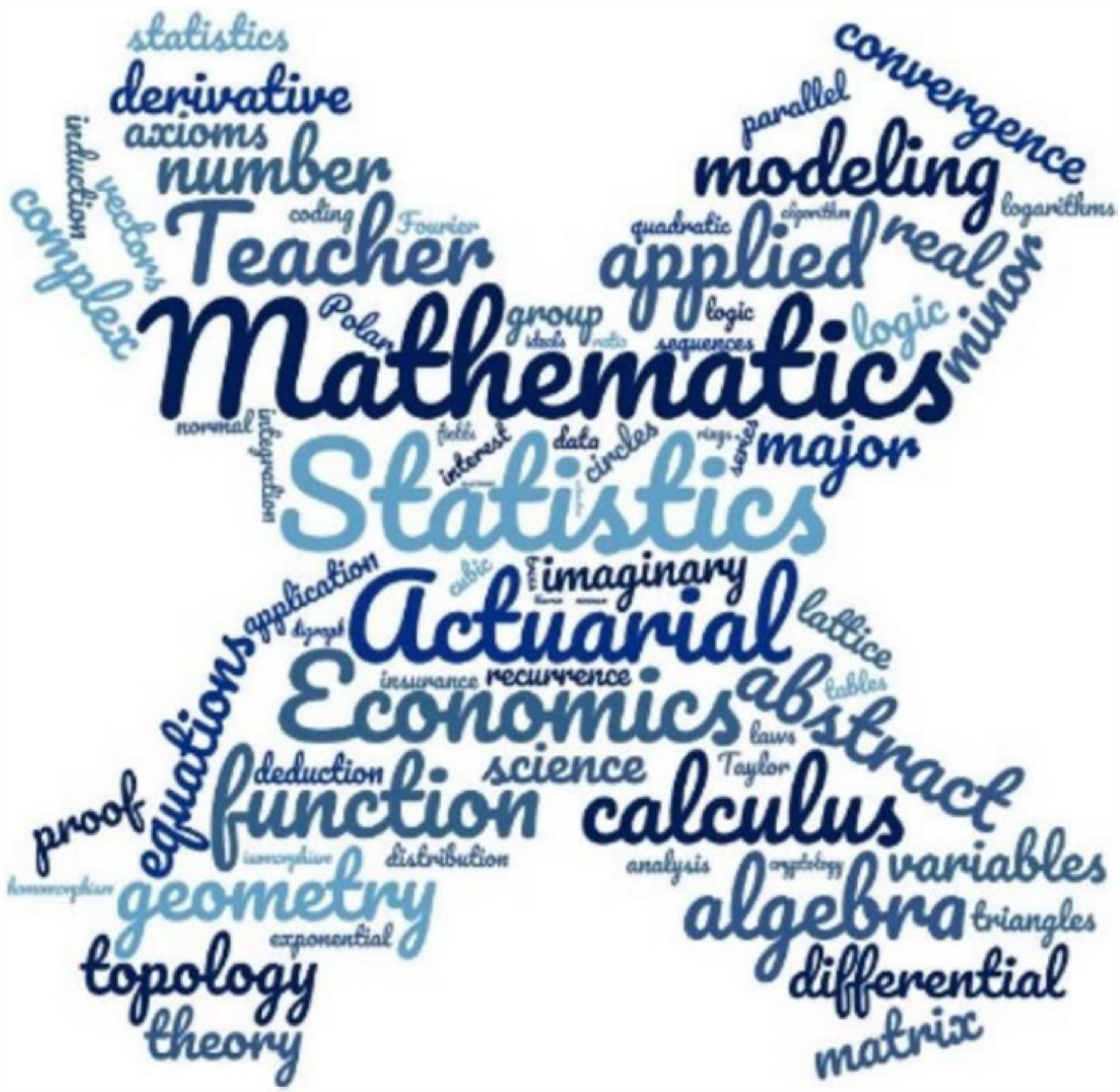
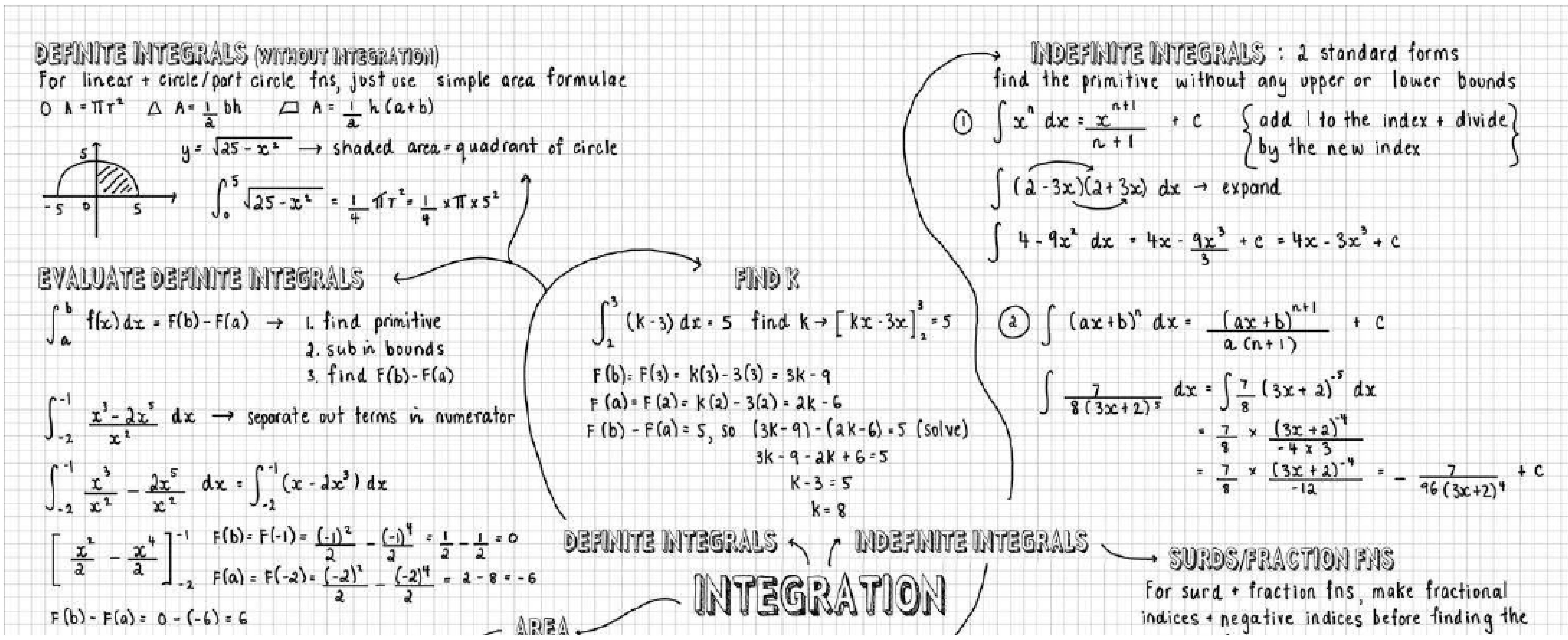


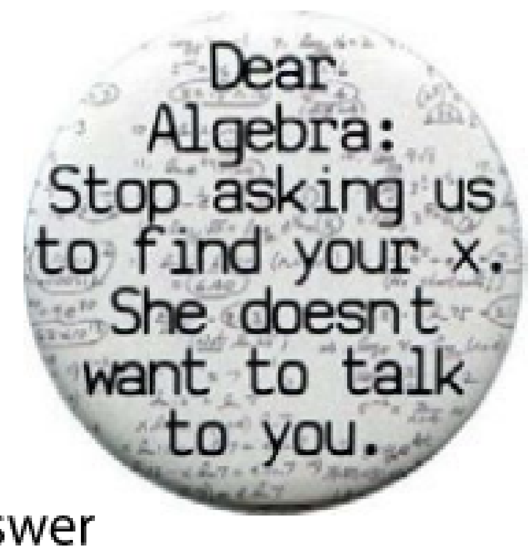
PREPARING A MARK BOOSTING SET OF REFERENCE MATERIALS FOR MATHEMATICAL METHODS EXAM 2

Preparing a Mark Boosting Set of Reference Materials for Mathematical Methods Exam 2



Students studying **Mathematical Methods** or **Specialist Mathematics** are required to sit two examinations at the end of the year. Between them they cover the whole year's work.

Examination 1 is a one hour technology free paper (no calculators or notes allowed) and consists of a series of short answer and some shorter extended answer questions. This examination is worth 22% of the marks used to determine each mathematics Study Score. A formula sheet will be provided to students in this examination.



Examination 1 is reasonably straight forward, but requires that students possess strong algebraic and arithmetic skills (including fractions and surds), as well as excellent graphing skills. The new Study Design for 2016-2020 has changed the nature of Exam 1 so that some shorter multi step extended answer questions could be included to be done tech free in Exam 1.

Examination 2 is a two hour "open book" examination (calculators and notes allowed) and consists of a series of multiple choice and extended answer questions. This examination is worth 44% of the marks used to determine each mathematics Study Score. Students are allowed to bring into this examination, unlimited pages of pre-prepared notes or a text book. These materials are referred to as **"Reference Materials"**. The same formula sheet that was provided in Exam 1 will also be provided to students in this examination. You will find that the formula sheet for Mathematical Methods is much briefer than that for Specialist Mathematics.

