A
VICTORIAN CERTIFICATE OF EDUCATION

2016

MATHEMATICAL METHODS (CAS)

MATH
WARS

SOLUTIONS

Episode V – The Exponential Strikes Back

Question 1 (11 marks)

a. The temperature, H in ^{o}C, over a 24 hour cycle on the planet $H \circ th$ can be modelled by the function

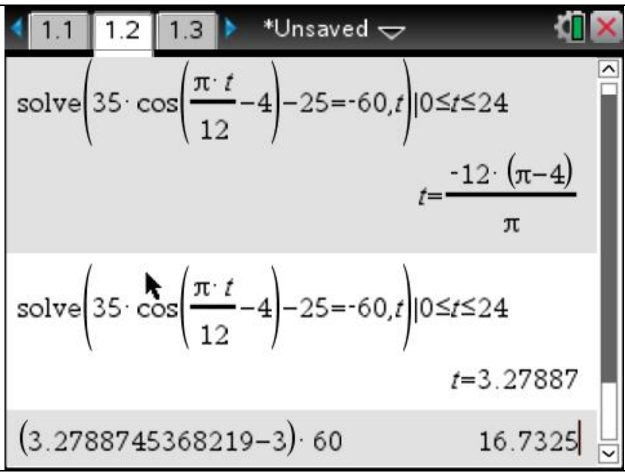
$$H:[0,24] \to R,\ H(t) = 35\cos\left(\frac{\pi t}{12} - 4\right) - 25$$

where t is the number of hours after midnight.

i. What is the minimum temperature that is reached on the planet $H \circ th$? 1 mark

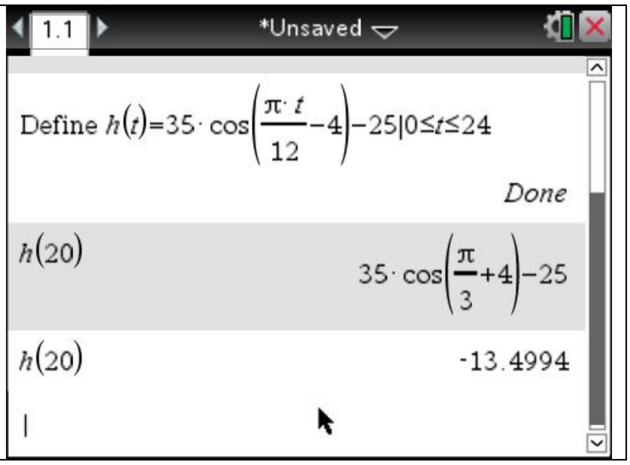
$-25-35=-60^{\circ}\text{C}$ Units are required <u>unless</u> student states H(t)=-60 as H has already been defined with units	**A1**

ii. At what time of day, to the nearest minute, does this minimum occur? 2 marks

$-60 = 35\cos\left(\frac{\pi t}{12} - 4\right) - 25$	**M1**	
$t = \frac{-12(\pi-4)}{\pi}$ OR $\frac{48}{\pi} - 12$ $t = 3.29$ $t = 3:17$ am	**A1**	

b. Harnath Solo finds Colin Shnierwalker wandering through the snow at $t = 20$ hours. What is the temperature of Planet $H \circ th$ at this time? Express your answer correct to 2 decimal places. 1 mark

