VICTORIAN CERTIFICATE OF EDUCATION

2016

STUDENT NAME:

MATHEMATICAL METHODS (CAS)



Episode IV – A new graph

2016

Reading Time: 15 minutes Writing time: 120 minutes

QUESTION AND ANSWER BOOK

Number of questions	Number of questions to be answered	Number of marks
5	5	59
	Total	59

- Students are permitted to bring into the test room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved CAS Calcuator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.
- Students are NOT permitted to bring into the examination: blank sheets of paper and/or correction fluid/tape.

Materials supplied

• Question and answer book of 24 pages.

Instructions

- Write your name in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the test room.

Along time ago in a galaxy far, far away... ...this page was left blank.

Instructions

Answer **all** questions in the spaces provided In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown. Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.



Episode IV

A NEW GRAPH

It is a period of civil war. Quadratic spaceships, striking from a hidden base have won their first victory against the evil Galactic Exponential.

During the battle, some Quadratic spies managed to steal a set of secret plans to the Exponential's ultimate weapon, the POLYNOMIAL STAR, an armored space station with enough "power" to destroy an entire planet.

Pursued by the Exponential's sinister agents, Princess Lisa races home aboard her starship, custodian of the stolen plans that can save her people and restore freedom to the galaxy...

4



The Exponential's most sinister agent, Darth Kermond, successfully manages to capture Princess Lisa, but not before she manages to give the plans to an astromech droid named Range2Domain2 and its companion protocol droid named C3B4MePO, who are able to evade capture and escape to the nearby desert planet of Tan(2)ine.

Unfortunately for Range2Domain2 and C3B4MePO, they are captured by scavengers on Tan(2) ine who then sell them to a young moisture farmer by the name of Colin Shnierwalker. After cleaning the droids and explaining to them their duties, Colin goes to bed. Whilst he is asleep, Range2Domain2 decides to leave the moisture farm and continue the mission to find a way to return the stolen plans to the Quadratics.



Range2Domain2 can travel at 3km/h across sand.

Question 1 (10 marks)

Assuming that Range2Domain2 travels in a straight line, how far will it travel before a. Colin Shnierwalker wakes up and notices Range2Domain2 is missing 8 hours later?

1 mark

b. Colin immediately gets into his speeder and pursues Range2Domain2, travelling at a constant speed 7km/h. Given Range2Domain2 left 8 hours before Colin does, at what time, to the nearest hour, will Colin catch up to Range2Domain2?

c. 14km away from the moisture farm, in the direction in which Range2Domain2 was heading, the terrain changes from sand to rocky hills. This affects the speed at which Range2Domain2 and Colin Shnierwalker's Speeder can travel. Over the rocky hills, Range2Domain2 can travel at 2km/h and Colin Shnierwalker's Speeder can travel at 4km/h.

Assume that the time at which Range2Domain2 leaves the moisture farm is t = 0 hours.

i. Show that the time at which Range2Domain2 reaches the rocky hills is $t = \frac{14}{3}$ hours.

ii. Show that the time at which Colin Shnierwalker reaches the rocky hills is t = 10 hours.

1 mark

1 mark

6

The distance of Range2Domain2 from the moisture farm, t hours after leaving, can be described by the hybrid function R(t) shown below:

$$R(t) = \begin{cases} 3t , 0 < t \le \frac{14}{3} \\ \frac{2}{3}(3t+7) , t \ge \frac{14}{3} \\ 0 , elsewhere \end{cases}$$

iii. Find the rule for the function f(t) such that the hybrid function, C(t), shown below, accurately describes Colin Skywalker's distance, in km, from the moisture farm at time *t*.

$$C(t) = \begin{cases} 7t - 56 , 0 < t \le 10 \\ f(t) , t \ge 10 \\ 0 , elsewhere \end{cases}$$





Colin successfully manages to catch up with Range2Domain2 in amongst the rocky hills. Whilst trying to collect the runaway astromech droid he is attacked by a Truncus Raider (a member of a nomadic tribe that lives in the desert). Luckily for Colin, an old hermit named Obi-Wan Kebohni arrives and scares the Truncus Raider away.