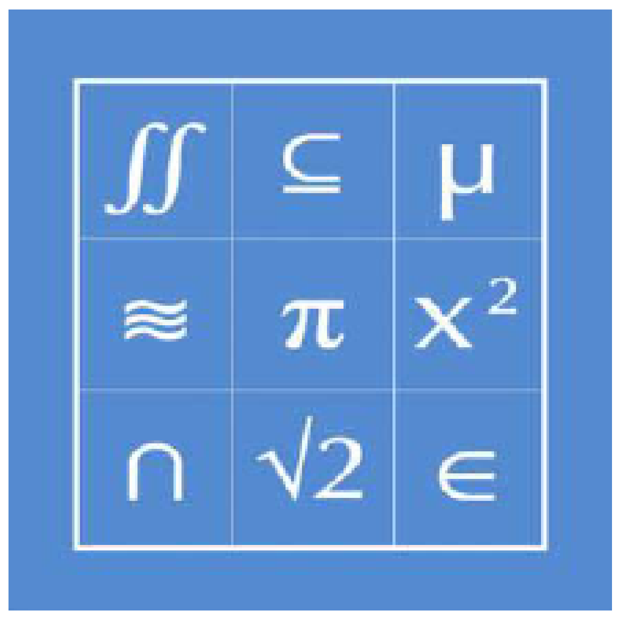
The multiple-choice questions in Examination 2 are **usually** written so that the use of a CAS calculator is not particularly of great benefit, although a strong familiarity with the calculator provides an enormous benefit to short cutting a solution.

The answers to questions in both examinations must be given in exact values (unless otherwise specified). This means that once an exact value is found **DO NOT** continue on to decimalise it. When more than 1 mark is allocated to a question, appropriate working must be shown. This means that students will be often be required to show working steps in order to obtain full marks.

The extended answer questions incorporate applications that are not usually detailed in text books, and will require students to apply their knowledge of taught mathematical principles to applications they have not previously been exposed to.

Criteria for Acceptable Reference Materials

The new Study Design for 2016-2020 states that the reference materials ***may be a textbook (which can be annotated), a securely bound lecture pad, a permanently bound student-constructed set of notes without fold-outs or an exercise book.***



(a) Materials must be in book format of A4 size or smaller when closed.

(b) There is no page limit for the Reference Materials. As long as the materials are contained in one book or bound item – they will be accepted.

(c) **ONLY ONE** bound set of notes is allowed into the open-book examinations. A single textbook is allowed as the reference material, although not advised.

(d) Materials must be held together via a single horizontal or vertical spine.

(e) Pages must be permanently bound and securely attached to the spine. They **must not** be able to be detached from the Reference Materials during the exam. This means that you cannot use Reference Materials that are presented in a form that will enable you to modify the content by adding or removing pages or tabs. Be careful particularly of the A-4 lecture pad that has a fine perforated line near the spine. These are NOT allowed.

Forms of collation/binding that are not acceptable include:

- Ring-binder folders.
- Plastic A4 slips (permanent or removable) into which pages may be inserted or removed.
- Manila folders and similar folders with clip, clamp, slide and metal prong style binding of loose-leaf material.
- Glued lecture pads.
- Bound books that have perforations designed so as to allow pages to be detached. If one or more pages can be or are detached from the Reference Materials, the entire Reference Materials will be confiscated by the examination supervisor, and the incident will be reported to the VCAA as a breach of rules.

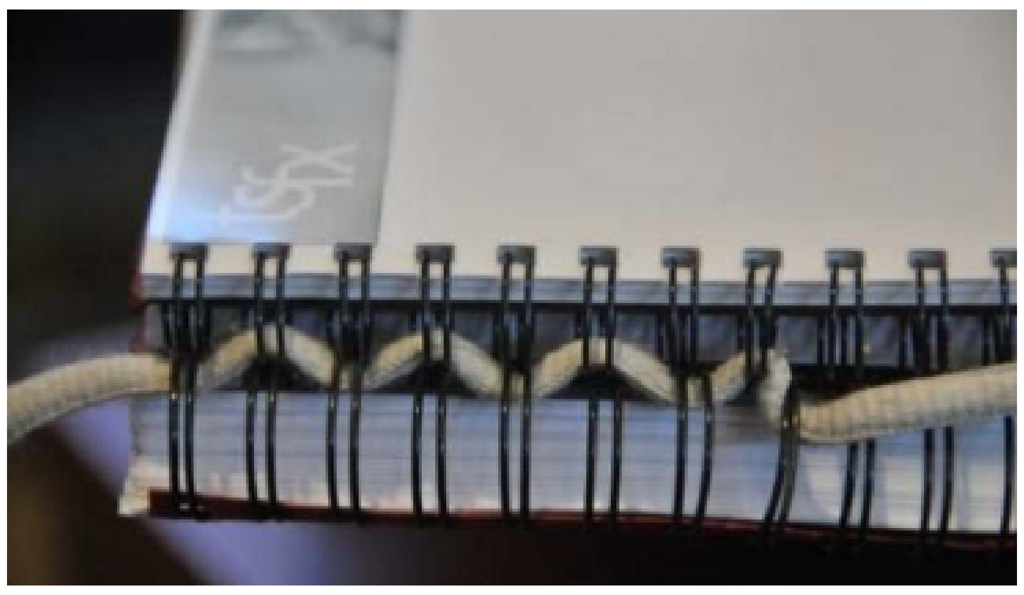
Acceptable "Reference Materials" include:

Text books or your own prepared permanently bound notes. For example:

- A securely bound lecture pad.
- A permanently bound student-constructed set of notes without fold-outs or sticky tabs.
- An exercise book with cloth, glue or staple binding.
- An A4 (or smaller) sized spiral bound exercise books (without perforations).
- Lecture pads that have been permanently bound by stapling or some other means.
- A4 sheets of paper that have been appropriately bound (plastic comb binding is acceptable). Materials can be bound at "Officeworks"

Note:

Students **MAY** securely bind individual units together to make an acceptable set of Reference Materials. For example, a series of exercise books may be permanently glued together and then secured/joined with a single spine such as wide industrial tape that is **PERMANENTLY** glued to the spines of each book. Multiple coil bound books can be transformed into acceptable Reference Materials by **PERMANENTLY** gluing the covers together, and then "sewing" the spines of the books together with thick string or wire, as illustrated below.



Further Notes:

Different schools have different interpretations as to what constitutes an "acceptable set of Reference Materials". Therefore, it is important that students confirm what is "acceptable" before the exams.

(a) No fold-outs, maps or brochure style components are permitted.

(b) You cannot attach removable tabs, post-it notes or other non-securely bound items to the Reference Materials.

(c) You are allowed to firmly attach (e.g. by glue, adhesive tape or staples) additional materials to the pages in the bound reference. Be careful to glue every side and corner. No curling pages allowed.

(d) Reference Materials can be annotated (have your handwritten comments/additional notes written in). They can also include photocopied pages of notes from your teacher.

(e) You are allowed to draw attention to sections of the Reference Materials by folding or cutting page corners, or by colour coding pages.

(f) Students are allowed to insert coloured dividers into their prepared materials.

The rules relating to acceptable "Reference Materials" continue to be refined and clarified by the VCAA. We therefore recommend that students read the advice provided by VCAA at the beginning of each school term.
Many schools check and 'sign off' the notes in the week of the exams. This makes entering into the exam room much easier.

Preparing an Effective Set of Reference Materials

The "open book" style of testing in the mathematics subjects has resulted in more difficult questions and applications in the technology active examinations.

These examinations usually emphasise applications that are **NOT** typically covered in text books, and hence, we strongly recommend that students prepare their own Reference Materials rather than taking their text book into the open-book exams.

The fact that the Reference Materials have no page limit and there are no restrictions regarding the type of materials that can be included suggests that VCAA is confident that no amount of notes or resources will be of significant benefit to those students who do not know their course materials.

But is this really the case?

Each year, the majority of students derive only a small benefit from their Reference Materials in the actual exams. However, a carefully and cleverly constructed set of Reference Materials can make a huge impact on examination scores.

So what should students include in the mathematics Reference Materials in order to make these notes powerful resources in the final exams?

Your materials should include the following:

- Summaries of vital concepts and rules.
- Guidelines as to when each rule can be applied.
- Step-by-step instructions and clear methodologies with sample calculations that you can fall back on if you develop "mind blanks" in the examination.
- Limitations of rules and methodologies.
- Exceptions to rules and methodologies.
- Flow charts that describe how to tackle complex analytical style questions.
- Summaries of every examinable technique (including skills and concepts from prerequisite Units) as well as clear explanations as to what each technique is used for. This summary will serve as a valuable prompt that may help determine which approach to take when asked to solve questions that students have not been previously exposed to.
- Details of the common errors made by students, reminders as to how exam writers try to trick/trap students, as well as lists of watch-outs by topic. Students may then peruse the appropriate "Tricks and Watch-out List" before attempting questions on the examination. This will ensure that valuable marks are not lost during the course of the exam(s).

For example:

The Logarithmic Functions section of the Mathematical Methods materials should include a reminder to "check the validity of answers" before leaving logarithmic equation questions. Many students lose marks in this topic as they forget to substitute the calculated values of x into the given equation so as to determine whether the answers are defined (the log of zero or a negative number is undefined). In fact – students should check the validity of all answers, as anomalies may arise with every function type.

- Difficult/challenging sample extended answer examination questions that have been used in recent year's CAS-active examinations.

- CAS calculator methodologies and usual syntax, particularly syntax that has given you trouble in the past. This may be, for Maths Methods, how to use inverse Binomial CDF to find a probability of success or how to use factor or rfactor to find a root of a graph. Or in Specialist Maths, how to use your CAS to find partial fractions, use propfrac to long divide, or to solve a DE in one step.

Detailed Instructions Regarding the Preparation of Reference Materials

For ease of use, Reference Materials should consist of separate "Theory Sections" as well as "Question Banks".

The Theory Section(s)

The Theory Section(s) should incorporate the relevant theory, carefully selected worked examples (to protect you from mind blanks and stress moments), methodologies/step-by-step instructions, prompts (to help you determine a solution process), watchouts and common errors lists (to protect you from losing valuable marks in the exam), as well as a clear and detailed index that will enable you to quickly and easily locate information. Notes that have been prepared throughout the year and not at the last moment are much easier to use because of familiarity.

Suggestion:

As you are studying/revising each topic in mathematics this year, write, type, or scan summary notes, methods (in a step-by-step fashion), as well as worked examples into a WORD document. This will allow you to edit and update your notes without having to re-write large slabs of materials before the end of the year examinations.

