Camberwell Girls Grammar School
An Anglican School

CREATE YOUR TOMORROW

STUDENT NUMBER

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| **Student Name** | | |
| **Teacher:** | Mr. Woodlock | Mr. Trufitt |

# MATHEMATICAL METHODS – UNIT 4

## Analysis Task – Calculus

Wednesday, 19 June, 2019

**Reading time: 10 minutes**
**Writing time: 2 hours**

| *Number of questions* | *Number of questions to be answered* | *Number of marks* |
|---|---|---|
| 7 | 7 | 72 |

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one set VCAA-approved notes, one CAS, (optional: scientific calculator)
- Students are not permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

**Materials supplied**
- Question and answer book of 13 pages.
- Working space is provided throughout the book.

**Instructions**
- Write your name in the space provided above on this page.
- All responses must be written in English.

**Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room. Students must not disclose the contents of the task; to do so will be a breach of VCE guidelines and will be dealt with according to VCAA regulations.**

**Question 1: (8 marks)**

The Augusta Golf Course is bordered by two straight boundaries and a river. The shaded area in the diagram below shows the golf course.

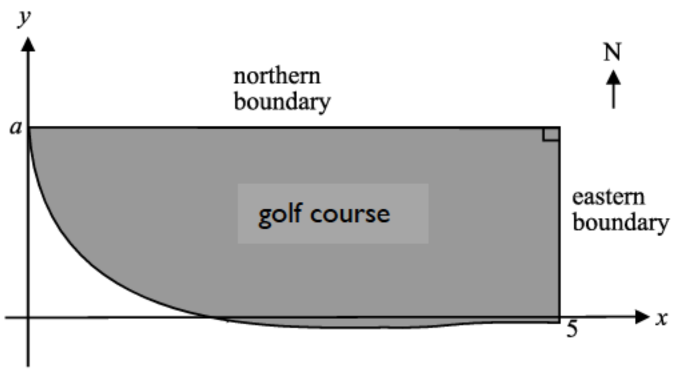
The $y$-axis runs in a north-south line and in relation to the set of axes the northern boundary of the golf course is described by the rule $y = a$. The eastern boundary is described by the rule $x = 5$.

The river boundary is described by the function:

$$f(x) = \frac{1 - log_e(x+1)}{(x+1)^2}, \quad x \in [0, 5]$$

where: 1 unit represents 1 kilometre.

The north-west corner of the golf course is located at the point $(0, a)$.




**(a)** Show that $a = 1$. **1 mark**

**(b)** Show algebraically that the function $f$ crosses the $x$-axis at $x = e - 1$. **2 marks**

**(c)** Find the length of the eastern boundary of the golf course. Express your answer in kilometres correct to four decimal places. **2 marks**

**(d)** **(i)** Evaluate $\int_{e-1}^{5} f(x)dx$. **1 mark**

**(ii)** Write down an integral that would give the area of the golf course and hence find this area, correct to the nearest square kilometre.

**2 marks**

**Question 2: (9 marks)**

**(a)** Find the gradient of the tangent to the curve $y = x^3 \cos(3x)$ at the point where $x = \pi$.

**2 marks**

**(b)** Find, exactly, the average value of the function $y = 4 - 3e^{-\frac{x}{2}}$ over the interval $[0, 6]$.

**2 marks**

(*For this part of the question, you must show working for full marks*)

**(c)** **(i)** Differentiate $(x-1)\log_e\left((x-1)^2\right)$ with respect to $x$. **2 marks**

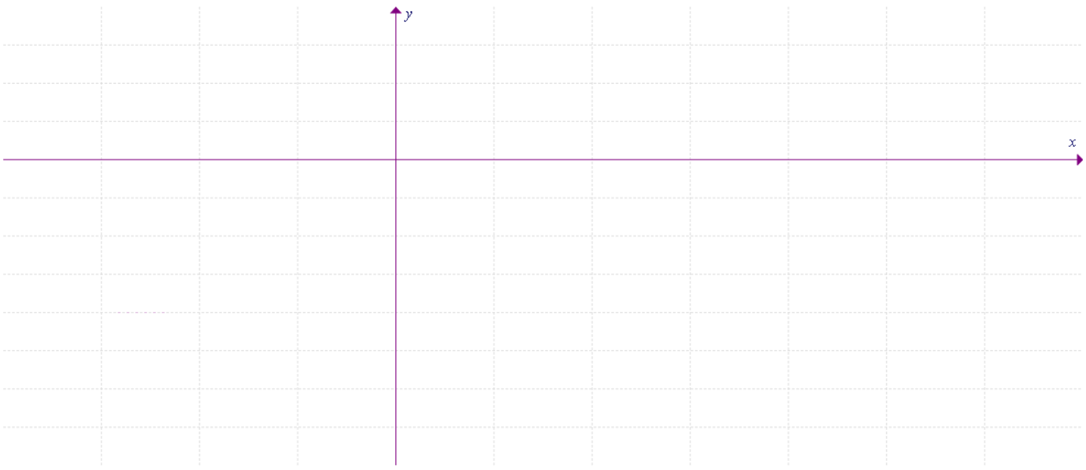
**(ii)** Hence or otherwise evaluate $\int_2^3 \frac{1}{2}\log_e\left((x-1)^2\right)dx$. **3 marks**

**Question 3: (6 marks)**

Consider the function $f(x) = x - x^2,\ x \in [-1, k]$ where $k \in (1, 3]$.

**(a)** Sketch the graph of $f$. Use shading, or similar, to indicate the area enclosed by the $x$-axis, and the bounds $x = -1$ and $x = k$.

**3 marks**



**(b)** Find the total area enclosed by the graph of $f$, the $x$-axis and the line $x = k$. Leave your answer in terms of $k$.