Obi-Wan Kebohni then brings Colin back to his home and reveals that he was actually a good friend of Colin's father and that they fought together in the Cos Wars many years ago. Obi-Wan Kebohni reveals that Colin's father was actually a member of an ancient order of warriors known as the Jedpi, who could harness the power of "The Maths", a mystical force that surrounds and penetrates all living things, binding the galaxy together.

He then gives Colin an old lightsaber that used to belong to his father, "This is the weapon of a JedPi Knight" he says, "not as clumsy or as random as a blaster. An elegant weapon for a more civilised age". Obi-Wan and Colin talk at length about Colin's father and how a Jedpi Knight named Darth Kermond hunted down and killed all of the Jedpi Knights, including Colin's father.

Suddenly, Range2Domain2 produces a holographic message for Obi-Wan from Princess Lisa requesting his aid in returning the droid and stolen plans to the Quadratics on the planet AldeRan. Obi-Wan convinces Colin to come with him to AldeRan and together they travel to the Tan(2)ine spaceport known as [Eisley].

Whilst at |Eisley| Obi-Wan and Colin meet the smuggler Harnath Solo and his first mate, Cvetkovsbacca (a hairy alien called a Wookie from the planet Kashyyyk). They agree to take Obi-Wan, Colin and the droids to AldeRan for a price...

Harnath Solo and Cvetkovsbacca are desperate for work as they are deep in debt to a dangerous crime lord known as Corkill the Hutt. Originally, they borrowed 10,000 credits from Corkill the Hutt but lost the money that they were going to use to repay him when they had to ditch their illegal cargo to avoid being caught by the Exponential Authorities.



Question 2 (6 marks)

Now that their debt is overdue, Corkill the Hutt has started to charge Harnath interest on the original 10,000 credit loan.

a. Harnath learns from a bounty hunter named Greedo that Corkill the Hutt is charging a cumulative interest rate of 2.5% per day on the amount that Harnath owes.

A cumulative interest rate means that the additional interest being charged on any day is calculated on the total amount owing from the previous day. This can be expressed by the equation $P_n = P_0(1+I)^n$, where P_n is the amount owing after *n* days, P_0 is the original loan amount and *I* is the interest rate (expressed in decimal form).

Solo is 11 days late on his repayment to Corkill the Hutt.

i. How much will Harnath Solo owe Corkill the Hutt after the first day of being late? 1 mark

b. Harnath Solo knows that it will take 3 days to take Obi-Wan and Colin to AldeRan and an additional 3 days to return to Tan(2)ine. If the interest continues to accumulate in the same manner, how much will he owe Corkill the Hutt when he returns to Tan(2)ine?

2 marks

8

Obi-Wan offers to pay Harnath Solo 2,000 credits in advance and an additional 15,000 credits when they arrive at AlderRan. Harnath agrees and they all board Harnath's ship, the Millennium Falcon, and begin their journey.

Once on their way, Cvetkovskbacca decides to work out how much time 17,000 credits will buy them, in case something unexpected happens on their trip and they are held up.

c. How many days can they afford to lose before Obi-Wan's 17,000 credits no longer covers their debt to Corkill the Hutt?

2 marks

Meanwhile, Darth Kermond arrives with his prisoner, Princess Lisa, at the Exponential's brand new space station, the Polynomial Star. A space station with such a high order of power that it can destroy and entire planet. Darth Kermond brings Princess Lisa before his commander, Grand Moff Cheah-Johnson.

Grand Moff Cheah-Johnson asks Princess Lisa to tell her the location of the Quadratic Alliance's Secret base but Princess Lisa refuses to tell her anything. In order to try and persuade Princess Lisa, the Grand Moff decides to demonstrate to her the true power of the Polynomial Star by destroying Princess Lisa's home planet of AlderRan.

