Camberwell Girls Grammar School
An Anglican School

CREATE YOUR TOMORROW

STUDENT NUMBER

									Letter
Figures									
Words									

Student Name			
Teacher:	Ms Tan	Ms Bergamin	Mr Trufitt

MATHEMATICAL METHODS – UNIT 4

Analysis Task – Calculus

Wednesday, July 29, 2020

Reading: 2:30pm – 2:40pm

Writing: 2:40pm – 4:40pm

Number of questions	*Number of questions to be answered*	*Number of marks*
7	7	65

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one set VCAA-approved notes, one CAS, (optional: scientific calculator)
- Students are not permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied

- Question and answer book of 13 pages.
- Working space is provided throughout the book.

Instructions

- Write your name in the space provided above on this page.
- All responses must be written in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room. Students must not disclose the contents of the task; to do so will be a breach of VCE guidelines and will be dealt with according to VCAA regulations.

Mathematical Methods - Modelling Assessment Task
(Functions & Graphs, Algebra and Calculus)

Introduction:

Lying just two metres above sea level at its highest point, the island nation of TootiFrooti is an early indicator of climate change, with predictions that many of the islands in the group could be lost to the sea by year 2050.

One of the islands in this group is Tanville, often called the Venice of the South Pacific. Estimates suggest around 110,000 people live here, and half of these live in South Tanville - a chain of small islands sharing a lagoon and coral reefs.

Sea level is expected to continue to rise in TootiFrooti. By 2030, under a high emission scenario, this rise is projected to be in the range of 5 – 14 cm. The sea-level rise combined with natural year-to-year changes will increase the impact of storm surges and coastal flooding.

SECTION 1: (35 marks)

A simple model of the tidal pattern at Tanville is:

$h(t) = 0.5 \times \cos\left(\frac{\pi}{6}(t+3)\right) + 0.70$, where: t is the number of hours after midnight, and

$h(t)$ is the height above sea level at time t, in meters

Question 1: (10 marks)

(a) Starting from midnight, sketch a fully-labelled graph of a 24-hour period of $h(t)$ v t.

(3 marks)

(b) What is the height above sea level of the high tide(s) and **at what times** do these occur?

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(2 marks)

(c) What is the height of the tide at 4 p.m, to 2 dec. places?

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(1 mark)

(d) St. Ken's Plaza is an ancient entertainment area on the island. It's floor is 0.95 meters above sea level. At what time(s) does the plaza begin to flood?

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(1 mark)

(e) How high does the water rise in the plaza?

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(1 mark)

(f) **(i)** Using the equation for $h(t)$, calculate the rate of change of the height of the tide at 12 noon. (Give your answer to 2 dp).

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(1 mark)

(ii) Find the **average** rate of change of the height of the tide between 11 am and 1 pm. (Give your answer to 2 dp).

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(1 mark)

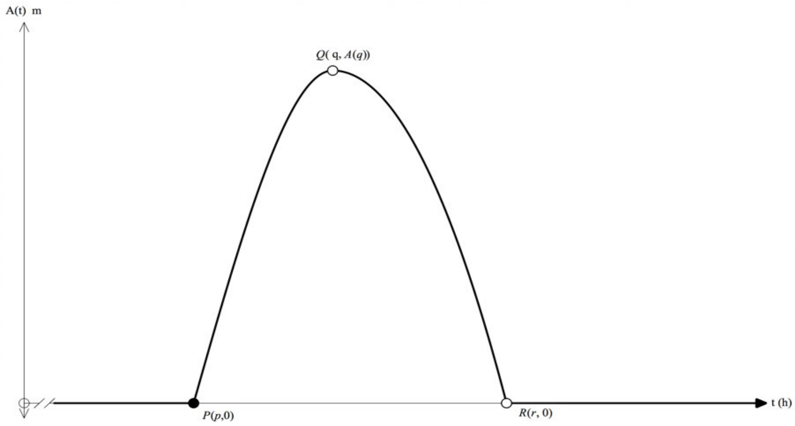
Question 2: (8 marks)

The governing body of St Ken's Plaza have decided they would like to try to minimize water damage from the tides rising in the plaza. They decide to build temporary "walls" around the canals. However, St Ken's Plaza is extremely old and has a poor drainage system. When the tide rises, the water also rises through a circular drain located in the centre of the plaza, forming a large "puddle" in the plaza.

The area of St Ken's Plaza is 15 000 m^2. At high tide, the worst of the flooding, one third of the area of the plaza is underwater.

