Camberwell Girls Grammar School
An Anglican School

CREATE YOUR TOMORROW

Student Name			
Teacher	Mr Trufitt	Ms Tan	Ms Bergamin

MATHEMATICAL METHODS UNIT 3

Application Task – Curve fitting

PART 4 – Using a power series

Date: May 2020

Reading time: 10 minutes
Writing time: 75 minutes

Structure of Task

Section	*Number of questions*	*Number of questions to be answered*
Part 4	4	4

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one CAS calculator and/or one scientific calculator, and one approved bound reference.
- Students are not permitted to use: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 9 pages.
- Working space is provided throughout the book.

Instructions
- Write your name in the space provided above on this page.
- All responses must be written in English.

Stu dents are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Students must not disclose the contents of the task; to do so will be a breach of School guidelines.

PART 4

Introduction

When a scientific calculator gives a very accurate rational approximation to, say, $e^{\frac{1}{2}}$ or $\cos(1.5)$, it does so by using a ***power series approximation*** of e^x and $\cos(x)$ respectively.

For example: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

This is called a ***power series.*** A power series is a collection of terms in increasing powers of x according to a specific pattern. In fact, most mathematical functions can be approximated (about a particular value of x) by a power series.

Evaluating this series up to a certain number of terms gives an approximation for $\sin\left(\frac{\pi}{6}\right)$. For example:

1 term: $\sin\left(\frac{\pi}{6}\right) \; ; \; \frac{\pi}{6} \quad (; \; 0.523599)$

2 terms: $\sin\left(\frac{\pi}{6}\right) \; ; \; \frac{\pi}{6} - \frac{\left(\frac{\pi}{6}\right)^3}{3!} \quad (; \; 0.499674)$

3 terms: $\sin\left(\frac{\pi}{6}\right) \; ; \; \frac{\pi}{6} - \frac{\left(\frac{\pi}{6}\right)^3}{3!} + \frac{\left(\frac{\pi}{6}\right)^5}{5!} \quad (; \; 0.500002)$

Note: You know from your exact value table that: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. What do you notice about the accuracy of the estimation as the number of terms used increased?

A power series for a particular function is also called a **Maclaurin Series**, when centered about $x = 0$. The particular Maclaurin series for e^x is:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$ i.e. e^x is closely approximated by a series of polynomial terms.

Previously, we looked at how well the line $y = x + 1$ approximated the curve of $f(x) = e^x$ for $x \in [-0.2, 0.2]$

Question 1

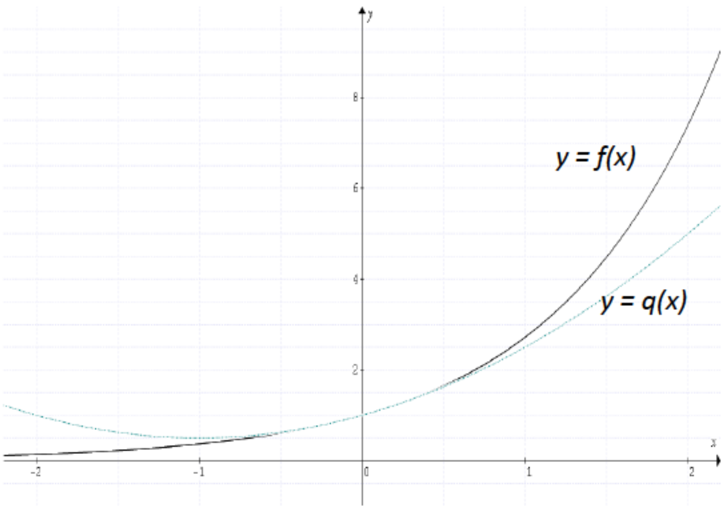
(a) Using the Maclaurin series for $f(x) = e^x$ above, evaluate:

(i) the linear approximation $e^x = 1 + x$ (first 2 terms) for $e^{1.1}$

(ii) the quadratic approximation $e^x = 1 + x + \frac{x^2}{2!}$ (first 3 terms) for $e^{1.1}$.

