# UNIT 3 MATHEMATICAL METHODS
# SAC 1, 2019 - PART C

Reading time: 5 minutes
Writing time: 85 minutes
Marks available: 49 marks

**Name:** ______________________________

**Teacher:** **APW** **JDR** **MRC** **NJM** **REC**

**Instructions**

During the assessment there should be no items on the desk other than those noted here:

- Pens
- Pencils
- Highlighters
- Eraser
- Sharpener
- Ruler
- This Assessment paper
- Scientific calculator
- CAS calculator
- A single bound reference

Unless stated, diagrams shown are not to scale.

Write your **name** in the space provided on the front page of this paper and **circle your teacher's initials**.

All questions are to be answered in the spaces provided.

Answers must be in blue or black ink or grey lead pencil.

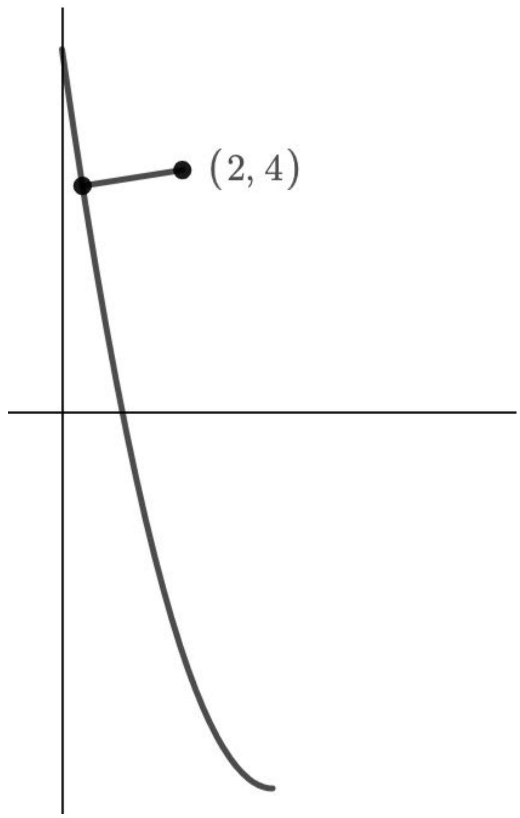
Students should NOT bring mobile phones and/or any other electronic communication devices into the assessment room.

## Part C (49 marks)

**Question 1** (8 marks)
The engineers at the Boring Company have made another tunnel, this time according to the quadratic rule

$$t(x) = x^2 - 7x + 6, \quad 0 \le x \le w$$

where $x$ and $t(x)$ are distances measured in kilometres. The graph of $t$ is shown below.




a. If $w$ is the value of $x$ at which the rule $t(x)$ has a local minimum, state the value of $w$.

1 mark

b. The engineers are also building a Control Station, to monitor the operation of the transport sled. The station will be located at the point (2, 4). Consider a point on the track with coordinates $(a,\ t(a))$.

Write down a formula $D(a)$ that gives the distance between the Control Station and this point, in terms of $a$.

2 marks

c. The engineers want to build a straight-line corridor from the Control Station to the tunnel. On the axes above, sketch a possible corridor that goes between the Control Station and the point (1,0), and write down its length.

2 marks

d. The engineers have drawn on the axes above another possible corridor so that the length of this corridor is **as short as possible**. Find, to 4 decimal places, the coordinates for the point where the tunnel and this corridor meet, and the length of the corridor.

3 marks

**Question 2** (9 marks)

Let *f* be a piecewise function which models yet another proposed tunnel. *f* is defined as

$$f:(-2,3)\to R, f(x)=\begin{cases}\sqrt[3]{x}, & -2<x\leq 1\\ a(x-b)^3, & 1<x<3\end{cases}$$

a. Find the values of *a* and *b* such that the tunnel joins and joins smoothly when $x=1$.

4 marks

b. State the domain for which $f'(x)$, the derivative of *f* is defined.

1 mark

c. There is a straight road that crosses the path of the tunnel. This road goes through the point (0,0), and the angle between the road and the positive direction of the $x$-axis is $\frac{\pi}{5}$ radians.

Find the coordinates for the other points where the road crosses over the tunnel, to 3 decimal places.