The water level in the plaza rises with the tide, but then it takes longer for the level to dissipate than the tidal time due to the poor drainage.

Shown is a graph of the area of water in the plaza verses time, for a 24 hour period.



(a) Using information given, state the coordinates of P and Q.

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(2 marks)

(b) The equation for the area flooded is modelled on a hybrid equation of the form:

$$A(t) = \begin{cases} a(t-7)(t^2-14t+37) & p \le t \le q \\ b(t-6)(t-12) & q < t \le 12 \\ 0 & elsewhere \end{cases}$$

Find the values of a and b, and state the equation for $A(t)$ in full.

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(3 marks)

Assume that the "puddle" formed by the floodwater in the plaza is a flat cylinder.

(c) To the nearest metre, what is the puddle's diameter at high tide?

(2 marks)

(d) Find the volume of the puddle at high tide. (Give your answer in m^3).

(1 mark)

Question 3: (7 marks)

The council is considering adding an overflow drainage pipe that opens when the flood water level rises to more than 5 cm above the level of the plaza, and stays open until the plaza is no longer flooded. Remember that the floodwater level rises at the same rate as the tide.

(a) Show that the overflow pipe first opens at about 7:14 am.

(1 mark)

For parts **(b)** and **(c)** use $t = 7.229$.

(b) Find the volume, in m^3, of the puddle when the pipe first opens (to 2 dec places).

(2 marks)

Now assume that the rate of increase of the volume of floodwater in the plaza is a constant 625 m^3/hr. The area and the height of the puddle both vary as functions of time, $A(t)$ and $h(t)$ respectively.

(c) By applying the product rule to the formula for the volume of the puddle, $V = A.h$, find the rate of change of the height of the puddle when the pipe first opens (to 2 dec. places).

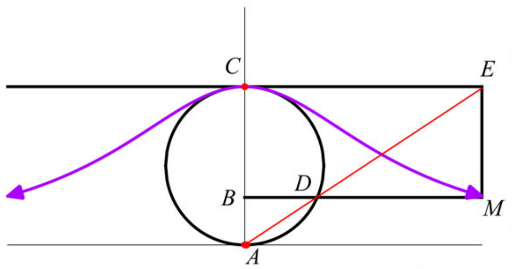
(4 marks)

TootiFrooti's climate varies considerably from year to year due to the El Nino effect. This is a natural climate pattern that occurs across the tropical Pacific Ocean and affects weather around the world. El Nino events tend to bring wetter, warmer conditions than normal. The Bureau of Meteorology predicts that this year there will be a strong El Nino event.

Long term rainfall predictions for July indicate possible monsoonal rainfalls that can be easily modelled.

The **daily** monsoonal rainfall for TootiFrooti is given by an equation known as the Witch of Agnesi.

The general equation of this curve is : $y = \frac{a^3}{x^2+a^2}$



Question 4: (10 marks)

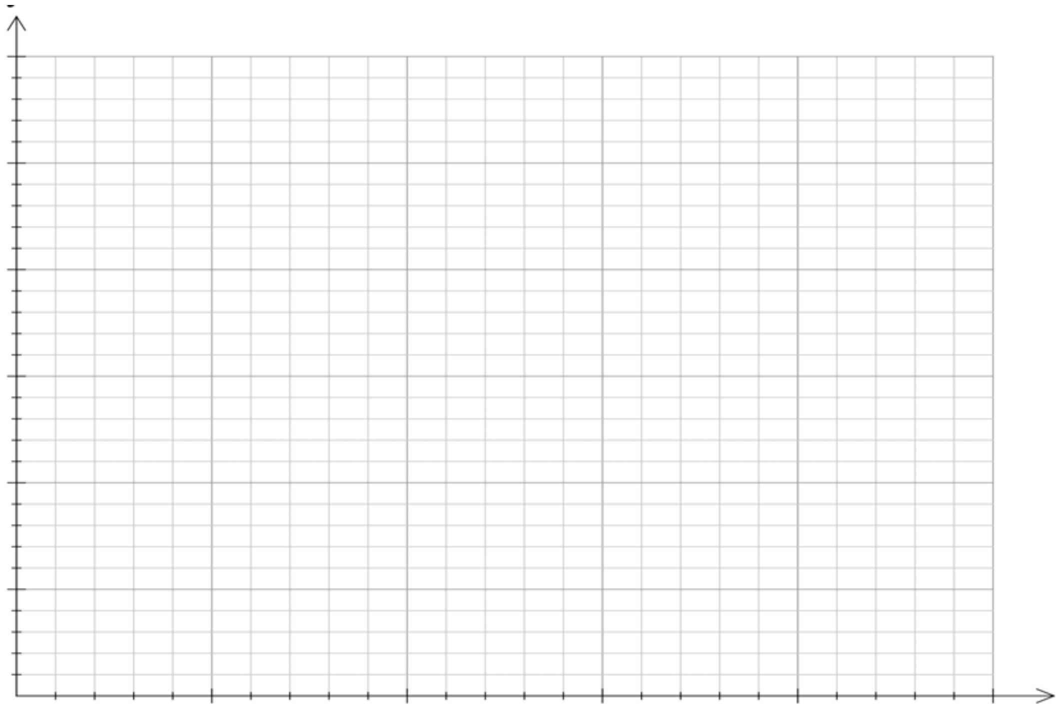
Regular measurement shows that the rule for the daily monsoonal rainfall is given by:

$$r(t) = \frac{k.a^3}{(t-12)^2 + a^2}$$ where $r(t)$ represents the amount of rainfall **in metres**,

t represents the number of hours after midnight, and

k is a scaling factor. Choose $k = 2$ and $a = 1.5$

(a) Starting from midnight, sketch a fully-labelled graph of a 24-hour period of $r(t)$ v t.



(3 marks)

(b) On the same axis above, draw the resulting effect from the combination of the two events, high tide and monsoonal rain, i.e. $h(t) + r(t)$.

(3 marks)

(c) What is the maximum combined height of the water? (Assume that the new drain is not installed). Give your answer to 2 dec. places.

(1 mark)

(d) Use CAS to find the maximum rate of increase of the combined height of water and the time of day when this occurs. (Give the rate to 2 dec. places).

(3 marks)

***SECTION 2*: (30 marks)**

Question 5: (10 marks)

The velocity v km/h of a train moving along a straight track from station A to station B is given by $v(t) = kt[1 - \sin(\pi t)]$, where t (hours) is the time measured from when the train left station A and k is a positive constant.

A**ssume** that the train began at station A from rest and did not stop until it reached station B.

(a) Show that the time it takes to travel from A to B is half an hour.

(2 marks)

(b) **(i)** Use calculus to differentiate $t\cos(\pi t)$ with respect to t.

(2 marks)

(ii) Hence find $\int t\sin(\pi t)\,dt$

(3 marks)

(c) Given that the distance from A to B is 20 km, use previous results, or otherwise, to find, to the nearest integer, the value of k.

(3 marks)

Question 6: (10 marks)

The position of a particle, in cm, travelling along a straight number line (as shown below) is given by the rule:

$$x(t) = (2t^2 - 3t)e^{at}, \ t = \text{number of seconds}, \ t \geq 0$$

(a) Show that the value of $'a'$ is $\log_e\left(\frac{\sqrt{6}}{2}\right)$ if the particle is 3 cm to the right of the origin after 2 seconds.

(2 marks)

(b) The particle is accidently bumped so its position is now given by the rule:

$$x(t) = (2t^2 - 3t)e^{-2t},\ t \geq 0.$$

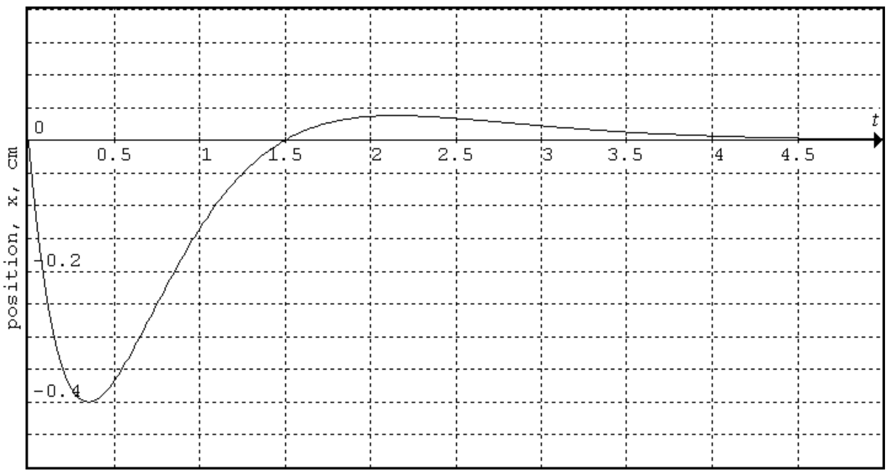
(i) At what value(s) of time t is the particle instantaneously at rest? (Give exact answers).