<table>
<tr>
<td>$h(20) = -13.50^{o}C$

*Students must have answer expressed with 2 decimal places.</td>
<td>A1</td>
<td>
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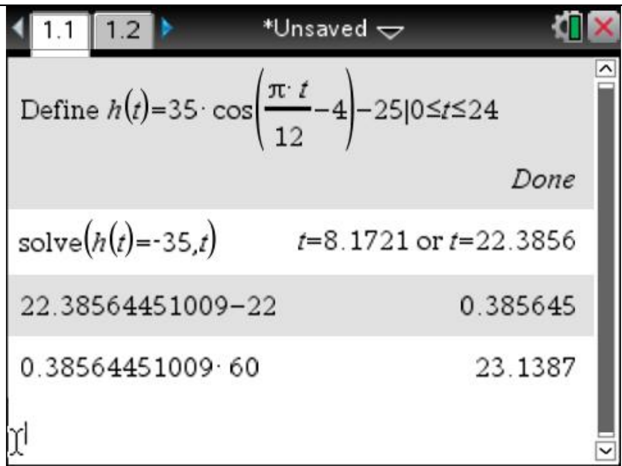
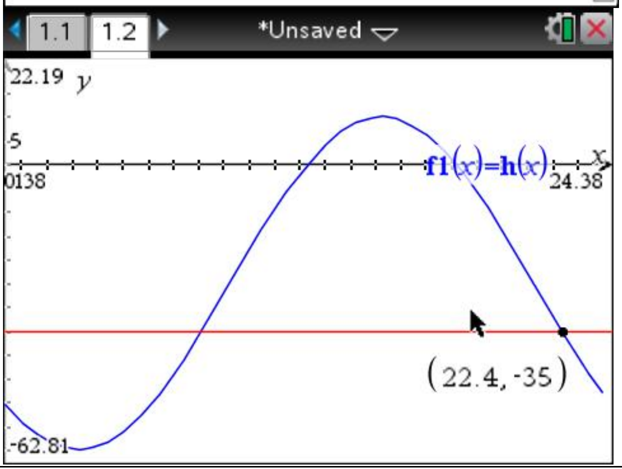
Whilst checking on Colin who appears to be suffering from a mild case of hypothermia, Harnath Solo's Tan-Tan dies from exhaustion. With no way to transport Colin back to the Quadratic's Base, Harnath sets to work constructing a shelter for him and Colin to spend the night in.

c. Harnath Solo needs to finish building the shelter before the temperature drops below $-35°C$ or else both he and Colin will freeze to death. How long, to the nearest minute, does Harnath Solo have after starting at time ($t = 20$ hours) to build the shelter? 2 marks

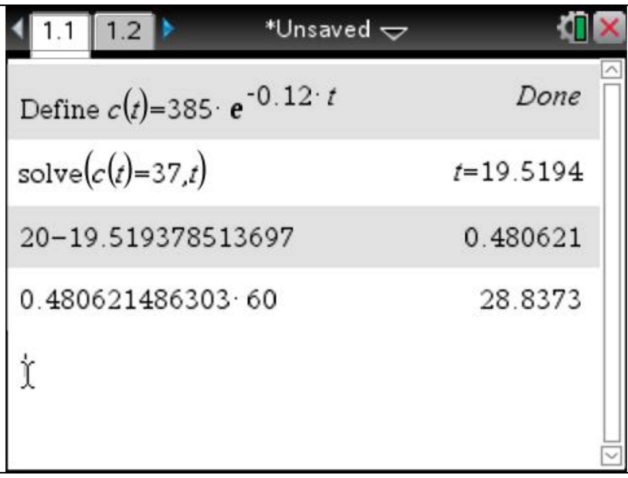
<table>
<tr>
<td>$h(t) = 35\cos\left(\frac{\pi t}{12} - 4\right) - 25 = -35$
$t = 8.1721$ or 22.3856
reject 8.1721 as Harnath only
finds Colin after 20 hours
$t = 22.3856$ hours</td>
<td>A1</td>
<td rowspan="2">


</td>
</tr>
<tr>
<td>time to build shelter $= 22.3856 - 20$
$= 2.39$ hours
or
= 2 hours and 23 minutes
or
=143 minutes</td>
<td>A2</td>
</tr>
</table>

d. Colin's body temperature is dropping rapidly. Harnath measures Colin's body temperature to be 34.9°C at $t = 20$ hours.

Harnath knows that Colin's body temperature has been and will continue to decrease according to the rule $C(t) = \begin{cases} 37 & , \quad 0 \le t < a \\ 385e^{-0.12t} & , \quad a \le t \le 24 \end{cases}$.

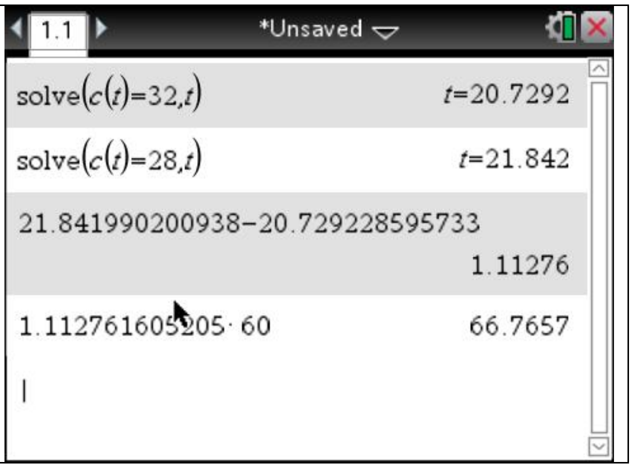
i. If Colin's body temperature when he was first exposed to the cold was a normal 37°C, find the value of a, to the nearest minute. 3 marks