It is also advisable to take notes as the year progresses, rather than writing them all at the end, because if you are familiar with your own notebook it will be easy in the exam to quickly know where to look. A teacher's greatest disappointment is to see a student 'borrow' a friend's notes the day of the exam because theirs haven't been well prepared!

The Question Bank(s)

The Question Bank(s) should include the challenging questions that you came across throughout the year (including past extended answer exam questions) and a variety of worked examples that will assist in addressing questions that you are not confident with. This includes guidelines and examples relating to solving the difficult and strange applications that are not detailed in your text books but that can appear in the exams (this will require a bit of research of past exams, both VCAA and other publishing companies).

Commercially Produced Material

It is quite common for students to copy sections of important work from their textbook or from the shared One-Note that the class may use throughout the year. This is quite reasonable as long as these sections are selected purposefully and with careful decision. There is no point in hurrying to copy slabs of material just before the exams.

An example of books that can be added as part of your bound notes are the Nelson A+ exam 1 and 2 books that you have worked on throughout the year, annotated personally to give you hints for questions you have found particularly challenging. You may include comments about questions that you know the examiners remarked on in the previous year that this topic was done badly. These A+ books are of A4 size purposefully so they can be 'stuck' to a book or lecture pad and bound together. The Essential series of Cambridge Mathematics have well developed and detailed explanations at the beginning of each section and these could be copied and stuck into your book. The new Nelson VCE series for 2016, in contrast, have more succinct explanations but include all past exam questions and their solutions included that are relevant to each topic, all graded for difficulty and the data of student success in the years these questions appeared in the exams.

It used to be in long ago years that the bound book had to be all hand written. Now that any form of type, handwritten, photocopied or stuck in pages are all allowed, it is suggested that you use all forms available to you to produce the most useful notes FOR YOU.

Organising the Reference Materials

Students can choose to construct their Reference Materials in one of two ways:

- By grouping questions and theory by topic.
- By creating one "Theory Section" and one "Question Bank".

Option 1: Grouping Questions and Theory by Topic

Separate the notes for each topic using plastic or paper dividers, as illustrated on side.

For Example:

Section 1 could be dedicated to Algebra.

Section 2 could be dedicated to Trigonometric Functions etc.

Then collate questions by topic, and add them to the back of each individual set of theory notes.

Suggestion:

Dedicate one section for analysis questions (as these integrate multiple topics) as well as a separate section for additional questions. Insert any questions that cannot fit into the appropriate topic bank as well as last minute question into the "Additional Questions" section.

Option 2: Generic Format

Pool all the theory notes at the front of the Reference Materials and create one generic Question Bank behind the Theory Section.

The Question Bank(s)

Throughout the year, as you are working through your various resources, identify those questions you would like to take into the exams, and incorporate these in your Question Bank(s).

The questions in the Question Bank(s) do not need to be organised in any particular fashion. As you come across a new question, just add it to the relevant Question Bank (i.e. Topic Question Bank or Generic Question Bank) and annotate the corresponding Theory Section with the relevant question number. This will help you easily locate worked examples when necessary, as well as save on valuable time!

In other words – Collate questions in the order you paste them into your Reference Materials.

If you have decided to adopt the "Generic Format" (Option 2), then Questions 1 – 10 could relate to Trigonometry, Questions 11 – 50 could relate to Differentiation, Question 51, 59, 77 could relate to Trigonometry again.

If you have decided to adopt Option 1 (Grouping questions and theory by topic), then using the Algebra Section as an example, Questions 1, 2 and 3 could relate to factorising quadratic expressions, Questions 4 – 29 could relate to factorising linear expressions, Questions 30, 102, 5467 could relate to factorising cubic expressions.

Creating a Referenced Question Bank

As you add each question to your Bank(s), reference these questions in the appropriate section of the given notes.
For example: The Theory Section could include the following materials:

Evaluating Limits Algebraically

The manner in which we evaluate limits algebraically depends on whether the **denominator** of the function carries terms involving x.

Given $\lim\limits_{x \to a} f(x)$ **:**

(a) If the equation does not carry a denominator or if the denominator has **NO** terms involving x, simply substitute the value of a into the given equation and simplify.

For example: $\lim\limits_{x \to 1}(5 - 2x - x^3) = 5 - 2(1) - (1)^3 = 2$

(b) If the denominator carries terms involving x, we need to first determine whether the value of x (or a) in question causes the function to be undefined. Substitute the value of a into the denominator of the equation.

If the answer is equal to zero, we need to eliminate the term(s) that is/are causing the function to become undefined. Factorise the given equation and eliminate these terms by cancellation. Before evaluating the limit, substitute the value of a into the new denominator to ensure there are no other terms present that will make the function undefined.

For example: $\lim\limits_{x \to 3}\left[\dfrac{(x^2 - 5x + 6)}{(x-3)}\right] = \lim\limits_{x \to 3}\left[\dfrac{(x-3)(x-2)}{(x-3)}\right] = \lim\limits_{x \to 3}(x-2) = 3 - 2 = 1$ ***(See Question 4)***

The Question Bank would include the **additional examples** you wish to take into the exams (in this case, Question 4):

QUESTION 4

Evaluate $\lim\limits_{x \to 1}\left(\dfrac{x-1}{x^2 + x - 2}\right)$.

Solution

Substitute $x = 1$ into the denominator to determine whether the function will become undefined. As the denominator is equal to zero, factorise the given expression and simplify.

Remember to state any restrictions on the values of x (*remember to define the restrictions using the ORIGINAL (given) equation):*

$$\lim_{x \to 1}\left(\frac{x-1}{x^2 + x - 2}\right) = \lim_{x \to 1}\frac{(x-1)}{(x-1)(x+2)} = \lim_{x \to 1}\left(\frac{1}{x+2}\right) = \frac{1}{1+2} = \frac{1}{3}, \quad x \neq 1, -2$$

Additional Suggestions

- Regularly review and update your materials to incorporate changes and additional worked examples that you have encountered throughout the year.

- If your school teacher suggests you use an exercise book to produce your Reference Materials, only write on one side of each sheet. You will then be able to cut out the sections you wish to retain and paste these materials into your final/official Reference Materials.

- Students may choose to highlight each section of their notes using different coloured text.

  For example: Theory could be typeset using black text, general methodologies in blue text and common errors and watchouts in red text, so that appropriate materials can be located quickly and with ease.

  Alternatively, use frames, images and icons to highlight the different sections of your Reference Materials.

- Students adopting the "Generic Format" may choose to print each topic on different coloured paper so that relevant materials can be quickly located. **Note:** Choose pale pastel coloured paper so that the notes are easy to read.

- We recommend that students print two pages of notes onto one side of a sheet of paper (in landscape format) so that materials are not bulky, and so that sections of interest can be located with speed.

- Do not bind the Reference Materials until a week or two before the exams. This will enable you to add newly discovered materials to your Reference Materials whilst you are preparing for the exams.

- Once you have completed your Reference Materials, add a few sheets of blank paper to the back of each "Theory Section" and "Question Bank" if you have opted to group questions and theory by topic (Option 1), and at least 20 sheets of blank A4 paper to the back of the generic Theory Section and Question Bank (if you opted for the generic Format). This way, if you do come across additional questions/notes between the time of binding and the exam, you will have the opportunity to incorporate these resources into your Reference Materials.

- Add a **detailed index** to the front of your Reference Materials. Use this index as prompts/suggestions when you come across a question that you do not know how to solve.

- Very importantly, refer to your Reference Materials regularly in the weeks leading up to the examinations so that you can learn to locate relevant sections quickly (you will not have time to search for materials during the exam).

Regards, TSFX

COMPLIMENTARY MATHEMATICAL METHODS REFERENCE MATERIALS

EXTRACT FROM THE NOTES THAT ARE BEING HANDED OUT AT THE

TSFX – UNIT 3 EXAM REVISION LECTURES IN JULY & SEPTEMBER

DIFFERENTIATION REFERENCE MATERIALS

IMPORTANT NOTES

Specialist Mathematics and Further Mathematics students will also be issued a complete set of Reference Materials for the exams.

Note: All mathematics students will also receive a comprehensive collection of exam style questions to include in their question banks.

CALCULUS

LIMITS AND DERIVATIVES

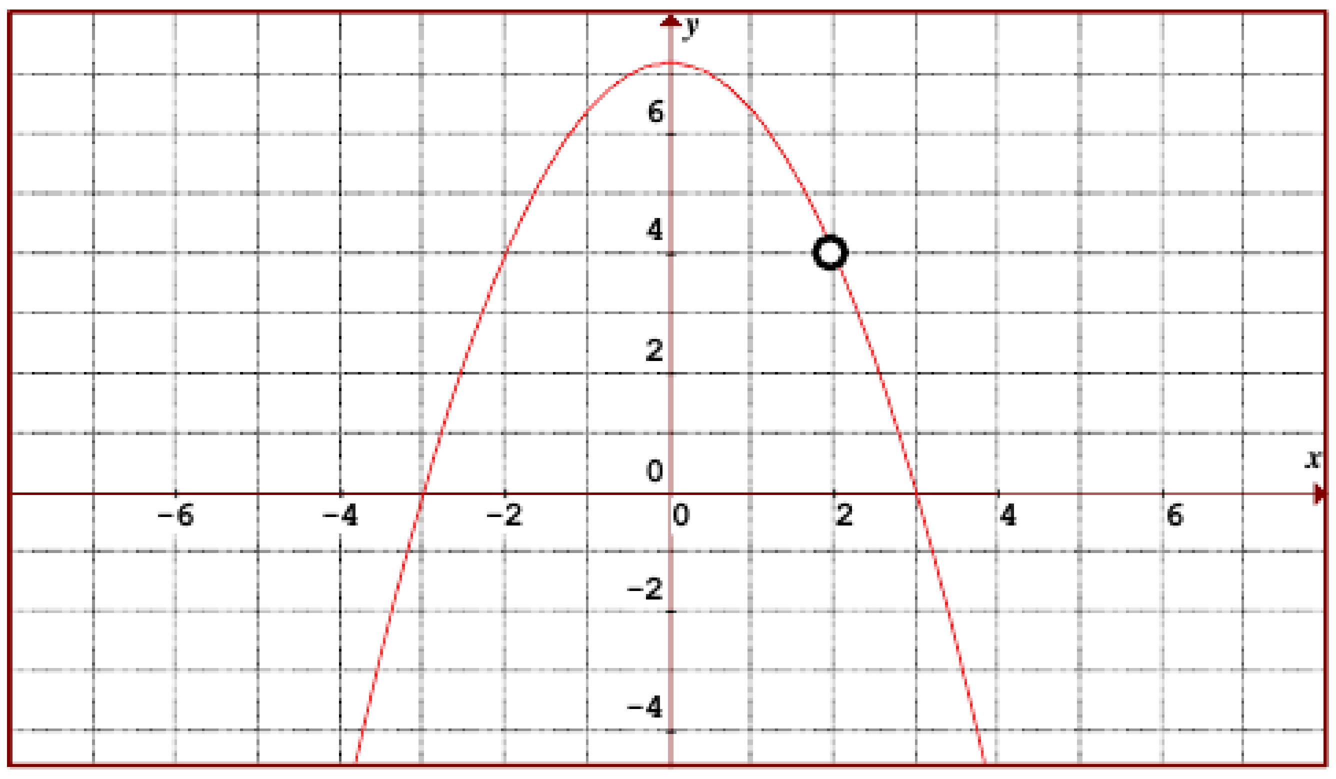
The limit of a function is the value of y that the function approaches as x gets closer to a particular value.