(3 marks)

(ii) State $\lim_{t \to \infty} x(t)$.

(1 mark)

The graph of $y = x(t) = (2t^2 - 3t)e^{-2t}$ for $t \in [0, 5]$ is shown below:



(iii) State the values of t, $t \in [0, 5]$ when the particle will be in a negative position.

(1 mark)

(iv) Using previous results. state the values of t, $t \in [0, 5]$ when the particle will be **moving** in a negative direction. (Give exact answers).

(3 marks)

Question 7: (10 marks)

The grass in a nearby parkland is dying off and rangers suspect that it is due to the level of carbon monoxide from the exhausts of passing traffic. The grass is found to have a carbon monoxide content given by the rule: $C = \frac{A}{x} + B$, where C = carbon monoxide content (mg/kg), x = metres from the road, and A, B are constants.

(a) At a distance of 10m from the road, the carbon monoxide content is measured at 50mg/kg and $\lim_{x \to \infty} C = 2$. Explain why the values of A and B are 480 and 2 respectively. Show your working.

(2 marks)

It is also found that the pollution from passing traffic affects how much grass can grow. The thickness of the grass can be modelled by: $G = e^{0.01x}$, (mg/m^2), x = metres from the road.

The overall concentration of the carbon monoxide in the grass is given by:

$$T(x) = C \times G = \left(\frac{480}{x} + 2\right)e^{0.01x}, \ x \in (0, 200], \ T(x) \text{ is measured in mg/m}^2.$$

(b) Give the **exact** co-ordinates of the furthest endpoint for this domain.

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(1 mark)

(c) Calculate the average rate of change of the overall concentration of carbon monoxide in the grass as the distance from the road increases from 50 m to 75 m. (Answer correct to 3 dec pl).

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(units are mg/m^3)

(1 mark)

(d) Calculate the average overall concentration of carbon monoxide in the grass as the distance from the road increases from 50 m to 75 m. (Answer correct to 3 dec pl).

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(units are mg/m^2)

(2 marks)

(e) At what distance from the road is the overall concentration, $T(x)$, a minimum? Give your answer to 2 dp and justify (by any valid technique) that it is, in fact, a local minimum.

(2 marks)

(f) As with most models, there are limitations. Looking at the graph of $T(x)$, what is the most obvious limitation of this model?

(2 marks)

END OF PROBLEM-SOLVING TASK

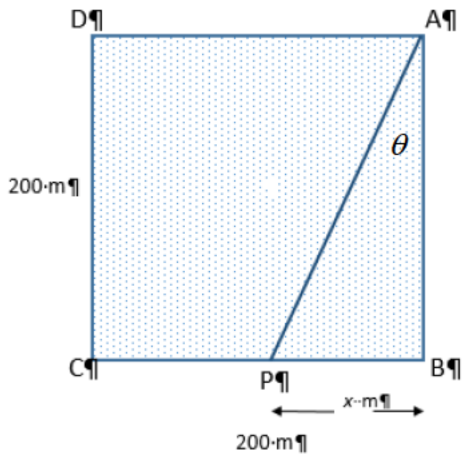
QUESTION OMITTED (TIME ??)

Question 8: (9 marks)

Angela the athlete is training for the next Olympics. Part of her training program is to run "square laps" around a 200 m square track. On a particularly wet afternoon she decides to "cheat".

Rather than running the last 400 metres along the perimeter of the track, she decides to run from the vertex A onto the muddy field and cut across, in a straight line, to a point P somewhere on the last 200 m stretch, as shown.

Angela can run through the muddy field at a constant speed of 5 m/s and on the track at a constant speed of 8 m/s.



(a) Given $\angle PAB = \theta$ where $0 \leq \theta \leq \frac{\pi}{4}$ and that PB = x, find an expression for x in terms of θ.

(1 mark)

(b) Show that the time it takes Angela to run from A to P can be written as $\dfrac{40}{\cos(\theta)}$.

(2 marks)

(c) Find an expression in terms of θ for the time it takes Angela to run from P to C .

(1 mark)

(d) **(i)** Hence show that the total time taken for Angela to run from A to C via P can be expressed in the form: $T(\theta) = 200[a(1 - tan\theta) + \frac{b}{cos\theta}]$.

(2 marks)

(ii) Write down the values of *a* and *b*.

(1 mark)

(e) Find the angle, θ, to the **nearest tenth of a degree**, for which Angela's running time will be a minimum.

(2 marks)