**3 marks**

## Question 4: (10 marks)

The acceleration, $a$ m/s$^2$, of a particle travelling in a straight line is given by the rule: $a(t) = 6 - \frac{2}{(t+1)^2}$, $t \geq 0$, where $t$ is the time in seconds from the start. Initially the particle is at rest.

**(a)** Show algebraically that the expression for the velocity, $v$ m/s, of the particle at time, $t$ seconds, is given by the rule $v(t) = 6t + \frac{2}{t+1} - 2$.

**3 marks**

**(b)** Sketch a graph of $v$ against $t$ for $t \in [0, 5]$, showing the coordinates of the endpoints **exactly**.

**2 marks**


**(c)** Find the average rate of change of the velocity in the first 3 seconds.

**1 mark**

**(d)** Find the average velocity in the first 3 seconds. State your answer correct to 2 dp.

**2 marks**

**(e)** Briefly explain why the distance the particle travels in the first 5 seconds and the displacement of the particle after the first 5 seconds are the same.

**2 marks**

**Question 5: (14 marks)**

A fish farm is trialling a new method of sustainable fishing/breeding. They find that the rate of change of the population of fish over a 6-week period is given by the rule:

$$\frac{dP}{dt} = \begin{cases} \frac{1}{\sqrt{e}}t - \frac{1}{\sqrt{e}} & , \quad t \in [0, 2] \\ e^{-\frac{(t-3)^2}{2}} & , \quad t \in (2, 6] \end{cases}$$

where $P$ is the number of fish **in thousands (1000's)** and $t$ is the number of weeks. Initially, there were 1000 fish in the farm.

**(a)** **(i)** At the start of the trial (to the nearest integer), what was the rate of fishing (removal of fish)? Give units in your answer.

**1 mark**

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**(ii)** After 1 week of the trial, (to the nearest integer), how many fish were there in the farm?

**2 marks**

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**(iii)** What was the maximum increase in fish population and after how many weeks this occur?

**2 marks**

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**(iv)** After 6 weeks, (to the **nearest hundred**), what was the final fish population?

**2 marks**

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To address the problem that the fish population was initially decreasing, an alternate model is trialled. It is found that the new rate of change of population is modelled by the rule:

$$\frac{dP}{dt} = \frac{\sin(t-2)}{(t-2)} - 0.2, \quad t \in [0, 6], t \neq 2.$$ Initially, there were 1000 fish in the farm.

**(b)** **(i)** Does this new model meet the requirement? Justify your answer.

**2 mark**

**(ii)** (To the nearest hundred), what is the maximum population of fish?

**3 marks**

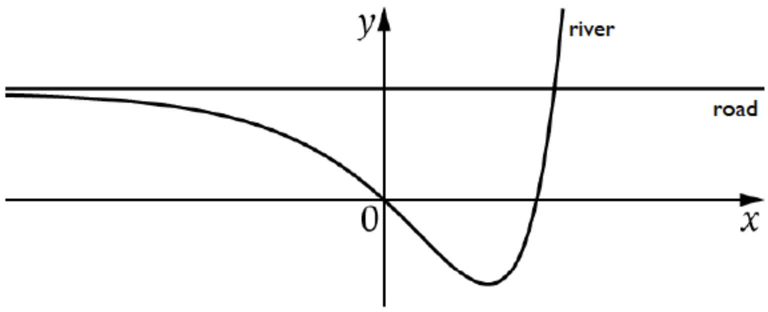
**(iii)** In the long term, $(t \to \infty)$, is this new model of fish farming sustainable? Briefly explain.

**2 marks**

**Question 6: (13 marks)**

A river runs close to an east-west road as shown in the diagram below. The river then slowly veers away from the road before turning back and eventually crosses **under** the road. The road can be modelled by the straight line with equation $y = c$ and the river by the curve with equation: $y = e^{2x} - 2ke^{x} + 5$, where each unit on the $x$ and $y$ axes represents one kilometre.

The river passes through the origin.




**(a)** If the equation modelling the road is an asymptote to the curve modelling the river, find the value of $c$.

**1 mark**

**(b)** Show that $k = 3$.

**1 mark**

**(c)** Find the exact coordinates of the second point where the river crosses the x-axis.

**2 marks**

**(d)** Find the exact coordinates of the point where the river passes under the road.

**1 mark**

**(e)** Find the exact coordinates of the point where the river turns back towards the road. **Hence** state the furthest distance the river is south of the road in kilometres.

**3 marks**

**(f)** Find the exact area of the region bounded by the river and the x-axis.

**2 marks**

A developer wishes to purchase the land enclosed by the road, the river and the line with equation $x = -1$.

The land is valued at \$60 000 per square kilometre.

**(g)** Find the amount, to the nearest \$100, that the developer will need to pay for the land.

**3 marks**

**Question 7: (12 marks)**

Let $f(x) = \log_e\left(x^2\right),\ x \neq 0$ and $g(x) = \dfrac{kx}{x+1},\ x \neq -1$ where $k$ is a real number.

**(a)** State

**(i)** $f'(x)$

**1 mark**

**(ii)** $g'(x)$

**1 mark**

Let the function $h$ be given by: $h(x) = f(x) - g(x)$, $x \neq 0$ and $x \neq -1$.

**(b)** Show that $h'(x)$ can be expressed in the form: $\dfrac{2x^2 + (4-k)x + 2}{x(x+1)^2}$.

**2 marks**

**(c)** Find the values of $k$ for which the equation $h'(x) = 0$ has

**(i)** no solutions

**3 marks**

**(ii)** one solution

**1 mark**

**(iii)** two solutions

**2 marks**

**(d)** If the graph of $y = h(x)$ has exactly one point of zero gradient, find the coordinates of this point.

**2 marks**

**END OF TASK**