Let's investigate how well the **quadratic** polynomial from the Maclaurin series part **(a) (ii)** approximates the curve: $f(x) = e^x$ for $x \in [-0.4, 0.4]$. **Note the changed domain!**

The graph shows the functions $f(x) = e^x$ and $q(x) = 1 + x + \frac{x^2}{2!}$. You can see that $y = q(x)$ is a reasonable approximation to the curve $y = f(x)$ for values of x close to 0.



(b) Now complete the table of values for the quadratic approximation: $q(x) = 1 + x + \frac{x^2}{2!}$. Give all values in this table to 4 decimal places.

x	$f(x) = e^x$	$q(x) = 1 + x + \frac{x^2}{2!}$	Magnitude of the error $R(x) = \lvert f(x) - q(x) \rvert$	Percentage Error $\frac{R(x)}{f(x)} \times 100$
- 0.4				
- 0.2				
0				
0.2				
0.4				

(c) **(i)** Write down the cubic polynomial that would approximate the function $f(x) = e^x$ using the Maclaurin series.

(ii) The function $f(x) = e^x$ is transformed by a reflection in the y-axis and a dilation factor 0.5 from the y-axis. Write down the rule of the new function.

(iii) Transform the cubic polynomial function above so that it would approximate the new exponential curve. Write down the equation of the transformed function.

The $\cosh(...)$ function is a mathematical function that has similar properties to the familiar trigonometric $\cos\mathit{ine}$ function. It is, however defined in terms of e^x, $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$.

The shape of the curve described by the $\cosh(...)$ function is called a **catenary curve**. The famous St Louis Gateway Arch in the USA (shown) is actually an inverted catenary curve, not a parabola.

(d) Use previous results to write down the quartic polynomial that would approximate the $\cosh(x)$ function.

(e) Write down the quartic polynomial that would approximate the function: $g(x) = 2e^{-3x}$.

(f) **(i)** Write down the quartic polynomial that would approximate the function $g(x) = e^{\sqrt{x}}$ using the Maclaurin series.

(ii) Use your answer to **(f) (i)** to evaluate $g'(1)$.

(iii) Given $f(x) = e^{\sqrt{x}}$, use CAS (or calculus) to find $f'(x),\ f'(1)$.

(iv) How good do you think the Maclaurin approximation to $e^{\sqrt{x}}$ is? How could you improve it?

Recall that: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$ and $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

(g) A skateboard ramp is modelled on a portion of the function $h(x) = e^x \sin(2x)$. Find a power series that would approximate the function $h(x)$.
In your answer, **do not list any terms** with degree higher than 4, ie do not include terms containing $x^5, x^6,$ etc.

You are now probably wondering where the above (and many other) power series came from .....

Question 2

Given $f(x) = \log_e(1+x)$, **use CAS** to write down expressions for:

(a) $f'(x),\ f''(x),\ f'''(x)$.

(b) Evaluate: $f'(0),\ f''(0),\ f'''(0)$.

In general, given a function $f(x)$, the Maclaurin Series for that function can be written as:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ \dots \ + \frac{f^{(n)}(0)}{n!}x^n + \dots\dots,$$ where f', f'', f''' and $f^{(n)}$

are the first, second, third and n^{th} derivatives of that function.