The limit of $f(x)$ as x approaches a is written as $\lim\limits_{x \to a} f(x)$.

CONDITIONS FOR THE EXISTENCE OF A LIMIT

A limit may exist at $x = a$ if:

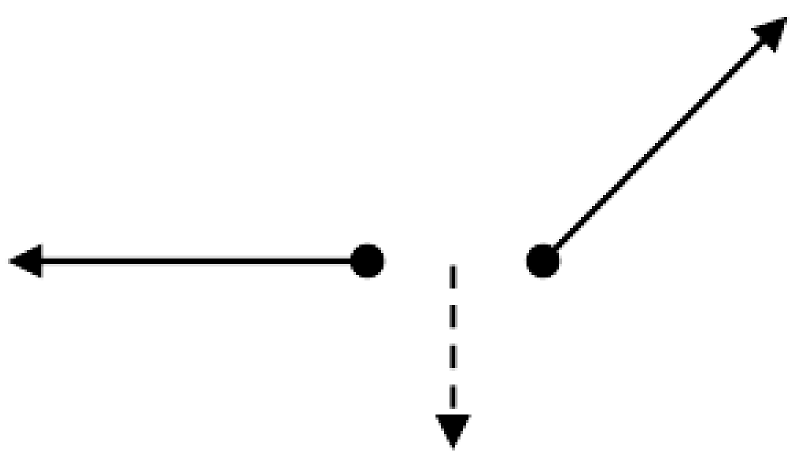
- If the function $f(x)$ is continuous at $x = a$.
- If there is point discontinuity at $x = a$.

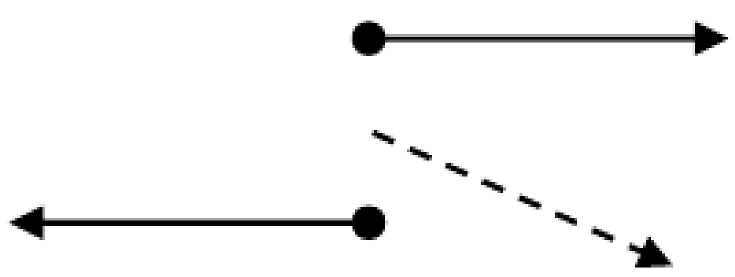


$\lim\limits_{x \to 2} f(x)$ exists and is equal to 4.

A limit will not exist at $x = a$ if:

- There is discontinuity across an interval and $x = a$ lies within this interval.





EVALUATING LIMITS GRAPHICALLY

SINGLE FUNCTIONS

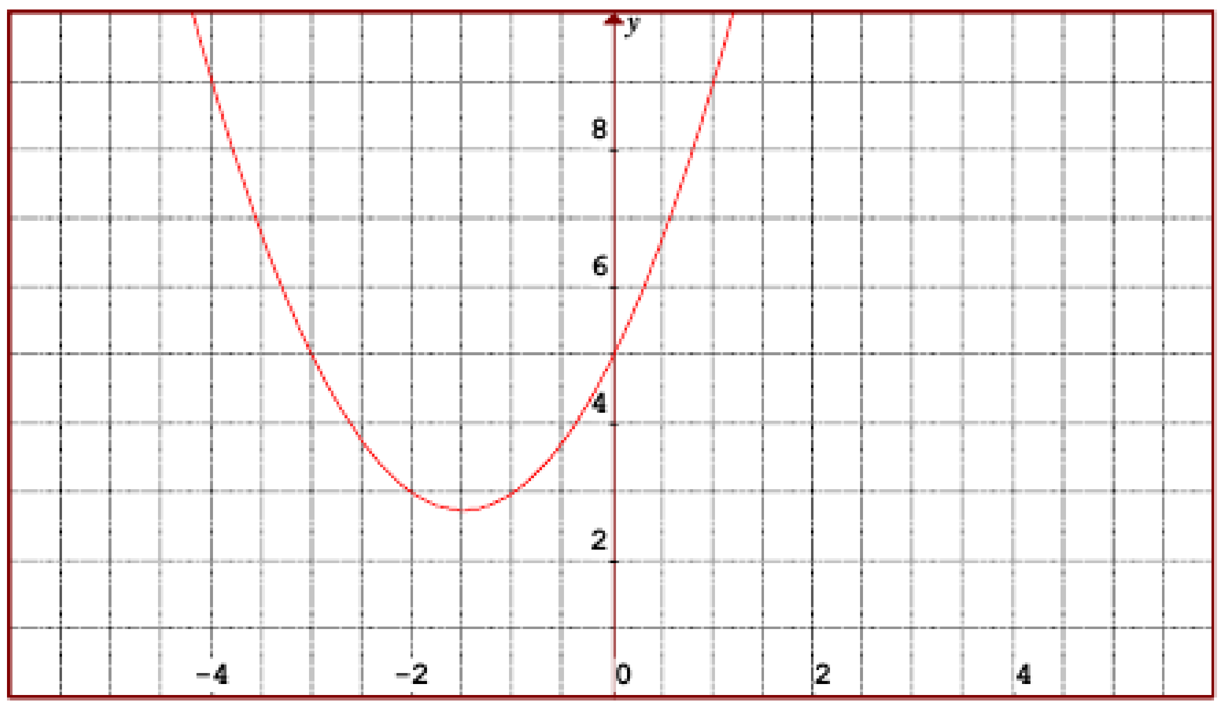
Step 1: Observe the value of y as x approaches a value below the point of interest.

Step 2: Observe the value of y as x approaches a value above the point of interest.

Step 3: If the left hand limit is equal to the right hand limit a limit exists at $x = a$.

The limit is simply equal to the value of y that the curve is approaching on either side of the point of interest.

For example: Find $\lim_{x \to -1}(x^2 + 3x + 5)$.



As x approaches -1 from below
(or from the left hand side), $f(x)$ approaches 3.

As x approaches -1 from above
(or from the right hand side), $f(x)$ approaches 3.

$$\therefore \lim_{x \to -1}(x^2 + 3x + 5) = 3$$

HYBRID FUNCTIONS

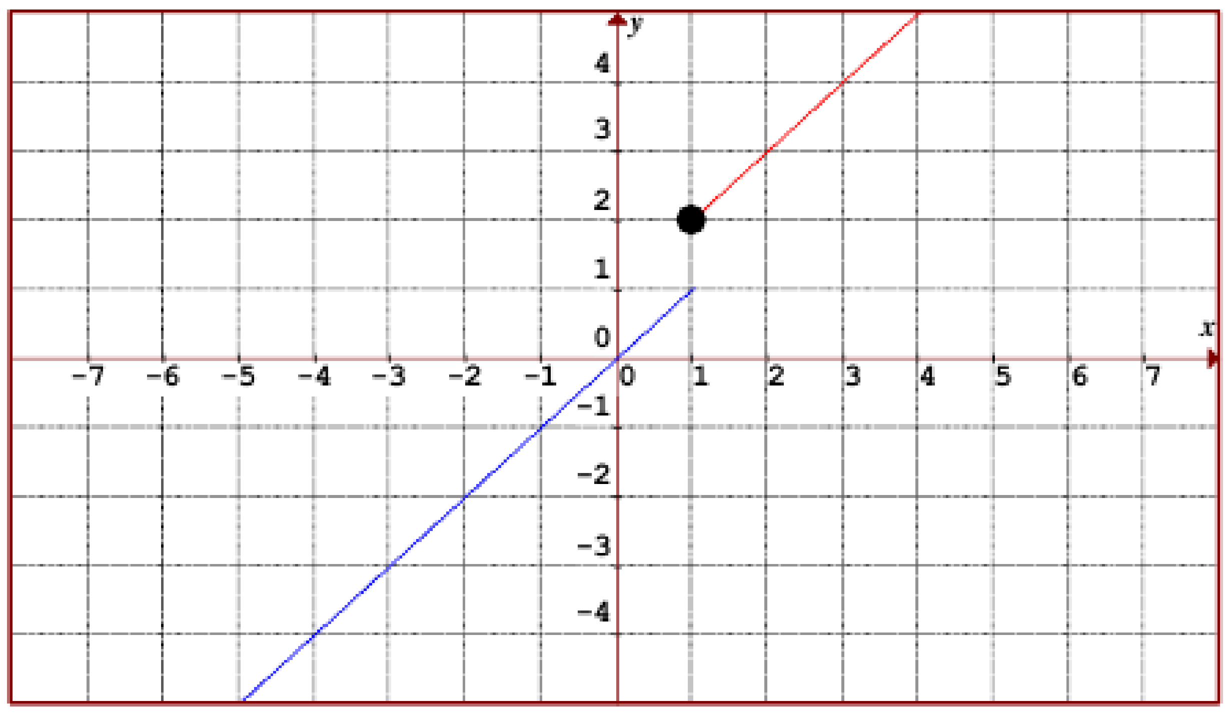
Limits of hybrid functions are evaluated in the same manner as previously described.