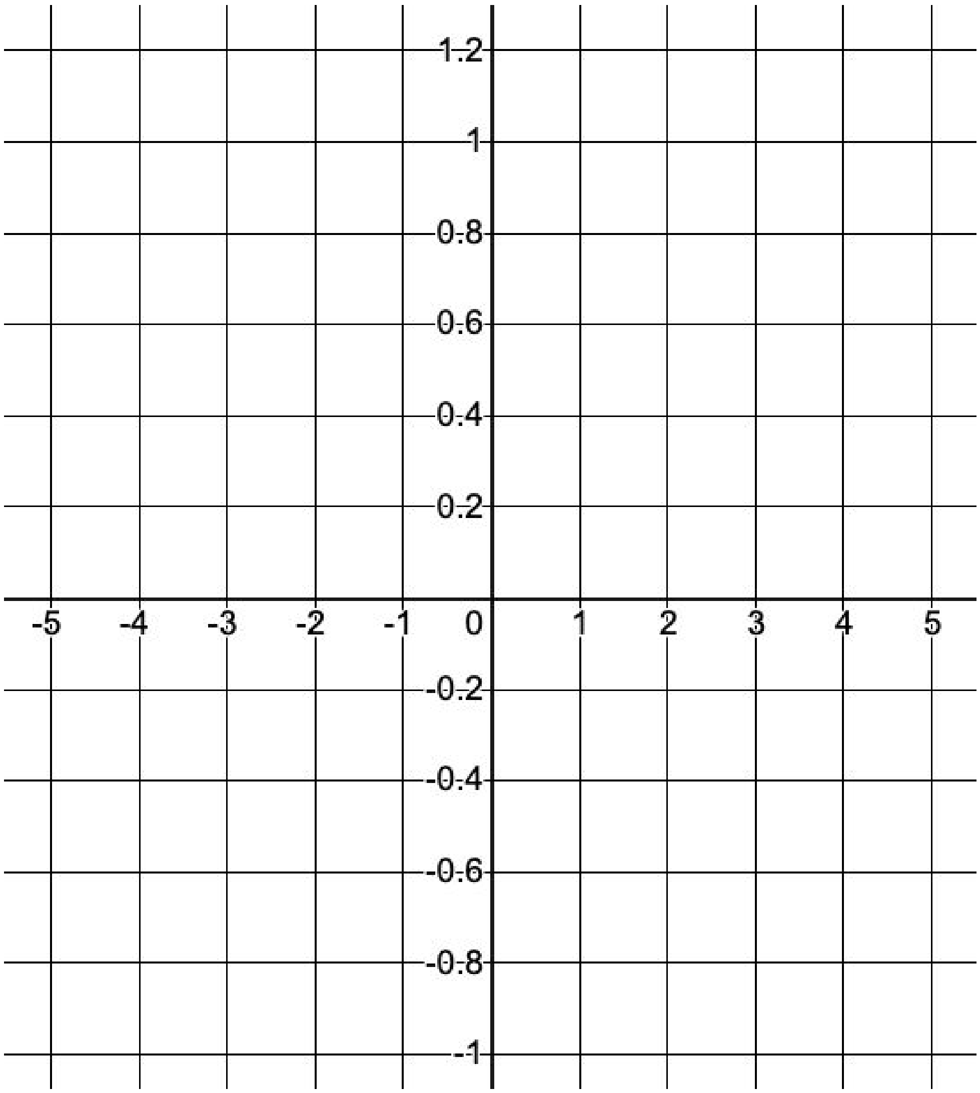
4 marks

**Question 3** (19 marks)

Why not build another tunnel? That's what the engineers were thinking to themselves…

Here is a new model they are working with:

$$s : [-4, 4] \rightarrow R,\ s(x) = \frac{e^x}{e^x+1}$$




a. Sketch the graph of this function on the axes above, labelling all significant points to 3 decimal places.

3 marks

b. Showing working, find the rule for the inverse function of $s$, $s^{-1}(x)$. Also state its domain, to 3 decimal places.

3 marks

c. Write down the coordinates, to 3 decimal places, when the graphs of $s$ and its inverse intersect.

1 mark

d. Using the fact that $\frac{1}{1+e^x} = \frac{e^{-x}}{e^{-x}+1}$ or otherwise, show that $z(x) = s(x) - 0.5$ is an odd function.

3 marks

e. The tunnel defined by $s(x)$ runs next to another existing tunnel whose graph matches the rule

$$u : [-4, 4] \rightarrow R,\ u(x) = 0.2(x + \cos(x))$$

On the same axes above, sketch $u$ alongside $s$, labelling important features to 3 decimal places.

3 marks

f. Engineers will dig a corridor to link the two tunnels. This corridor must run directly parallel to the $y$-axis. The length of this corridor can be written as $L(x) = s(x) - u(x)$.

