Camberwell Girls Grammar School
An Anglican School - Educating Tomorrow's Woman

STUDENT NUMBER

Figures									Letter:
Words									

Student Name			
Teacher:	Ms. Bergamin	Mr. Woodlock	Mr. Trufitt

MATHEMATICAL METHODS – UNIT 4

Problem-Solving Task – Calculus

Tuesday, 12 June, 2018

3:45 – 5:45pm
Reading time: 10 minutes
Writing time: 2 hours

Number of questions	*Number of questions to be answered*	*Number of marks*
7	7	74

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one set VCAA-approved notes, one CAS, (optional: scientific calculator)
- Students are not permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 13 pages.
- Working space is provided throughout the book.

Instructions

- Write your name in the space provided above on this page.
- All responses must be written in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room. Students must not disclose the contents of the task; to do so will be a breach of VCE guidelines and will be dealt with according to VCAA regulations.

Question 1: (6 marks)

The function $f(x) = 2 + 20e^{-4x},\ x \in [0,\ 6]$, is used to model the concentration of particulate matter (basically pollution and pollen) in the air, x km distance from a city centre. Particulate matter is measured in $\mu g/m^3$, micrograms per cubic metre.

a) (i) Find $f'(x)$.

(1 mark)

(ii) Hence, explain briefly why *f(x)* is a decreasing function for all values of *x*.

(1 mark)

(iii) In this situation, what could the constant term (2) in the rule for $f(x)$ represent?

(1 mark)

Let P be the point on the graph of *f* where $x = \frac{1}{2}$, i.e. half a kilometre from the city centre.

b) (i) Find the particulate matter concentration at P, **exactly**. Give units for this value.

(1 mark)

(ii) Find the rate of change in the concentration of particulate matter at P **exactly**. Give units for this value.

(1 mark)

(iii) Find the distance from the city centre when the concentration of particulate matter first drops below $\frac{11}{5}\ \mu g/m^3$. (Give answer to 2 dec pl).

(1 mark)

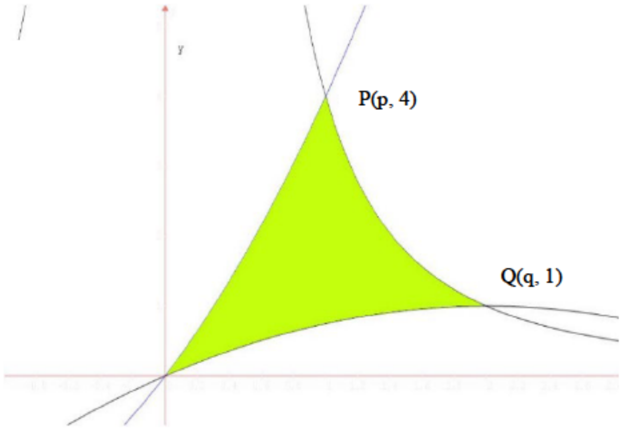
Question 2: (6 marks)

The local council wants to sow a small area of parkland with new lawn seed. The diagram opposite shows the area enclosed by three existing paths with equations. All distances are in metres.

$y = x(x+3)$,

$y = \dfrac{4}{x^2}$ and

$y = x - \dfrac{1}{4}x^2$



a) P and Q have coordinates $(p, 4)$ and $(q, 1)$ respectively. Find values for p and q.

(3 marks)

b) Write an expression, in terms of a number of definite integrals that gives the shaded area, i.e. (the area to be re-planted).

(2 marks)

c) Calculate the area to be re-planted, in m^2.

(1 mark)

Question 3: (10 marks)

The graph of the function $f(x) = ax^3 + bx^2 + cx$ has a stationary point at $(2, d)$ and the equation of the tangent at the point $x = 1$ to the curve is $12x + y = -1$.

a) Find the values of a, b, c and d.

(4 marks)

b) Find the coordinates of the point where the tangent *cuts* the curve of $f(x)$.

(2 marks)

c) Given $g(x)=f(x)+k$, where k is a real constant, find all values of k such that $g(x)=0$ has only one solution.