Step 1: Find the limit of each individual function at the given value of x.

Step 2: If the limiting values are the same, the limit exists at that value of x.

If one or more of the limiting values are different, the limit does not exist at that particular value of x.

For example:



$\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x+1 & for \;\; x \geq 1 \\ x & for \;\; x < 1 \end{cases}$

As x approaches 1 from above,
$f(x)$ approaches 2.

As x approaches 1 from below,
$f(x)$ approaches 1.

As the two limiting values are different, we say that the $\lim_{x \to 1} f(x)$ does not exist.

EVALUATING LIMITS ALGEBRAICALLY

The manner in which we evaluate limits algebraically depends on whether the **denominator** of the function carries terms involving x:

Given $\lim\limits_{x \to a} f(x)$**:**

(a) If the expression does not carry a denominator or if the denominator has **NO** terms involving x, simply substitute the value of a into the given expression and simplify.

For example: $\lim\limits_{x \to 1}(5 - 2x - x^3) = 5 - 2(1) - (1)^3 = 2$

(b) If the denominator carries terms involving x, we need to first determine whether the value of x (or a) in question causes the function to be undefined.

Substitute the value of a into the denominator of the given expression.

- If the answer does not equal zero, substitute the value of a into the expression.

  For example: $\lim\limits_{x \to -2}\left(\dfrac{x^2+1}{x-2}\right) = \dfrac{(-2)^2+1}{(-2)-2} = \dfrac{4+1}{-4} = -\dfrac{5}{4}$

- If the answer is equal to zero, we need to eliminate the term(s) that is/are causing the function to become undefined. Factorise the given expression and eliminate these terms by cancellation. Before evaluating the limit, substitute the value of a into the new denominator to ensure there are no other terms present that will make the function undefined.

  For example: $\lim\limits_{x \to 3}\left[\dfrac{(x^2-5x+6)}{(x-3)}\right] = \lim\limits_{x \to 3}\left[\dfrac{(x-3)(x-2)}{(x-3)}\right] = \lim\limits_{x \to 3}(x-2) = 3-2 = 1$

Note:

When there are terms involving x in the denominator of a fraction, there may be some values of x which cause the function to become undefined. These values of x must be stated with your answer.

To determine which values of x will make the function undefined, before the function is factorised and terms are eliminated by cancellation:

Step 1: Let the denominator of fraction equal zero .
Step 2: Solve for x.

For Example: $\lim\limits_{x \to -1}\left(\dfrac{x^2-3x-4}{(x+1)(x+3)}\right)$

Restrictions: $x \neq -1$ or $x \neq -3$.

Note: Even though this function is undefined at $x = -1$, we may still investigate what happens to the value of y as x approaches this value (i.e. the limit).

LIMIT THEOREMS

(a) The limit of a constant is equal to the value of the constant:

If $f(x) = k$ then $\lim_{x \to a} f(x) = k$.

For Example: $\lim_{x \to 2} (5) = 5$

(b) The limit of the sum and/or difference of a series of terms is equal to the sum and/or difference of the limits of each individual term.

$$\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

For Example: $\lim_{x \to 2} \left(x^2 - 3x - 1 \right) = \lim_{x \to 2} \left(x^2 \right) - \lim_{x \to 2} (3x) - \lim_{x \to 2} (1) = 4 - 6 - 1 = -3$

(c) The limit of the product of two functions is equal to the product of the limits of each individual function.

$$\lim_{x \to a} \left[f(x).g(x) \right] = \lim_{x \to a} \left(f(x) \right) \times \lim_{x \to a} \left(g(x) \right)$$

For Example: $\lim_{x \to {}^{-}1} \left(x^2 - 1 \right)(4x) = \lim_{x \to {}^{-}1} \left(x^2 - 1 \right) \times \lim_{x \to {}^{-}1} (4x) = 0 \times -4 = 0$

(d) The limit of the quotient of two functions is equal to the quotient of the limits of each individual function.

$$\lim \left[\frac{f(x)}{g(x)} \right] = \frac{\lim f(x)}{\lim g(x)}$$

Note that $g(x) \neq 0$.

For Example: $\lim_{x \to 1} \left(\frac{(x^2 - 5x + 6)}{(x - 3)} \right) = \frac{\lim_{x \to 1} (x^2 - 5x + 6)}{\lim_{x \to 1} (x - 3)} = \frac{1 - 5 + 6}{1 - 3} = -\frac{2}{2} = -1$

OR $\lim_{x \to 1} \left(\frac{(x - 3)(x - 2)}{(x - 3)} \right) = \lim_{x \to 1} (x - 2) = -1$

WATCHOUTS

- A point does not need to exist in order for a limit to exist.
- If the function is discontinuous across an interval and the value of x lies in this interval, the limit cannot be evaluated.

DIFFERENTIATION

The derivative $\frac{dy}{dx}$ or $f'(x)$ describes the gradient of the tangent to a curve at any value of x. The process of finding $f'(x)$ or $\frac{dy}{dx}$ is referred to as **differentiation**.

WATCHOUTS

- Whenever you see the following words/phrase, automatically think of differentiation: gradient, gradient function, gradient of the tangent.
- $\frac{dy}{dx}$ is an operation – it is not a quotient (it is not the same as dy ÷ dx).
- $\frac{d}{dx}(\ )$ reads as "the derivative of $(\ )$ with respect to x.

DERIVATIVES FROM FIRST PRINCIPLES

The derivative from first principles is obtained by applying the rule:

$$\text{Derivative} = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h} = f'(x) = \frac{dy}{dx}$$

Step 1: Write expressions for $f(x)$ and $f(x+h)$.

Step 2: Substitute the expressions for $f(x)$ and $f(x+h)$ into the rule

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}$$

Step 3: Expand and collect like terms.
Step 4: Remove h as a common factor and simplify.
Step 5: Substitute $h = 0$.

Note:

- To obtain $f(x+h)$ from $f(x)$, replace x in the given equation with $(x+h)$.
  For Example: If $f(x) = x^2 - 6x + 1$, then $f(x+h) = (x+h)^2 - 6(x+h) + 1$.
- To find the gradient at a specific point, we substitute the value of x into the derivative.
  Note: The gradient at $x = 2$ is denoted as $f'(2)$.

For example: Differentiate $f(x) = 1 - 2x^3$ using first principles.

$$f'(x) = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{[1 - 2x^3 - 6x^2h - 6xh^2 - 2h^3] - [1 - 2x^3]}{h}$$

$$= \lim_{h \to 0} \frac{[-6x^2h - 6xh^2 - 2h^3]}{h} = \lim_{h \to 0} \frac{h[-6x^2 - 6xh - 2h^2]}{h} = \lim_{h \to 0}\left(-6x^2 - 6xh - 2h\right) = -6x^2$$

Note: The $\lim\limits_{h \to 0}$ notation is included in each step, until such time that $h = 0$ is substituted into the equation.

WATCHOUTS

- If the limit from first principles rule is correctly applied, you will need to cancel out the terms in the original equation.

  If this does not occur, you have probably incorrectly expanded the second set of brackets (remember to multiply each term in these brackets by negative 1).

- You may check your answers by differentiating the given equation by rule. If the two answers are different, it is likely that an error has been made when expanding the brackets in the limit formula. Remember to multiply every term in the second set of brackets by negative one!

- Be careful to ensure that the gradient rule is correctly applied taking careful note of the limiting value. For example, to find the gradient of the tangent to $f(x)$ at $x = -4$ we must find the limit as $h \to 0$ **NOT** $h \to -4$.

  i.e. $\lim\limits_{h \to 0} \dfrac{f(h-4) - f(-4)}{h}$

DIFFERENTIATING BY RULE

DERIVATIVES OF POLYNOMIAL AND RATIONAL FUNCTIONS

The derivative of an algebraic term is obtained by multiplying the coefficient by the power and then lowering the power by one. This rule applies for all algebraic expressions, providing that $n \neq 0$.

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

For example: If $y = 2x^6$ then $\frac{dy}{dx} = 2 \times 6 \times x^{6-1} = 12x^5$

The derivative of a constant (a term that does not contain any variables) is equal to zero.

If $y = k$ then $\frac{dy}{dx} = 0$

For example: Given $y = 5a$ then $\frac{dy}{dx} = 0$

The derivative of the sum and/or difference of terms is equal to the sum/difference of the derivatives of each individual term.

If $y = ax^2 + bx + c$ then $\frac{dy}{dx} = \frac{d}{dx}(ax^2) + \frac{d}{dx}(bx) + \frac{d}{dx}(c) = 2ax + b$

For example: Given $y = x^2 - 5x + 6$ then $\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(6) = 2x - 5$

SUMMARY OF DIFFERENTIATION TECHNIQUES

There are a variety of techniques available to find derivatives. Which technique is employed simply depends upon the manner in which the rule of the function is presented.

Given a function in terms of x

Simplify:

Rewrite expressions as a series of terms Separated by addition and subtraction or reduce the expression to 1 term.

- Expand simple products.
- Apply index laws.
- Express surds as rational powers.
- Factorise and simplify.
- Rewrite terms over individual denominators.

Differentiate

If the expression is a composite function, apply the **Chain Rule**.

Given an expression/equation consisting of two separate functions in terms of x

Rewrite expressions as a series of terms separated by addition and subtraction or reduce the expression to 1 term.

Differentiate

If written as:

$$\frac{\text{expression in terms of } x}{\text{expression in terms of } x}$$

and quotients cannot be simplified, apply the **Quotient Rule.**

If written as:

(expression in terms of x) $\times$ (expression in terms of x)

and products cannot be simplified, apply the **Product Rule**.