Defeated, Princess Lisa gives in and tells Grand Moff Cheah-Johnson that the Quadratic's secret base in on Old Domain(2,ine). Grand Moff Cheah-Johnson thanks Princess Lisa and then proceeds to destroy AldeRan anyway, much to Princess Lisa's horror.

(The only comfort for Princess Lisa is that she didn't actually reveal the location of the Quadratic's Secret base. As it turns out Domain(2,ine) was an old location but there were no Quadratics there anymore).



Meanwhile, on board the Millennium Falcon, Obi-Wan Kebohni begins to train Colin Shnierwalker in the ways of the Jedpi. He explains to Colin that a Jedpi can feel The Maths flowing through them and then gets Colin to attempt to use the lightsaber to block laser blasts from a special training remote. At first, Colin struggles with this task due to the seemingly random nature of when the lasers are fired. With some encouragement from Obi-Wan, however, he clears his mind, closes his eyes and lets the Maths flow through him.



Question 3 (19 marks)

As soon as Colin closes his eyes he senses two different graphs in his mind. Colin trusts his instincts and realises that the times when the laser blasts are fired from the training remote correspond to when the graphs intersect. IT'S NOT RANDOM AT ALL! He quickly uses The Maths to find the points of intersection.

The two graphs that Colin senses in his mind are described by the following functions:

$$A:[0,3] \rightarrow \mathbb{R}, A(t) = 3\cos(2\pi t) + 2 \text{ and } B:[0,3] \rightarrow \mathbb{R}, B(t) = 5\sin\left(\frac{\pi t}{2}\right) + 3.$$

where *t* is the time in seconds after the training remote begins moving.

a. Sketch the graphs of y = A(t) and y = B(t) on the same set of axis provided below, labelling all endpoints and turning points. 2 marks



Find all points of intersection, correct to 2 decimal places, between $A(t)$ and $B(t)$.	2 n
	-

Now that Colin knows when the laser blasts will be fired, he just needs to determine where to position his lightsaber at each of these times in order to deflect the blasts and not get hit. Again, he clears his mind and feels the Maths around him. The training remote hovers in the air in front of him, following a seemingly random path. Colin visualises the space in front of him as a 2 dimensional, Cartesian plane and notices that the training remote is repeatedly moving along straight lines between coordinates* on this plane. The diagram below shows the path that Colin observes. The training remote begins at point A and moves through each other point alphabetically before returning to A.

* coordinates refer to the horizontal and vertical distances, in meters, from colin's eyes which can be assumed to be located at the single point (0,0).



c. What is the total distance, correct to 2 decimal places, travelled by the remote as it moves from *A* to *B* to *C* and back to *A* again?2

It ta mov	akes 2 seconds for the remote to travel completely around this path. Assuming that it ves with constant speed, show that it is travelling at 9.76ms ⁻¹ .	1 mark
At v loca	what time, <i>t</i> , correct to 2 decimal places, during the first 2 seconds will the remote be ited at:	
i.	point B?	2 mark
ii.	point C?	2 mark

13

- **f.** The first shot from the remote is fired at t = 0.13 seconds.
 - i. How far, correct to 2 decimal places, will the remote have moved from point *A* when it fires its first shot? 1 mark

ii. Find the equation of the line that joins point A and point B.

iii. The coordinates of the point Q, where this first shot is fired will be somewhere along the line joining point A and point B. If the coordinates of Q are (a,b), use your answer to part **f**. ii. to find an expression for b in terms of a.

iv. Hence find the coordinates of the point Q correct to 2 decimal places.

4 marks

1 mark



Colin successfully manages to use The Maths to deflect the laser blasts from the remote and earns a clap on the back from his proud mentor Obi-Wan Kebohni.

All of a sudden, Obi-Wan turns deathly pale and moves to sit down. Worried, Colin moves to his side and asks "Are you OK?"

Obi-Wan explains "I've felt a great disturbance in the Maths. As if millions of students forgot to use brackets and their teachers cried out in terror and were suddenly silenced... I fear something terrible has happened."