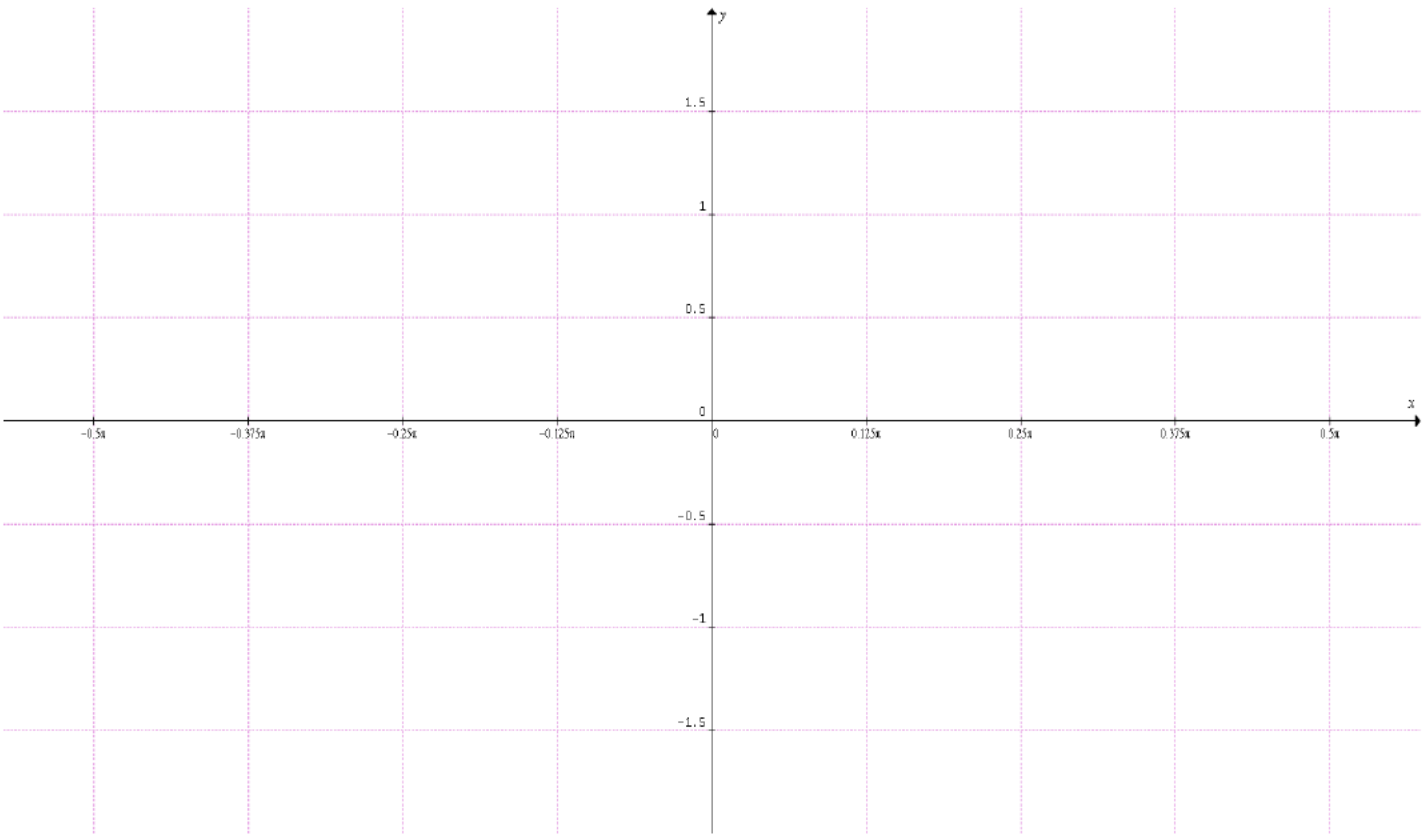
Question 3

(a) Consider $f(x) = \sin(x)$. Complete the table below using the Maclaurin series expansion (page 6.)

	$no. of\ terms$	$Maclaurin\ Series\ Polynomial\ Approximation$
$y_1(x)$	2	$\sin(x) \cong f(0) + f'(0)x =$
$y_2(x)$	3	$\sin(x) \cong$
$y_3(x)$	4	$\sin(x) \cong$

(b) On the same set of axes, quickly **sketch** $y = \sin(x)$, $y_1(x)$, $y_2(x)$ and $y_3(x)$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Use different colours or dashed lines to ensure that each graph is obvious.



(c) What do you notice about the graphs of $y_1(x)$, $y_2(x)$ and $y_3(x)$ in relation to that of $y = \sin(x)$? Is this what you would expect given the results in previous parts of this task?

Question 4

(a) Applying the Maclaurin series: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ \dots \ + \frac{f^{(n)}(0)}{n!}x^n \ + \dots..$

<u>and CAS</u>, find all the terms up to and including x^4 in the power series for $e^{\sin(x)}$.

(b) The first four terms of the Maclaurin series expansion of the function $f(x) = \dfrac{\sin(3x)}{x}$ are:

$f(x) = 3 - \dfrac{9x^2}{2} + ax^4 + bx^6$ where a, b are the coefficients of the x^4 and x^6 terms respectively. State a and b.

END OF TASK