FINDING DERIVATIVES – METHOD

Step 1: Rewrite all terms as powers on x.

$$\sqrt{x} = x^{\frac{1}{2}} \qquad \sqrt[3]{x} = x^{\frac{1}{3}} \qquad (\sqrt[p]{x})^q = (x^{\frac{1}{p}})^q = x^{\frac{q}{p}}$$

Step 2: Bring terms involving x in the denominator (bottom of a fraction) to the top, by changing the sign on the power.

For example: $\frac{1}{x^2} = x^{-2}$ **Note:** $\frac{1}{6x^2} = \frac{x^{-2}}{6}$

For example: Given $y = 2x + \frac{3}{\sqrt{x}} - 3\sqrt{x}$ then $y = 2x + 3x^{-1/2} - 3x^{1/2}$

Step 3: Simplify expressions so that terms are separated by addition and subtraction. Then differentiate each term individually. Alternatively, use algebraic techniques to reduce expressions to one term.

Products:

- Expand simple products rather than applying the Product Rule.
- Apply index laws to convert products involving the same base into single terms.

  For example: $x^{m+1}.x^{1-2m} = x^{2-m}$

- Apply logarithmic laws to simplify complex base expressions.

  For Example: $\log(expression \times expression)$ - use $\log_a(mn) = \log_a(m) + \log_a(n)$

Quotients:

- Remove common factors and simplify.
- Factorise and eliminate terms by cancellation.
- If there is only one term in the denominator, write each term in the numerator over the denominator so that individual fractions are produced. Then simplify each term by cancellation.

  For example: $\frac{x^2+1}{x^2} = \frac{x^2}{x^2} + \frac{1}{x^2} = 1 + x^{-2}$ (Don't use the Quotient Rule).

- Simplify expressions using index laws.

  For example: $y = \frac{5e^{(3x+4)}}{2e^{-(1-x)}} = \frac{5}{2}e^{3x+4+(1-x)} = \frac{5}{2}e^{3x+4+1-x} = \frac{5}{2}e^{2x+5}$

  therefore $\frac{dy}{dx} = 5e^{2x+5}$

- Simplify expressions using logarithmic laws.

  For example: Given $\log_a\left(\frac{\text{expression 1}}{\text{expression 2}}\right)$, use $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

Other Suggestions:

- Try to write expressions in terms of one trigonometric function and simplifying before differentiating.

  For example: $y = \tan(-1-2x)\cos(-1-2x)$

$$= \frac{\sin(-1-2x)}{\cos(-1-2x)}\cos(-1-2x) = \sin(-1-2x)$$

Step 4: Differentiate.

Step 5: Re-write the answer using positive powers. Bring terms with negative powers in the numerator (top of a fraction) to the bottom, by changing the sign on each power.

Step 6: State any restrictions on the values of x.

WATCHOUTS

- Differentiation is the opposite process to antidifferentiation. Use differentiation when given y and $\frac{dy}{dx}$ is required.
- Look carefully at what you are differentiating with respect to. We differentiate with respect to the last letter in the bottom of the derivative expression $\frac{d\#}{d*}$ i.e. " * ". Lower the power on this variable if the term is algebraic.

  For example: Given $y = 5a$ then $\frac{dy}{da} = 5$ and $\frac{dy}{dx} = 0$
- Always check each calculus question to confirm whether you are meant to differentiate or antidifferentiate.

IMPORTANT DERIVATIVES

$y = ax^n$, $n \neq 0$	$\frac{dy}{dx} = anx^{n-1}$
$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$
$y = \tan x$	$\frac{dy}{dx} = \sec^2 x$
$y = \log_e x$, $x > 0$	$\frac{dy}{dx} = \frac{1}{x}$
$y = e^x$	$\frac{dy}{dx} = e^x$

To differentiate more complex expressions involving trigonometric, logarithmic and exponential functions, we apply the Chain Rule.

THE CHAIN RULE

The chain rule states that: $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

The Chain Rule is used to differentiate complex expressions involving one function in terms of x. For example:

(a) Functions containing brackets with high powers, negative powers or fractional powers. **For example:** $y = (1-2x)^9$, $y = (1-2x)^{-3}$, $y = (1-2x)^{1/3}$

(b) Trigonometric functions whose angles are not linear expressions.
For example: $y = \sin(x^2)$, $y = \cos(\tan x)$

(c) Logarithmic functions where the base numeral is not a linear expression.
For example: $y = \log_e(1-x^3)$, $y = \log_e(\sin x)$

(d) Exponential functions whose powers are not linear expressions.
For example: $y = -5e^{\cos x}$, $y = e^{(x^2-6x+1)}$

METHOD:

If $y = f(u)$ where $u = g(x)$ then differentiate as follows:

Step 1: Write an expression for u (the inner function) in terms of x.

Bracket/power functions:	Let $u =$ contents of bracket
Trigonometric functions:	Let $u =$ the angle
Exponential functions:	Let $u =$ the power
Logarithmic functions:	Let $u =$ the base numeral

Step 2: Find the derivative $\dfrac{du}{dx}$ or u'.

Step 3: Write an expression for y in terms of u i.e. $y = f(u)$.

Step 4: Find the derivative $\dfrac{dy}{du}$ or $y = f'(u)$.

Step 5: Substitute the derivatives into the Chain Rule.
Step 6: Replace u with its original expression and simplify.

For example: Find the derivative of $y = (x^3-1)^5$.

Let $u = x^3 - 1$ $\qquad$ $y = u^5$

$\dfrac{du}{dx} = 3x^2$ $\qquad$ $\dfrac{dy}{du} = 5u^4$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 5u^4 \times 3x^2 = 15u^4x^2 = 15x^2(x^3-1)^4$$

QUICK WAYS OF APPLYING THE CHAIN RULE

BRACKET/POWER FUNCTIONS

Expressions that include brackets with high, fractional or negative powers

$\frac{dy}{dx} =$ Power $\times$ Derivative of contents of bracket $\times$

Given expression – but lower the power on the brackets by one

For Example:

If $y = (1-2x)^9$ then $\frac{dy}{dx} = 9 \times -2 \times (1-2x)^8 = -18(1-2x)^8$

If $y = \frac{1}{\sqrt{x+3}} = (x+3)^{-\frac{1}{2}}$ then $\frac{dy}{dx} = \frac{-1}{2} \times 1 \times (x+3)^{-\frac{3}{2}} = \frac{-1}{2(x+3)^{\frac{3}{2}}}$

TRIGONOMETRIC FUNCTIONS

If $y = a\begin{pmatrix} \sin \\ \cos \\ \tan \end{pmatrix}[f(x)]$ then $\frac{dy}{dx} = af'(x)\begin{pmatrix} \cos \\ -\sin \\ \sec^2 \end{pmatrix}[f(x)]$

i.e. $\frac{dy}{dx} =$ Coefficient of trig $\times$ Derivative of the angle $\times$ Derived trig function with the original angle

Note: $\sin\theta \to \cos\theta$ and $\cos\theta \to -\sin\theta$ and $\tan\theta \to \sec^2\theta$

For Example:

If $y = 4\sin\left(x^2\right)$ then $\frac{dy}{dx} = 4 \times 2x \times \cos\left(x^2\right) = 8x\cos\left(x^2\right)$

If $y = -\cos(1-3x^4)$ then $\frac{dy}{dx} = -1 \times -12x^3 \times -\sin(1-3x^4) = -12x^3\sin(1-3x^4)$

If $y = \cos(\tan x)$ then $\frac{dy}{dx} = 1 \times \sec^2 x \times -\sin(\tan x) = -\sec^2 x\sin(\tan x)$

WATCHOUTS

- **Do not** lower powers on trigonometric terms.
- Always re-write expressions in terms of 1 trigonometric function before differentiating.

For example: $y=-\frac{\sin(4x)}{2\cos(4x)}=-\frac{1}{2}\times\frac{\sin(4x)}{\cos(4x)}=-\frac{1}{2}\tan(4x)$

$\therefore \frac{dy}{dx}=-\frac{1}{2}\times 4\times\sec^2(4x)=-2\sec^2(4x)$

- Watchout: $y=\cos^4 x\neq\cos\left(x^4\right)$.

$y=\cos^4 x=\left(\cos x\right)^4$ which needs to be resolved using the bracket/power chain rule.

For example: $\frac{dy}{dx}=4\times-\sin x\times(\cos x)^3=-4\sin x(\cos x)^3$

$y=\cos x^4=\cos(x^4)$ which needs to be resolved using trig chain rule.

For example: $\frac{dy}{dx}=1\times 4x^3\times-\sin(x^4)=-4x^3\sin(x^4)$

EXPONENTIAL FUNCTIONS

$$\frac{dy}{dx} = \text{Derivative of the power} \times \text{Given term}$$

For Example: If $y = e^{(x^2-6x+1)}$ then $\frac{dy}{dx} = (2x-6)e^{(x^2-6x+1)} = 2(x-3)e^{(x^2-6x+1)}$

If $y = -5e^{\cos x}$ then $\frac{dy}{dx} = -\sin x \times -5e^{\cos x} = 5\sin xe^{\cos x}$

Given the product of two exponential functions with the **same base**, write the expression as one term using index laws. Do not apply the product rule.

For Example: If $y = \frac{e^{2x}}{2}.e^{1-x^2} = \frac{1}{2}e^{2x-x^2+1}$ then $\frac{dy}{dx} = (2-2x) \times \frac{1}{2}e^{2x-x^2+1} = (1-x)e^{2x-x^2+1}$

Given the quotient of two exponential functions with the **same base**, write the expression as one term using index laws. Do not apply the quotient rule.

For Example: If $y = \frac{-e^{(2x+1)}}{2e^{(1-3x)}} = -\frac{1}{2}e^{(2x+1)-(1-3x)} = -\frac{1}{2}e^{(2x+1-1+3x)} = -\frac{1}{2}e^{5x}$ then $\frac{dy}{dx} = -\frac{5}{2}e^{5x}$

WATCHOUTS

- **Do not** lower powers on exponential terms.
- The rule $\frac{d}{dx}\left(ae^{f(x)}\right) = af\,'(x).e^{f(x)}$ only applies to functions with the base **e**.