Find the length for the shortest possible corridor between the tunnels, that is also parallel to the $y$-axis. Also write down the value of $x$ for which this occurs. Write your answers to 4 decimal places.

3 marks

g. Write down and solve an equation to find the values of $x$ at which the two tunnels point in the same direction, i.e. have the same derivative? Write your answers to 4 decimal places.

3 marks

**Question 4** (18 marks)

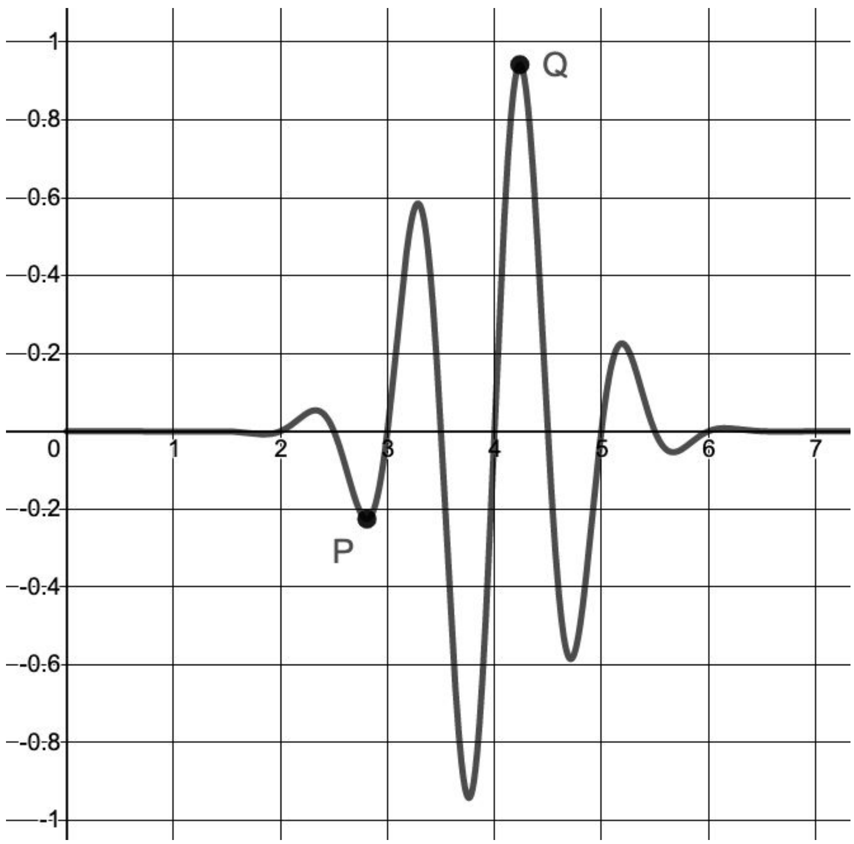
Engineers have constructed a test tunnel and have begun to use transport sleds to travel down the tunnel. The engineers install a vibration sensor in the wall of the tunnel to make sure that the tunnel is not shaken too much when the transport sled goes past.

When the sled goes past the sensor, it produces a vibration reading following the rule

$$v(t) = e^{-(t-k)^2} \cdot \sin(n\,\pi\,t), \qquad x > 0$$

where $v$ is the strength of the recorded vibration and $t$ is the time in seconds after the sensor is turned on.

Consider when $k = 4$ and $n = 2$. The graph for this scenario is shown below.




a. Write down the coordinates for points P and Q, to 3 decimal places.

2 marks

b. Vibrations are considered to be unacceptably high if the total time that the vibration strength is above 0.4 **or** below -0.4 is longer than 1.2 seconds. Use calculations to provide evidence that when $k = 4$ and $n = 2,$ vibrations will be unacceptably high.

3 marks

c. One of the engineers studying this model suggests that changing $k$ should not change whether vibrations are unacceptably high, because "all $k$ does is translate the graph of $v(t)$ left and right". A second engineer points out that this is wrong because when $k = 4.25$ and $n = 2$, that vibrations are no longer unacceptably high.

Show, using calculations, that the second engineer is correct. Also, explain why changing $k$ cannot be interpreted as translating the graph of $v(t)$.

4 marks

d. Consider when $n = 1$. Keep in mind that $v(t)$ can be interpreted as the product of two functions.

i. Explain with examples why the $x$-intercepts of $v(t)$ always occur at integer (whole number) values when $n = 1$, regardless of the value of $k$.

2 marks

ii. By referring to the graph of $y = e^{-x^2}$, explain why the range of $v(t) \subseteq [-1, 1]$ for all values of $k$.

2 marks