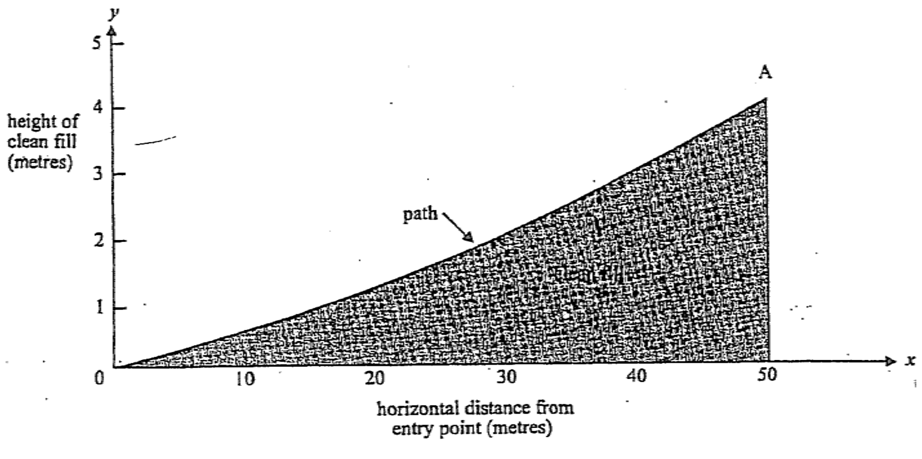
(2 marks)

d) State the values of x for which $f(x-2)$ is an increasing function.

(2 marks)

Question 4: (17 marks)

A rubbish tip is closed, then levelled flat and covered with clean fill in order to create public parkland. A straight path is constructed and runs from one of the entry points to the park to a seat at point A which is a horizontal distance of 50 metres from the entry point and is the highest point in the park. A cross-sectional view of this path is shown below.



Not to scale

With respect to the axes shown, the path follows a curve with the rule $y = \frac{x+1}{50} log_e(x+1)$.

a) Write down the coordinates of point A, giving the *y*-value to 2 decimal places.

(1 mark)

b) Show algebraically that the graph passes through the origin.

(1 mark)

c) Find the gradient of the path at point B which is a horizontal distance of 25 metres from the entry point to the park. Express your answer as a decimal correct to 3 decimal places.

(2 marks)

d) Determine whether the gradient of the path ever exceeds $\frac{1}{10}$. Explain your reasoning.

(2 marks)

e) Assuming that the path is 2 metres wide and that there is no change in the slope of the path from one side to the other, find the volume of clean fill lying directly beneath the path, to 2 decimal places. Assume: $x \in [0, 50]$.

(2 marks)

f) (i) **Use calculus** to find the derivative of $\frac{2}{(2x+1)^2}\log_e(2x+1) + e^{2x+1}$.

(4 marks)

(ii) **Use your answer** from part (i) to find $\int \frac{8\log_e(2x+1)}{(2x+1)^3}dx$

(5 marks)

Question 5: (12 marks)

The height, $h(x)$ metres, of a water slide above water level is given by the piecewise function:

$$h(x)=\begin{cases} 3-\frac{1}{4}x^2, & x\in[0,1] \\ Ae^{-k(x-1)}-0.5, & x\in(1,7] \end{cases}$$

where x is the horizontal distance in metres from the entrance point at the top of the slide and A and k are real constants.

a) Given that the two branches of $h(x)$ intersect at $x = 1$, show that $A=\frac{13}{4}$.

(1 mark)

Use $A = \frac{13}{4}$ for the remainder of the question.

b) (i) Find $h'(x)$.

(1 mark)

(ii) The water slide is continuous and smooth at the intersection between the two branches of $h(x)$. Use this information to show that $k=\frac{2}{13}$.

(2 marks)

c) Give the coordinates of the point on the slide that has the maximum negative slope. Explain your reasoning.

(2 marks)

d) On the axes below, draw a graph of $h(x)$ showing all key features.

(3 marks)

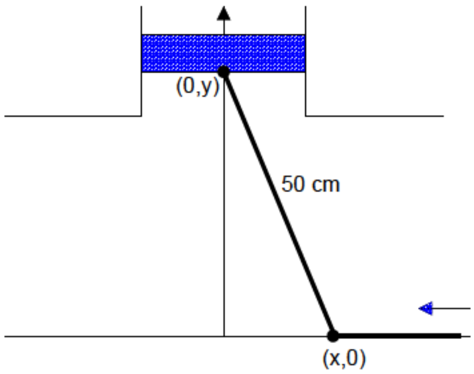
e) To gain maximum distance before entering the water, a person should jump off in a straight line at the end of the slide, where $x = 7$, in a direction that is perpendicular to the direction of the slide at $x = 4$. Find the equation of this line, giving any constants to 2 decimal places.

