h$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
volume of a sphere	$\frac{4}{3}\pi r^3$
area of a triangle	$\frac{1}{2}bc\sin(A)$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Circular (trigonometric) functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	
$\sin(2x) = 2\sin(x)\cos(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$

Circular (trigonometric) functions – continued

Function	$\sin^{-1}(\text{arcsin})$	$\cos^{-1}(\text{arccos})$	$\tan^{-1}(\text{arctan})$
Domain	$[-1, 1]$	$[-1, 1]$	R
Range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\text{cis}(\theta)$	
$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	$-\pi < \text{Arg}(z) \le \pi$
$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$
$z^n = r^n \text{cis}(n\theta)$ (de Moivre's theorem)	

Probability and statistics

for random variables X and Y	$\text{E}(aX + b) = a\text{E}(x) + b$ $\text{E}(aX + bY) = a\text{E}(x) + b\text{E}(Y)$ $\text{var}(aX + b) = a^2\text{var}(X)$
for independent random variables X and Y	$\text{var}(aX + bY) = a^2\text{var}(X) + b^2\text{var}(Y)$
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$
distribution of sample mean $\bar{X}$	mean $\text{E}(\bar{X}) = \mu$ variance $\text{var}(\bar{X}) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$	$\int x^n dx=\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$	$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$	$\int \frac{1}{x}dx=\log_e\lvert x\rvert+c$
$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$	$\int \sin(ax)\,dx=-\frac{1}{a}\cos(ax)+c$
$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$	$\int \cos\ (ax)\,dx=\frac{1}{a}\sin(ax)+c$
$\frac{d}{dx}\left(\tan(ax)\right)=a\sec^2(ax)$	$\int \sec^2(ax)\,dx=\frac{1}{a}\tan(ax)+c$
$\frac{d}{dx}\left(\sin^{-1}(x)\right)=\frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+c,\ a>0$
$\frac{d}{dx}\left(\cos^{-1}(x)\right)=\frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}}dx=\cos^{-1}\left(\frac{x}{a}\right)+c,\ a>0$
$\frac{d}{dx}\left(\tan^{-1}(x)\right)=\frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2}dx=\tan^{-1}\left(\frac{x}{a}\right)+c$
	$\int (ax+b)^n dx=\frac{1}{a(n+1)}(ax+b)^{n+1}+c,\ n\neq -1$
	$\int (ax+b)^{-1}dx=\frac{1}{a}\log_e\lvert ax+b\rvert+c$
product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$
Euler's method	If $\frac{dy}{dx}=f(x),\ x_0=a$ and $y_0=b,$ then $x_{n+1}=x_n+h$ and $y_{n+1}=y_n+hf(x_n)$
acceleration	$a=\frac{d^2x}{dt^2}=\frac{dv}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$
arc length	$\int_{x_1}^{x_2}\sqrt{1+(f'(x))^2}\,dx$ or $\int_{t_1}^{t_2}\sqrt{(x'(t))^2+(y'(t))^2}\,dt$

Vectors in two and three dimensions

$\underset{\sim}{r}=x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$
$\lvert\underset{\sim}{r}\rvert=\sqrt{x^2+y^2+z^2}=r$
$\dot{\underset{\sim}{r}}=\frac{d\underset{\sim}{r}}{dt}=\frac{dx}{dt}\underset{\sim}{i}+\frac{dy}{dt}\underset{\sim}{j}+\frac{dz}{dt}\underset{\sim}{k}$
$\underset{\sim}{r_1}.\underset{\sim}{r_2}=r_1r_2\cos(\theta)=x_1x_2+y_1y_2+z_1z_2$

Mechanics

momentum	$\underset{\sim}{p}=m\underset{\sim}{v}$
equation of motion	$\underset{\sim}{R}=m\underset{\sim}{a}$