LOGARITHMIC FUNCTIONS

$$\frac{dy}{dx} = \text{Coefficient of } \log_e \times \frac{\textit{derivative of base numeral}}{\textit{base numeral}}$$

For Example: If $y = 2\log_e(1 - x^3)$ then $\dfrac{dy}{dx} = 2 \times \dfrac{-3x^2}{1 - x^3} = \dfrac{-6x^2}{1 - x^3}$

WATCHOUTS

- Do not lower powers on logarithmic terms.
- The base numeral does not change when differentiating logarithmic expressions.
- The rule $\dfrac{d}{dx}\left(a\log_e f(x)\right) = \dfrac{af'(x)}{f(x)}$ only applies to functions with the base **e**.
- Given a logarithmic function where the base numeral consists of the product or quotient of two functions, write the expression as a series of individual terms by applying logarithmic laws. Do not apply the Product or Quotient Rule.

  For Example:

  If $y = \log_e(\cos x \sin x) = \log_e(\cos x) + \log_e(\sin x)$ then $\dfrac{dy}{dx} = \dfrac{-\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$

  If $y = \log_e\left(\dfrac{x}{x-1}\right) = \log_e(x) - \log_e(x-1)$ then $\dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{1}{x-1} = \dfrac{1}{x(1-x)}$

THE PRODUCT RULE

- The Product Rule is used to find the derivative of the product of two functions.

- Expressions are usually presented in the form:

$$(expression\ with\ x) \times (expression\ with\ x)$$

- The Product Rule states that given $y = u.v$, where u and v represent the individual functions, then the derivative is given by:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example: Find the derivative of $y = x^2(x^2 - 2x)^4$.

Let one function equal u:

Let $u = x^2$

Find the derivative $\frac{du}{dx}$:

$$\frac{du}{dx} = 2x$$

Let the other function equal v:

Let $v = (x^2 - 2x)^4$

Find the derivative $\frac{dv}{dx}$:

$$\frac{dv}{dx} = 4(x^2 - 2x)^3(2x - 2)$$

Substitute the relevant expressions into the Product Rule and simplify:

$$\frac{dy}{dx} = 2x(x^2 - 2x)^4 + 4x^2(x^2 - 2x)^3(2x - 2)$$

- **Quick Rule:**

$$\frac{dy}{dx} = \textit{(1st expression} \times \textit{derivative of 2nd)+(2nd expression} \times \textit{derivative of 1st)}$$

For example:

Given $y = (3x^2 - 1)\sin x$ then $\frac{dy}{dx} = [(3x^2 - 1) \times \cos x] + [\sin x \times 6x]$

$$= (3x^2 - 1)\cos x + 6x\sin x$$

Given $y = (1 - x^3)e^x$ then $\frac{dy}{dx} = (1 - x^3)e^x + e^x(-3x^2) = e^x(1 - x^3 - 3x^2)$

THE QUOTIENT RULE

- The Quotient Rule is used to find the derivative of the quotient of two functions.
- Expressions are usually presented in the form: $\dfrac{Expression\ with\ x}{Expression\ with\ x}$
- The Quotient Rule states that given $y = \dfrac{u}{v}$, where u and v represent the individual functions, then the derivative is given by:

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

For example: Find the derivative of $y = \dfrac{x}{x^2+1}$.

Let one function equal u:

Let $u = x$

Find the derivative $\dfrac{du}{dx}$**:**

$$\frac{du}{dx} = 1$$

Let the other function equal v:

Let $v = x^2 + 1$

Find the derivative $\dfrac{dv}{dx}$**:**

$$\frac{dv}{dx} = 2x$$

Substitute the relevant expressions into the Quotient Rule and simplify:

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

$$\therefore \frac{dy}{dx} = \frac{(x^2+1)1 - x.2x}{(x^2+1)^2}$$

$$= \frac{(x^2+1) - 2x^2}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2}$$

$$= \frac{1-x^2}{(x^2+1)^2}$$

- **Quick Rule:**

$$\frac{dy}{dx}=\frac{(Bottom\ expression\times derivative\ of\ top\ )\text{ - }(Top\ expression\times derivative\ of\ bottom\ )}{(Bottom\ expression)^2}$$

For example: If $y=\dfrac{\sin x}{e^x}$ then $\dfrac{dy}{dx}=\dfrac{\left[e^x\times\cos x\right]-\left[\sin x\times e^x\right]}{\left(e^x\right)^2}=\dfrac{e^x\cos x-e^x\sin x}{e^{2x}}$

$$=\frac{e^x\left(\cos x-\sin x\right)}{e^x.e^x}=\frac{\cos x-\sin x}{e^x}$$

WATCHOUTS

- Always let the numerator equal u .
- The order in which the products $v.\dfrac{du}{dx}$ and $u.\dfrac{dv}{dx}$ are subtracted is critical.

CONTINUITY AND DIFFERENTIABILITY

The derivative at a specific point may be evaluated if the following conditions are satisfied:

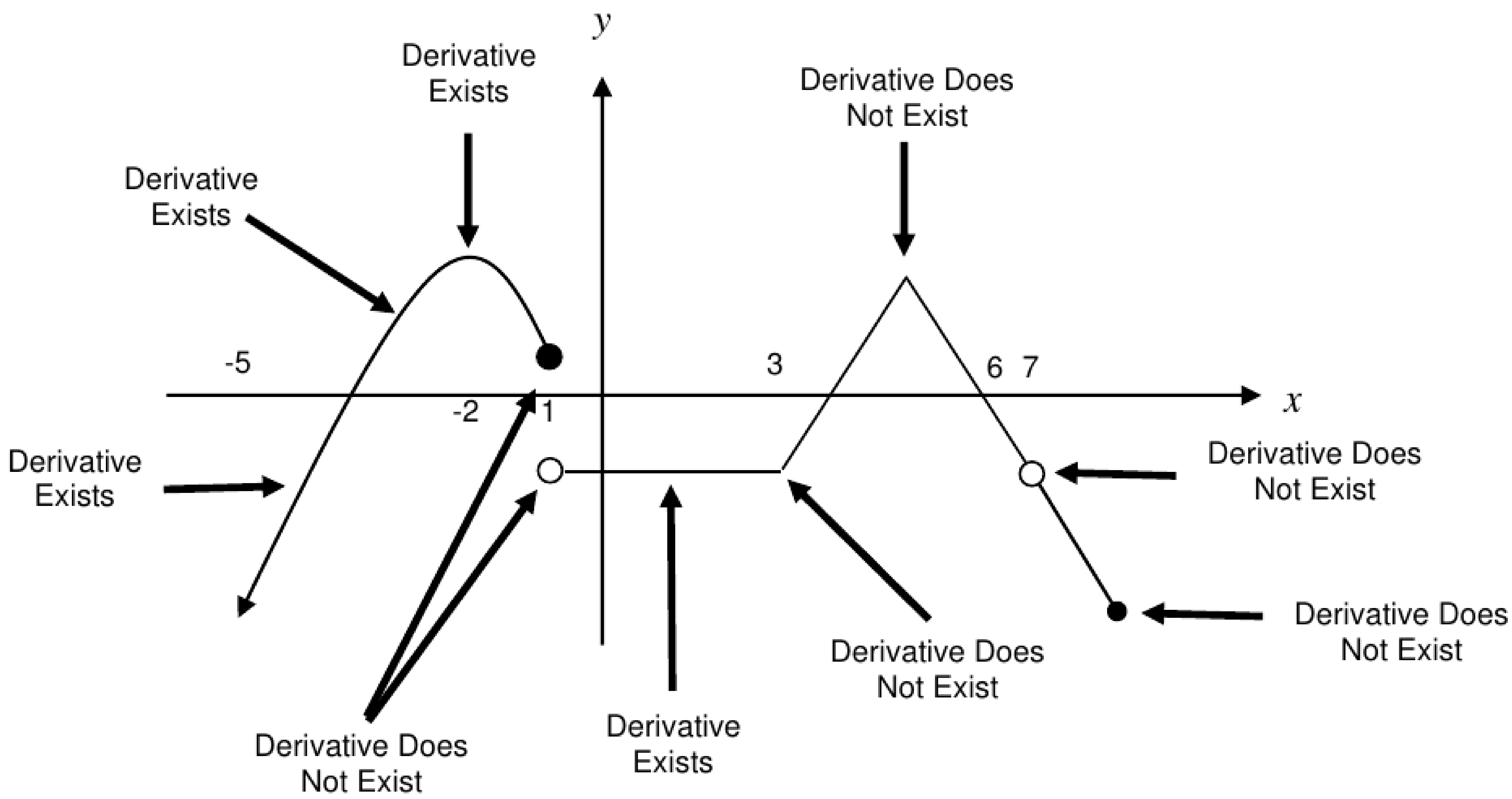
(a) The graph is continuous at that point.
(b) The gradient of the tangent on either side of the point $x = a$ converges to the same value.

Derivatives will not exist at:

- Points of discontinuity.
- Asymptotes.
- The end points of a domain.
- Sharp corners eg. Cusps (watch out when dealing with absolute value functions).

Note:

The derivative may not exist even though a limit exists at $x = a$.



WATCHOUTS

- The CAS will provide an answer even when a derivative does not exist at that point. Beware!
- When stating domains across which functions are differentiable – do not include the end points.
- To show that a function is differentiable at a point – show that the gradients on either side of that point gradually converge to the same value.

GRADIENTS AT SPECIFIC POINTS

The derivative in terms of x represents the gradient of the tangent at any value of x.

If the given equation is that of a straight line, the derivative will be a constant value independent of x (this should make sense as the gradient of a straight line is constant).

The gradient of a curve other than a line continually changes, and is dependent upon the value of x at which the tangent is drawn. Therefore, the derivative will be an expression in terms of x.

