Camberwell Girls Grammar School
An Anglican School

CREATE YOUR TOMORROW

Student Name			
Teacher	Mr Trufitt	Ms Tan	Ms Bergamin

MATHEMATICAL METHODS UNIT 3

Application Task – Curve fitting

PART 3 – Approximating a function by its tangent

Date: May 2020

Writing time: 60 minutes (one on-line class)

Structure of Task

Section	*Number of questions*	*Number of questions to be answered*
Part 3	2	2

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one CAS calculator and/or one scientific calculator, and one approved bound reference.
- Students are not permitted to use: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 6 pages.
- Working space is provided throughout the book.

Instructions
- Write your name in the space provided above on this page.
- All responses must be written in English.

Stu dents are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Students must not disclose the contents of the task; to do so will be a breach of School guidelines.

Continuing with the theme of modelling curves, part 3 of the task will require you to apply your knowledge of differential calculus and various functions to further model the curves of given functions, and reflect on your findings.

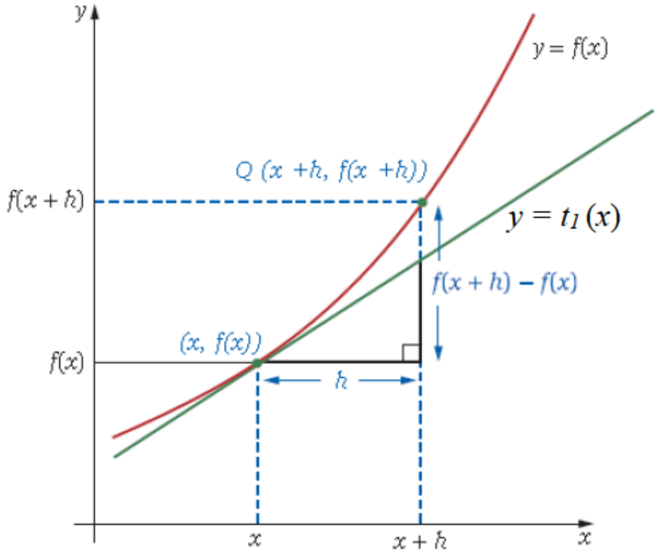
Approximating a function locally by its tangent

The linear approximation of any function $y = f(x)$ may be considered as using the tangent at a point to approximate the curve in an immediate neighbourhood "locally" of the point at which the tangent is determined.

In the diagram, $t_1(x)$ is the tangent to the curve at the point $(x, f(x))$.

$f'(x)$ is the gradient of the tangent at x.

Recall: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$



Question 1

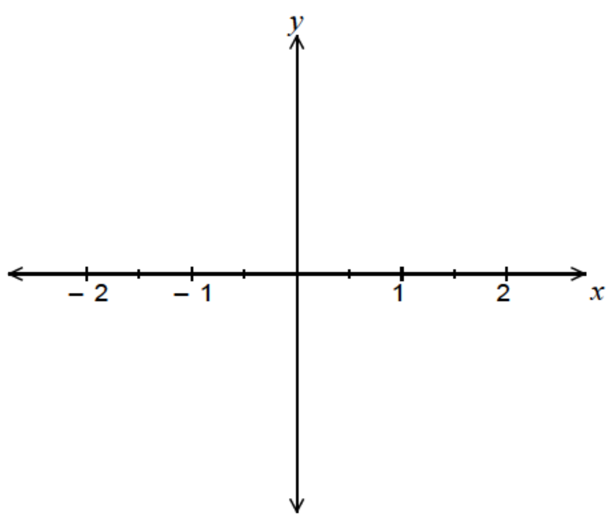
Consider the function $f(x) = e^x$, $x \in R$.

(a) Find $f(0)$ and $f'(0)$.

(b) Show algebraically that the equation of the tangent $t_1(x)$ to $f(x)$ at $x = 0$ is given by $t_1(x) = x + 1$.

(c) On the set of axes, sketch, so that the point of intersection is shown, the graphs of $y = f(x)$ and its tangent at $x = 0$, $x \in [-2, 2]$.