Whilst the crew discuss Obi-Wan's beliefs, the Millennium Falcon proceeds to drop out of hyperspace around the planet AldeRan. But, as they quickly discover, the planet is no longer there. Instead they are in the midst of an asteroid field and the only object in the immediate area is a small grey moon.

"That's no moon. That's a space station!" Obi-Wan remarks. Everyone on board the Millennium Falcon seems concerned. It appears as though the Evil Exponential had created a weapon that could destroy an entire planet. Everyone quickly agrees that they probably shouldn't hang around to find out what happened here. Harnath Solo attempts to turn the Millennium Falcon around but it is too late, the Millennium Falcon was caught in the space station's tractor beam and was being dragged on board.

The crew hide in the Millennium Falcon's hidden storage spaces and manage to sneak into the Space Station. Their plan is to disable the tractor beam and then make their escape. They sneak into a control room and locate the best way to disable the tractor beam. Obi-Wan Kebohni tells the others to stay put whilst he takes care it.

Whilst waiting, Range2Domain2 discovers that the space station is known as the Polynomial Star and that, being held captive on board is Princess Lisa, who sent the hologram message that started them on this trip. Colin, Harnath and Cvetkovskbacca decide to go rescue her whilst they wait for Obi-Wan to get back.

Colin, Harnath and Cvetkovskbacca successfully locate the cell holding the princess. Colin opens the cell door and Princess Lisa looks up at him. "Isn't your amplitude a bit low for a Sin Trooper?" she asks. Colin takes his helmet off and tells her who he is and that he is here to rescue her and together they leave the cell.

As they make their escape, they get cornered by some Exponential Sin Troopers and the only way to escape is by jumping into a garbage compactor. Unfortunately for our heroes, the garbage compactor walls begin to close in on them threatening to squash them unless they can find a way to stop the walls from moving.



Question 4 (5 marks)

As the garbage compactor walls close in upon our heroes, Colin attempts to use the Maths to help him find a solution to stop them from being squished. He closes his eyes and visualises the garbage compactor as shown in the diagram below.



The initial width of the garbage compactor is 3.5 meters and there is a 4 meter long metal pole that can be used as a wedge between the two walls.

a. The left end of the metal pole is fixed at the coordinates (0,0). Show, correct to two decimal places, that the value of q, the vertical component of the coordinate Q where the right end of the metal pole touches the other wall, is 1.94 meters.

1 mark

The walls proceed to move inwards, causing the metal pole to bend upwards as shown below. The metal pole remains fixed at (0,0) and at point Q which remains at the same height but whose horizontal coordinate value changes as the wall moves together.



Colin realises that the shape of the curved rod can be modelled as a parabola with equation $M(x) = ax^2 + bx + c$.

b. Colin is standing in the exact middle of the room. He estimates that the metal pole is 1.86 meters above the water at this point. Find the values of a, b and c, correct to 2 decimal places, and hence state the equation for M(x) when the room is 3 meters wide.

3 marks

Colin is just about to investigate the relationship between the curvature of the metal pole and its height about the water in order to determine the point at which the metal pole will break when he realises that Obi-Wan hasn't trained him to use integral calculus yet and so he doesn't know how to use the equation for the length of a curved line below:

$$D = \int_{a}^{b} \sqrt{1 + \left(f'(x)\right)^2} \, dx$$

Instead, Colin uses his communication device to contact C3B4MePO and Range2Domain2 and they stop the garbage compactor and help them escape being flattened to death.



18



Whilst everyone else was escaping the garbage compactor, Obi-Wan Kebohni successfully found and disabled the Polynomial Star's tractor beam. As he makes his way back to the Millennium Falcon to rejoin the others though, he senses through the Math, his old enemy Darth Kermond nearby.