(3 marks)

Question 6: (14 marks)

A machine uses a 50cm long moveable, connecting rod to move a piston up and down a vertical shaft. The piston is attached to the connecting rod. The coordinates of the ends of the rod are $(x,\ 0)$ and $(0,\ y)$ as shown in the diagram. The position, x cm, of the end of the rod on the x-axis at time t seconds is given by

$$x(t) = 30\cos\left(\frac{\pi}{8}t\right),\ t \geq 0.$$



a) Show that the distance y cm in terms of x is given by: $y = \sqrt{2500 - x^2}$.

(2 marks)

b) (i) Hence, show that a rule for $y(t)$, the distance y cm at time t seconds is given by:

$$y(t) = \sqrt{2500 - 900\cos^2\left(\frac{\pi}{8}t\right)}$$

(2 marks)

(ii) **Hence** find the average value of y over the first 8 seconds of the motion of the rod. (Answer correct to three decimal places).

(2 marks)

c) At what x value(s) will the piston be at its highest position on the y-axis? Explain briefly.

(2 marks)

d) For $t \in [0,16]$, solve the equations $x(t) = 15$ and $x(t) = -15$. Hence find, in the first 16 seconds of the motion of the rod, the exact length of time for which $-15 \le x \le 15$.

(2 marks)

e) Show for $t \in [0,16]$ that the **speed** of the end of the rod along the x-axis, when $x = 15$, is $\dfrac{15\pi\sqrt{3}}{8}$ cm/s.

(2 marks)

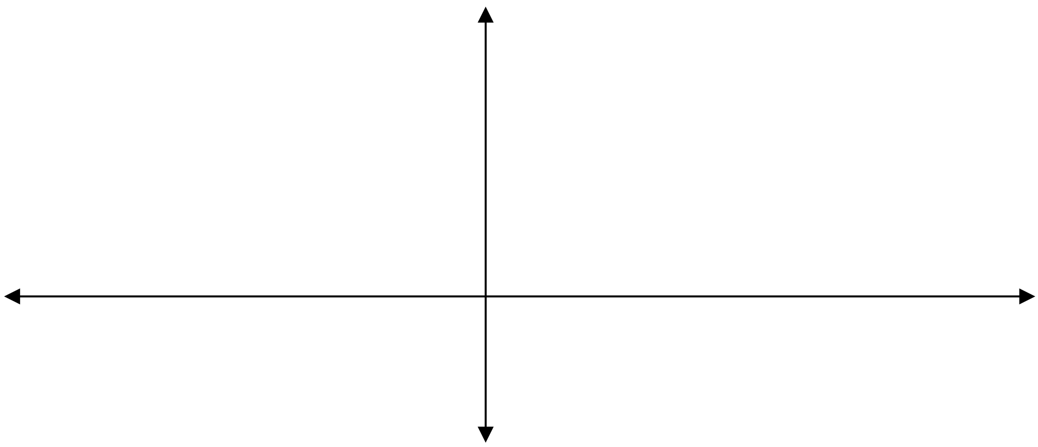
f) For $t \in [0,16]$, find the **exact** maximum speed of the end of the rod as it moves along on the x-axis.

(2 marks)

Question 7: (9 marks)

The $sinc(\,)$ function is defined as $sinc(x) = \dfrac{\sin(x)}{x}$, and is used widely in communications engineering theory.

a) Sketch the $sinc(\,)$ function for $x \in [-4\pi, 4\pi]$. Axial intercepts and stationary points do NOT need to be labelled; however ALL other key features should be.



(2 marks)

b) Can you evaluate $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x}$? Briefly explain.

(2 marks)

Looking at the curve you drew in part a), it is obvious that the limit **does**, in fact exist, as x approaches 0 from either side. We need another technique to evaluate this type of limit.

<u>L'Hôpital's rule:</u> states that if the limit has an indeterminate form, $\frac{0}{0}$ or $\frac{\infty}{\infty}$, all we need to do is differentiate **both** the numerator and the denominator **separately** and **then** take the limit.

For example: Evaluate $\lim\limits_{x \to 2} \frac{x^2 - 3x + 2}{x - 2}$ This limit reduces to the indeterminate form $\frac{0}{0}$.

Applying L'Hôpital's rule **first** gives: $\lim\limits_{x \to 2} \frac{2x - 3}{1} = \frac{2(2) - 3}{1} = 1$

L'Hôpital's rule can be applied as many times as required to get a limit that can be evaluated.

c) Now evaluate $\lim\limits_{x \to 0} \frac{\sin(x)}{x}$.

____

____

____

(2 marks)

d) Apply L'Hôpital's rule to evaluate the following limit:

$$\lim_{x \to 0} \frac{e^x - 1 - x - x^2/3}{\cos(x) - 1 + 4x^3}$$

____

____

____

____

____

____

____

(3 marks)

END OF TASK