Show end-points, the coordinates of any axial intercepts and point of intersection.



When "locally" approximating a function, two important issues are the accuracy of the approximation and the interval over which the approximation has a desired level of accuracy. These issues can be explored by considering how close the y-values on the tangent, $t(x)$, are to the actual y-values on the curve $f(x)$.

The error term, $R(x) = f(x) - t(x)$, gives the difference of the y-values of each function for a particular value of x. We will be using the $|....|$ sign, so that we work with the magnitude (size) of the error. So $|f(x) - t(x)|$ just removes the negative sign, if $f(x) < t(x)$.

The percentage error can be found by calculating $\frac{R(x)}{f(x)} \times 100$, where $f(x) \neq 0$.

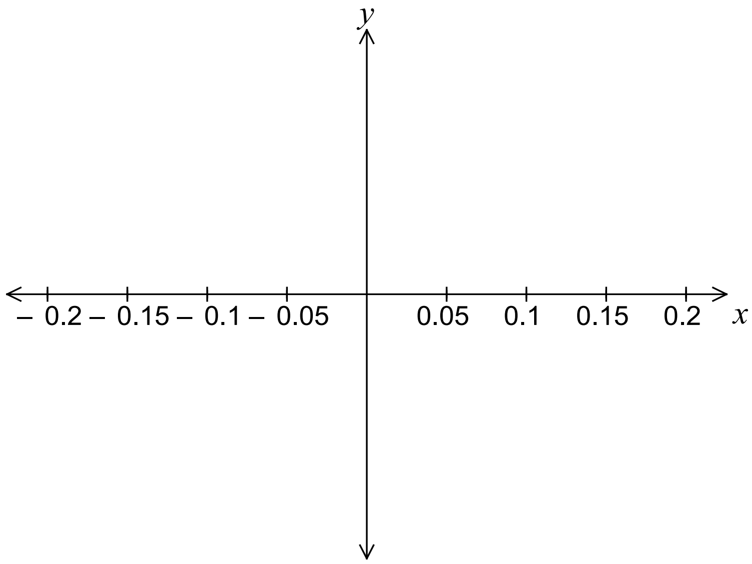
(d) Use CAS to complete the following table of values showing $f(x)$, $t_1(x)$, $R(x) = f(x) - t_1(x)$ and $\frac{R(x)}{f(x)} \times 100$, for $x \in [-0.2, 0.2]$ in increments of 0.05. **Note** that $t_1(x)$ is the rule for the tangent to the graph of $f(x) = e^x$ at $x = 0$. Give your answers correct to three decimal places.

x	$f(x) = e^x$	$t_1(x) = x + 1$	**Error** $R(x) = \|e^x - (x+1)\|$	**Percentage Error** $\frac{R(x)}{f(x)} \times 100$
- 0.2				
- 0.15				
- 0.1				
- 0.05				
0				
0.05				
0.1				
0.15				
0.2				

(e) The table gives an indication of how good an approximation the tangent at $x = 0$ is to $f(x) = e^x$, for $x \in [-0.2, 0.2]$. In your own words, do you think that this is a good approximation? If so, why?

(f) **(i)** Sketch the graph of the Percentage Error versus x for $x \in [-0.2, 0.2]$.

Label end-points correct to three decimal places.



(ii) Does this graph confirm your response to part **(e)**? Explain.

(g) Hence, or otherwise, find the interval, correct to four decimal places, for which the linear approximation has a percentage error of less than 1%.