$C(t) = 385e^{-0.12t} = 37$	**M1**	
$t = 19.519$ hours	**A1**	
$a = 19$ hours 32 minutes or $a = 1171$ minutes	**A2**	

Moderate hypothermia occurs when a person's body temperature falls below 32°C. Severe hypothermia occurs when a person's body temperature falls below 28°C.

ii. How long, to the nearest minute, will Colin experience moderate hypothermia before he starts to experience severe hypothermia? 2 marks

Moderate Hypothermia $C(t)=32$ $t=20.7292$ Severe Hypothermia $C(t)=28$ $t=21.842$	**M1**
$21.842-20.7292=1.11276$ $=67$ minutes OR 1 hour and 7 minutes	**A1**



Question 2 (12 marks)

The $H \circ th$ cold virus is a particularly dangerous virus and the Quadratic Medical Department are keen to return Colin to full health as quickly as possible. The number of $H \circ th$ cold viruses in Colin's body, V, will grow according to the rule $V(t)=V_0 \times 2^{kt}$ where t is the time in minutes after the initial infection.

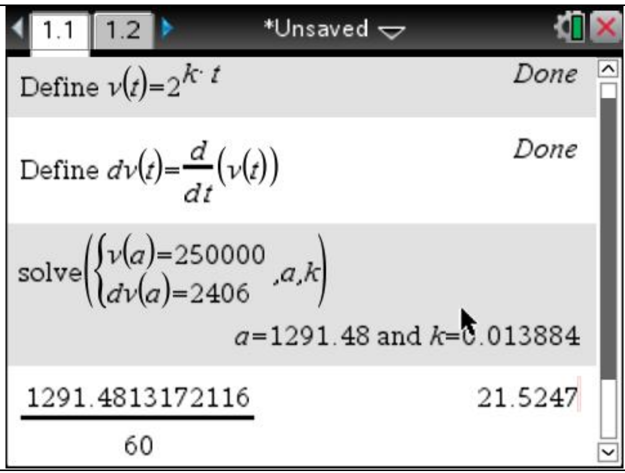
a. Initially, Colin was infected by a single $H \circ th$ cold virus.

i. Show that $V_0=1$ 1 mark

$V(0)=V_0 \times 2^{k \times 0}=1$ $V_0=1$ *Substitution of t=0 must be shown to gain this mark	**A1**

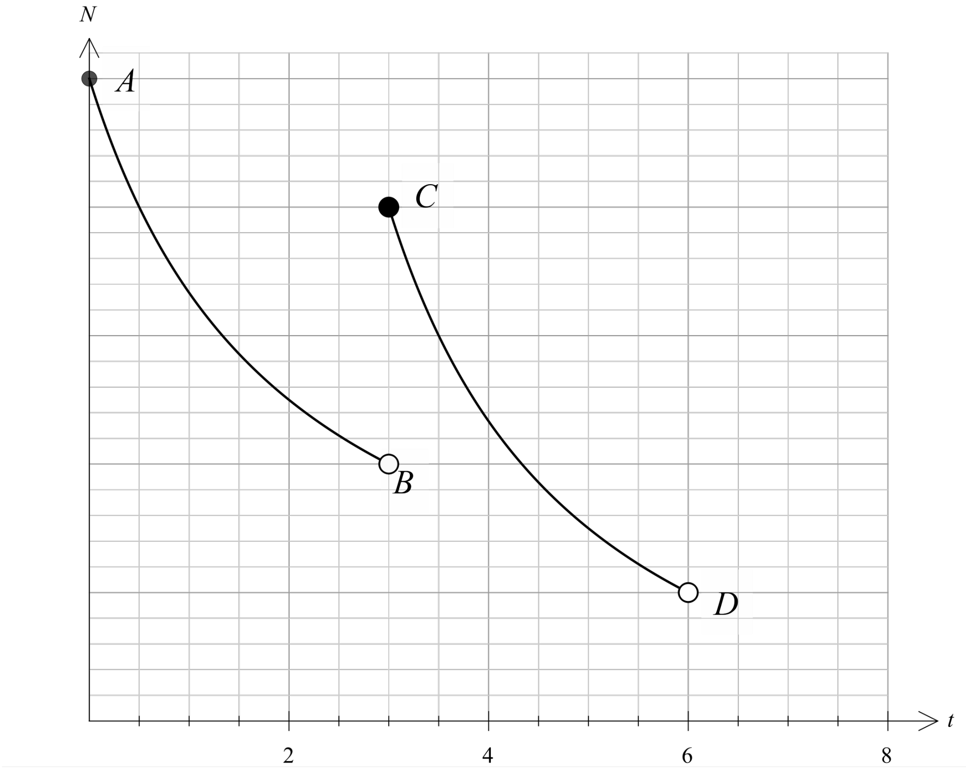
Upon first contact with the Quadratic Medical Department, a scan reveals that Colin has 250,000 viruses in his system. At this time, the number of viruses in his system are increasing at a rate of 2,406 viruses per minute.