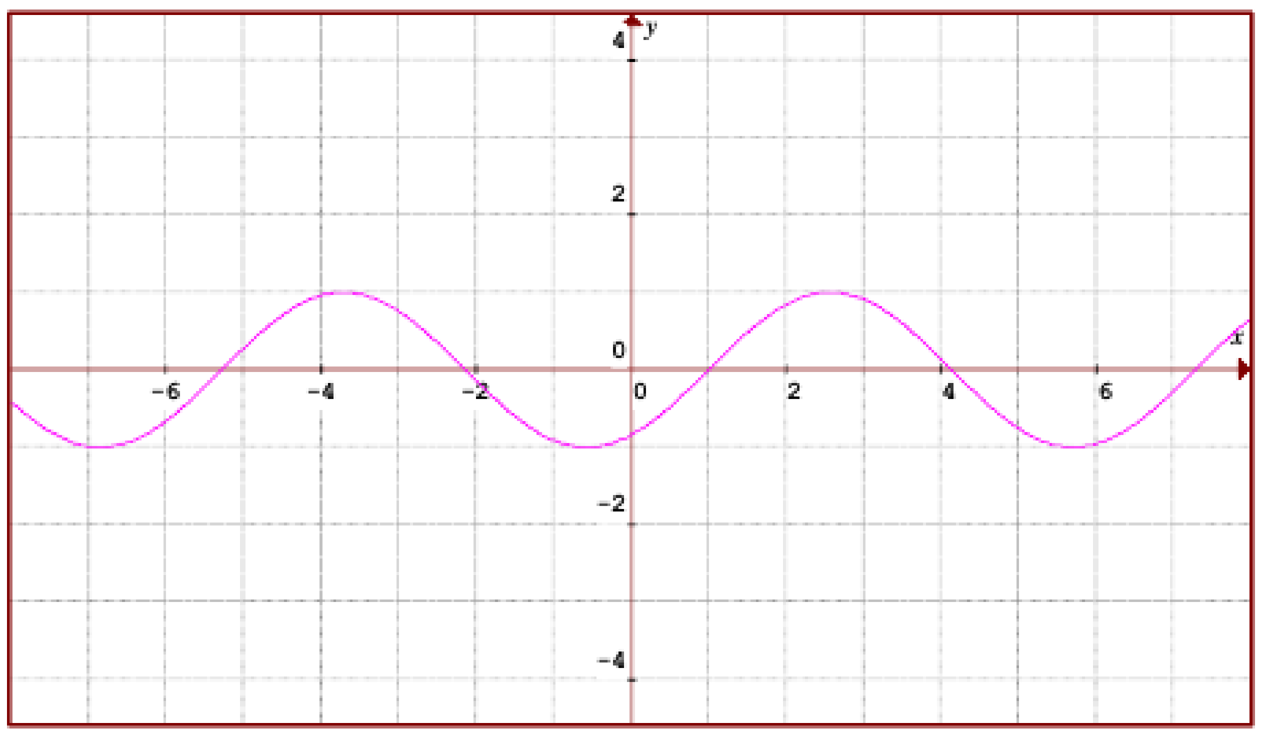
To find the gradient at a specific point:

Step 1: Determine whether the derivative can exist at the required point. (Check the Conditions for Differentiability and the given domain).

Step 2: If the derivative exists at the point, substitute the value of x into the derivative expression.

Note: The gradient at $x = 2$ is denoted as $f'(2)$.

For example: Given that $f(x) = \sin(x-1)$ does $f'(x)$ at $x = 1.5$ exist?



$f'(1.5)$ exists only if the derivative exists at that point. As:

(a) the graph is continuous at that point;
(b) the gradient of the tangent on either side of the point $x = a$ converges to the same value;

Then $f'(1.5)$ exists.

$$f(x) = \sin(x-1)$$

$$f'(x) = \cos(x-1)$$

$$f'(1.5) = \cos(1.5-1) = \cos(0.5) = \frac{\sqrt{3}}{2}$$

WATCHOUTS

- Before finding a derivative at a specific point, first determine whether a derivative can exist at that point.
- Remember to use single inequalities ($>$ or $<$) when stating the domain of the derived function at its end points.

DERIVATIVES OF HYBRID FUNCTIONS

METHOD:

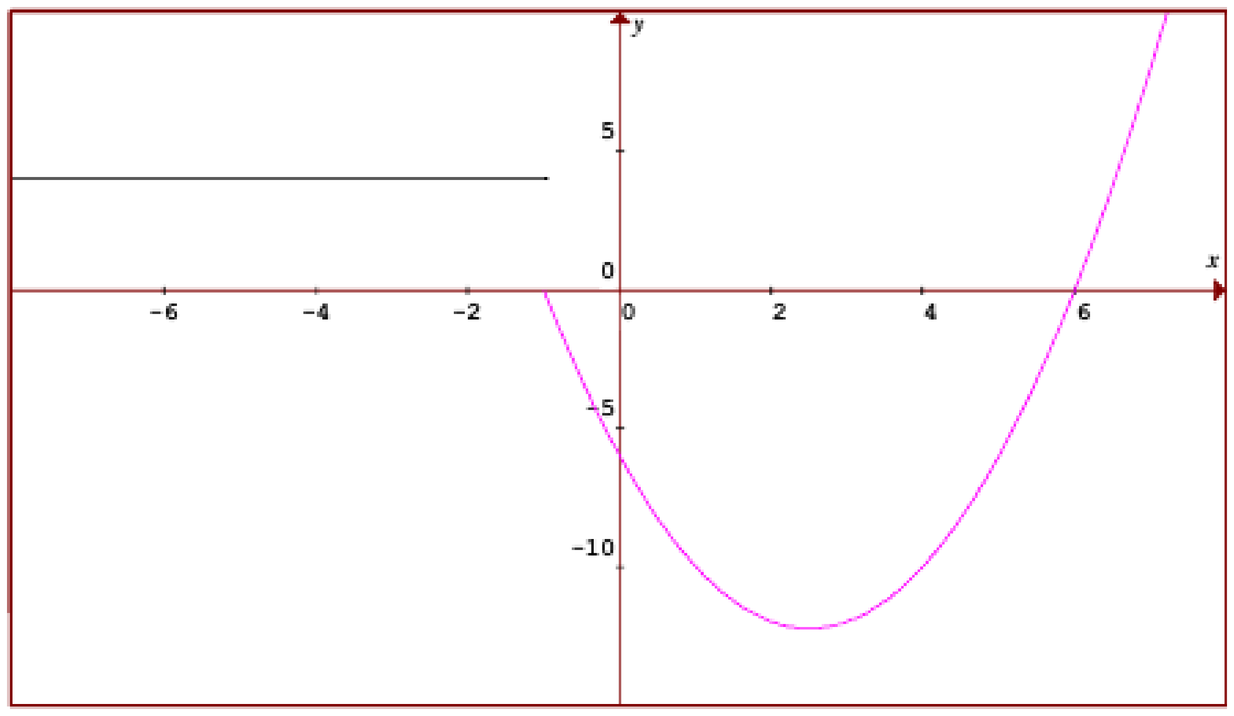
Step 1: Write/identify the rule describing each part of the domain.
Step 2: Find the derivative of each individual rule.

If asked to find the derivative at a specific point:

Step 1: Identify the equation describing the part of the curve that the point lies on.
Step 2: Determine whether the derivative is defined at the point in question.
Step 3: If the derivative exists, differentiate the appropriate equation.
Step 4: Substitute the required value of x.

For example:

Given that $f(x)=\begin{cases} x^2-5x-6 & for\ \ x\geq -1 \\ 4 & for\ \ x<-1 \end{cases}$



Then $f'(x)=\begin{cases} 2x-5 & for\ \ x>-1 \\ 0 & for\ \ x<-1 \end{cases}$

Note the use of the single inequality for $x>-1$ (the derivative does not exist at the end points of a domain and hence $\geq$ cannot be used).

To find $f'(3)$:

Determine whether the derivative can exist at the required point. (Yes)
Substitute $x=3$ into the rule describing the derivative across that part of the domain.

$f'(x)=2x-5$ for $x>-1$
At $x=3$, $f'(3)=1$

To find $f'(-1)$:

Determine whether the derivative can exist at the required point.
As the derivative is not defined at an end point, a derivative does not exist at $x=-1$.

To find $f'(-4)$:

Determine whether the derivative can exist at the required point. (Yes)
Substitute $x=-4$ into the rule describing the derivative for that part of the domain.

$f'(x)=0$ for $x<-1$
At $x=-4$, $f'(x)=0$

WATCHOUTS

- Before finding a derivative at a specific point, first determine whether a derivative can exist at that point.
- You can find the derivative of a composite function if that **composite function exists.**
- Before differentiating, confirm whether the value of x in question lies within the domain of the composite function. If the value of x does not fall inside the domain of the composite, the derivative cannot be found at that point.

  For example:

  If $f:[-1,1]\rightarrow R$ where $f(x)=\sqrt{1-x^2}$ and $g:R\rightarrow R$ where $g(x)=2+x^2$ $g'(f(x))$ exists as $r_f \subset d_g$.

  However, $g'(f(3))$ does not exist as $x=3$ lies outside the domain of $g(f(x))$, which is equal to $d_f =[-1,1]$.
- Remember to use single inequalities ($>$ or $<$) when stating the domain of the derived function at its end points.

DERIVATIVES OF COMPOSITE FUNCTIONS

- The derivative of a composite function may exist if that **composite function exists.**
- A composite function will exist if the range of the second function is equal to or a subset (part of) of the domain of the first function.

For $g\left(f(x)\right)$ to be defined $\text{range}_f \subseteq \text{domain}_g$

For $f\left(g(x)\right)$ to be defined $\text{range}_g \subseteq \text{domain}_f$

Before differentiating:

Always confirm whether the value of x in question lies within the domain of the composite function before evaluating the gradient at that point. If the value of x does not fall inside the domain of the composite, the derivative cannot be found at that point.