As the others all race aboard the Millennium Falcon, Obi-Wan and Darth Kermond come face to face. Some witty banter ensues as they fight each other with their lightsabers. Seeing that the others are waiting for him and that at the same time, the Exponential's forces are gathering to stop them from escaping, Obi-Wan lets Darth Kermond kill him, so that Obi-Wan can become one with the Maths.

Colin, seeing Darth Kermond strike Obi-Wan Kebohni down cries out in anger and starts to fire his blaster rifle at the Sin Troopers pouring out of the doorway. Then he hears Obi-Wan's voice in his head "Run Colin. Run!" it says. Colin quickly boards the Millennium Falcon and they make their escape from the Polynomial Star.

They head to the actual Quadratic Secret base on the moon $y = a(vin)^4$. Here, the stolen plans that Range2Domain2 has been carrying all this time are downloaded and analysed. The Quadratic technicians discover a weakness in the Polynomial Star's design. If a proton torpedo can be fired down a thermal exhaust port on the surface of the Polynomial Star, it will create a chain reaction that will destroy the entire space station.

Whilst they are discussing their plan of attack on the Polynomial Star, the space station arrives. It turns out Darth Kermond had hidden a tracker on board the Millennium Falcon so that the Exponential Empire could find the location of the Quadratic Base.

Knowing this, Quadratic pilots quickly jump into their X-Wing fighter ships and mount an attack on the Polynomial Star. Colin Shnierwalker, determined to avenge Obi-Wan Kebohni's death decides to join the Quadratic Alliance and jumps in an X-Wing to fight with them. Harnath Solo however, knowing that his debt to Corkill the Hutt is still owing, says his farewells and takes his reward for rescuing Princess Lisa and leaves the Quadratic Alliance behind.



Question 5 (20 marks)

As the Quadratics make their attack on the Polynomial Star, the x-Wing fighter pilot Christopher 'Biggs' Hartelighter makes an attempt to fire his proton torpedos into the thermal exhaust port. He uses the targeting computer to assist him in making the shot.



The targeting computer models the scenario as shown in the diagram below.

Hartelighter flies 10 meters above the surface of the polynomial star and initially Hartelighter's *x*-Wing is a horizontal distance of d = 3000 meters from the thermal exhaust port.

The targeting computer needs to work out the equation of the dashed line in the diagram that will represent the path of the proton torpedo. The computer models this path as a quadratic function of the form $T_1(x) = ax^2 + b$.

- **a.** At this initial moment in time, the targeting computer places Hartelighter's x-Wing at coordinates (0,10) and the thermal exhaust port at (3000,0).
 - i. Determine the values of a and b in $T_1(x)$ that the targeting computer would assign for the path of the proton torpedo.

2 marks

ii. What is the gradient of the path when the proton torpedo reaches the thermal exhaust port?

iii. In order for the proton torpedo to successfully enter the thermal exhaust port, it must enter at an angle of 45° to the horizontal. If fired from 3000 meters away, what is the angle at which the proton torpedo will enter the thermal exhaust port, correct to 3 decimal places?

2 marks

Unfortunately, the targeting computer takes too long to work out the equation and in the time that it was working it out, Hartelighter's *x*-Wing has moved closer to the thermal exhaust port.

In order to accommodate for this problem. The targeting computer works out the equation of the line in terms of *d* the horizontal distance of the *x*-Wing from the thermal exhaust port, where 0 < d < 3000.

- **b.** Assuming that the coordinates of the thermal exhaust port are at (d, 0);
 - i. Find the values of *a* and *b* in $T_2(x) = ax^2 + b$ that the targeting computer would find for the path of the proton torpedo in terms of *d*, the distance of the *x*-Wing from the exhaust port.

2 marks

ii. Hence show that the gradient of the proton torpedo's path when it hits the thermal exhaust port can be given by $T_2'(d) = \frac{-20}{d}$.

iii. How close does the x-Wing need to get to the thermal exhaust port before it is possible to fire the proton torpedo so that it will hit the thermal exhaust port at an angle of 45° ?