Question 2

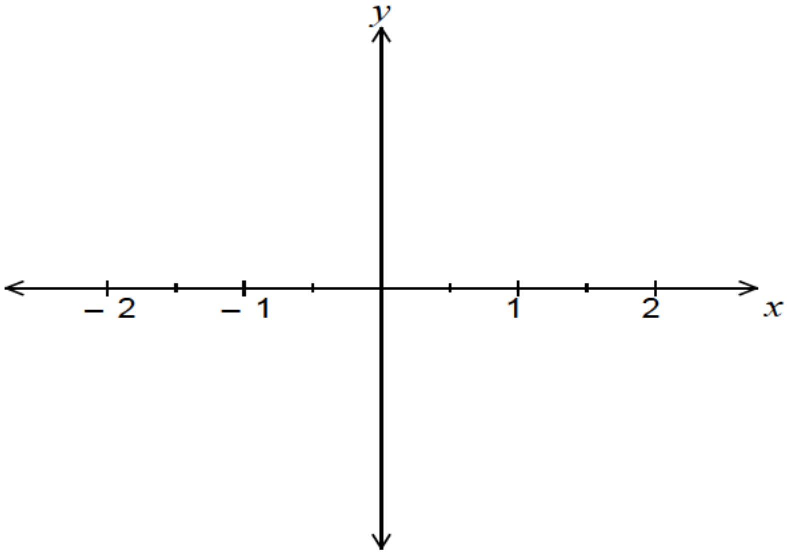
Consider the function $g(x) = \log_e(3 + 2x)$.

(a) Show that the equation of the tangent $t_2(x)$, to $g(x)$ at $x = 0$ is given by $t_2(x) = \frac{2}{3}x + \log_e(3)$.

(b) On the same set of axes, sketch, around their point of intersection, the graphs of $g(x)$ and the equation of the tangent to $g(x)$ at $x = 0$.

Use the domain $x \in [-2, 2]$ and show all end-points, coordinates of any axial intercepts and the point(s) of intersection.

Label any asymptote(s) with their equation(s).



(c) Investigate the accuracy of the tangent in approximating $g(x)$ in the vicinity of $x = 0$. This process is also referred to as finding the 'goodness of fit'. Complete the table, giving answers to 4 decimal places.

x	$g(x) = \log_e(3+2x)$	$t_2(x) = \frac{2}{3}x + \log_e(3)$	**Error**	**Percentage Error**
- 0.2				
- 0.15				
- 0.1				
- 0.05				
0				
0.05				
0.1				
0.15				
0.2				

(d) Compare the values in the previous table to the one in Question 1 part **(d)**. Considering only $x \in [-0.2, 0.2]$, which of the two options below gives the better approximation?

⇨ the tangent at $x = 0$ to the curve $f(x) = e^x$, or

⇨ the tangent at $x = 0$ to the curve $g(x) = \log_e(3 + 2x)$.

Justify your reason(s).

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Finally, one last bit of mathematical notation you will require for part 4 ….

<u>First, second, third and higher derivatives</u>

Consider $f(x) = 5x^4 - 7x^3 + 4x^2$:

$f'(x)$ is the **first derivative** of $f(x)$. For this example: $f'(x) = 20x^3 - 21x^2 + 8x$

$f''(x)$ is the **second derivative** of $f(x)$. For this example: $f''(x) = 60x^2 - 42x + 8$

$f'''(x)$ is the **third derivative** of $f(x)$. For this example: $f'''(x) = 120x - 42$.

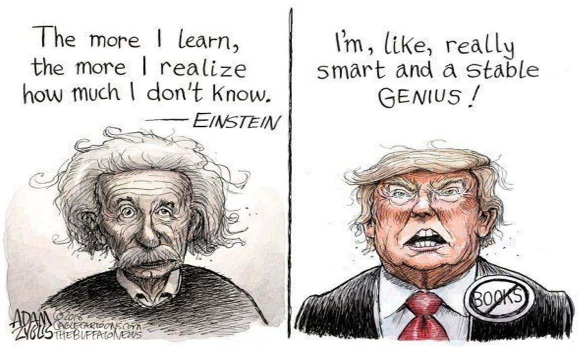
For higher order derivatives, typically greater than the third derivative, we use the notation $f^{(n)}(x)$ where n is the n^{th} derivative of f with respect to x.

For example, CAS can show that, given: $g(x) = 2x^6 + 3x^4 - 2x^3 + 8$:

the **fourth derivative** $g^{(4)}(x) = 720x^2 + 72$

the **fifth derivative** $g^{(5)}(x) = 1440x$.

….. **better than writing $g'''''(x)$!!**



End of part 3