2 marks



Hartelighter manages to take the shot successfully but unfortunately, the proton torpedo doesn't hit at the correct angle and so doesn't destroy the Polynomial Star. As he exits the trench, Darth Kermond flies in behind him and attempts to get a missile lock. Just as Darth Kermond fires, Hartelighter tells Colin Shnierwalker that he should make the next attempt.

Colin enters the trench leading up to the thermal exhaust port and swtiches on his targeting computer. As it begins calculations, he hears a familiar voice speak to him through the Maths. "Use the Maths Colin" the voice of Obi-Wan Kebohni tells him.

Colin closes his eyes and feels the Maths around him. Then, it occurs to him. The targeting computer is assuming that the surface of the Polynomial Star is flat when it is actually curved. Colin now knows why the last proton torpedo didn't go into the thermal exhaust port and what adjustments need to be made to get the next one to go in.

Colin remembers the schematic diagram of the Polynomial Star that was displayed at the mission briefing on $y = a(vin)^4$. It said that the Polynomial Star was a giant sphere with a radius of 75,000 meters.

Using the Maths, Colin models the curved surface of the Polynomial Star over which he is travelling by the equation $S(x) = -75,000 + \sqrt{5,625,000,000 - x^2}$ and determines that the coordinates of the thermal exhaust port will actually be at (2999,-60) when the targeting computer says the distance away is 3000 meters.

Colin knows that the angle that the proton torpedo must enter the thermal exhaust port must be greater than 45^{0} . So he sets about working out the place where he should fire the proton torpedo in order to ensure that it successfully enters the thermal exhaust port.



c. Colin has modelled the surface of the Polynomial Star by the equation $S(x) = -75,000 + \sqrt{5,625,000,000 - x^2}$. Through the maths he pictures the following diagram to represent the situation.



He reasons that the equation of his *x*-Wing through the trench can, therefore, be modelled by the equation $C(x) = -74,990 + \sqrt{5,625,000,000 - x^2}$.

i. Give an explanation for why Colin's x-Wing can be modelled by C(x). 1 mark

ii. Using the equation for the path of the proton torpedo, $T_3(x) = ax^2 + b$, find an expression for the gradient of proton torpedo at (2999,-60) in terms of *a*. 2 m

1 mark

1 mark

2 marks

iii. What is the gradient of the surface of the Polynomial Star at (2999,-60) expressed correct to 2 decimal places?

iv. Colin knows that the angle between two lines can be worked out using the gradients of the lines. In this case, he knows that $\theta = \tan^{-1}(S'(2999)) - \tan^{-1}(T_3'(2999))$. Determine the value of *a* for which $\theta = 45^\circ$ correct to 6 decimal places.

v. Hence determine the value of b in $T_3(x)$, correct to 2 decimal places, for this particular proton torpedo path.

vi. Using your answers to **part iv.** and **part v.** find the *x*-coordinate where Colin should fire the proton torpedo from in order to hit the thermal exhaust port at the correct angle.



Colin works out where to take the shot from and proceeds to travel along the trench as quickly as he can. Darth Kermond enters the trench behind him though and begins to catch up.

"The Maths is strong with this one" Darth Kermond remarks as he lines up Colin in his sights. Darth Kermond is about to fire when, out of nowhere, Harnath Solo and Cvetkovskacca appear in the Millennium Falcon and shoot down Darth Kermond's wing man. The explosion knocks Darth Kermond's Tan-Fighter out of the trench and sends him spinning off into space.

Now free to take the shot, Colin takes a deep breath and trusts in the Maths. He fires the proton torpedo and it goes straight in to the thermal exhaust port. All remaining Quadratics quickly fly away from the Polynomial Star as it is about to explode, destroying the Exponential's ultimate weapon and Grand Moff Cheah-Johnson in the process.

Back on $y = a(vin)^4$, Princess Lisa recognises the bravery of Colin Shnierwalker, Harnath Solo and Cvetkovskbacca in a lavish ceremony in front of the Quadratic Alliance.

Their bravery has made the Galaxy a much safer place... or has it?