ii. Determine the value of k and the length of time, in hours, that Colin has been infected with the $H \circ th$ cold virus at the time of this scan. Express both your answers correct to 3 decimal places. 4 marks

Let a represent the time of the scan $V(a) = 2^{ka} = 250000$	**M1**	
$V'(t) = k\log_e(2) \times 2^{kt}$ $V'(a) = k\log_e(2) \times 2^{ka} = 2406$	**M2**	
$k = 0.014$	**A1**	
$a = 1291.481$ minutes $a = 21.525$ hours	**A2**	

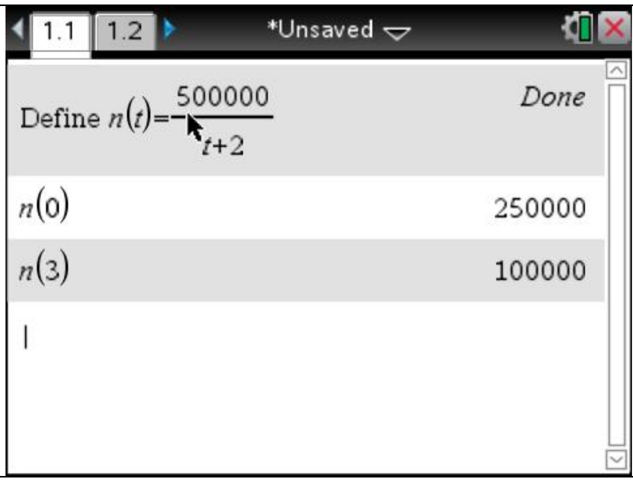
b. The Kaldunphloo tablet contains an anti-virus that prevents all viruses from replicating and that kills off viruses in Colin's system. Unfortunately, the antivirus stops working exactly after 3 hours and in between then and taking the next tablet, the number of viruses in Colin's system will double.

The graph of the number of viruses, N, remaining in Colin's system at time, t, hours after taking the first Kaldunphloo tablet looks like this. Each section of the curve has exactly the same shape as the curve AB.



The equation of the cuve AB is $N_{AB}(t)=\dfrac{500000}{t+2}$.

i. Find the coordinates of points A and B. 2 marks

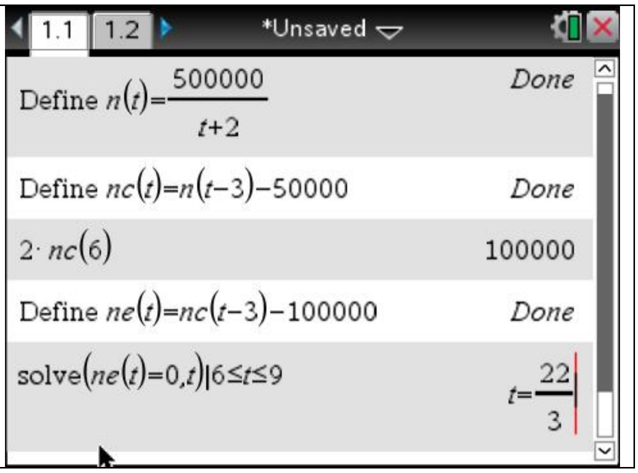
$N(0)=$ 250000 $A=(0,250000)$	**A1**	
$N(3)=100000$ $B=(3,100000)$	**A2**	

ii. After 3 hours, the effect of the Kaldunphloo tablet wears off and the number of viruses double before Colin takes another tablet to start the process again. Find the equation of the curve *CD*. 3 marks

$C = (3,200000)$ *recognition of vertical translation of -50000 **If finding the value through doubling the y-value of point B, it should be made clear by the student that they are finding a point on line CD. 2xB on it's own is not enough.	**M1**
$N_{CD} = \dfrac{500000}{t-1} - 50000$ *recognition of horizontal translation of -3	**M2**

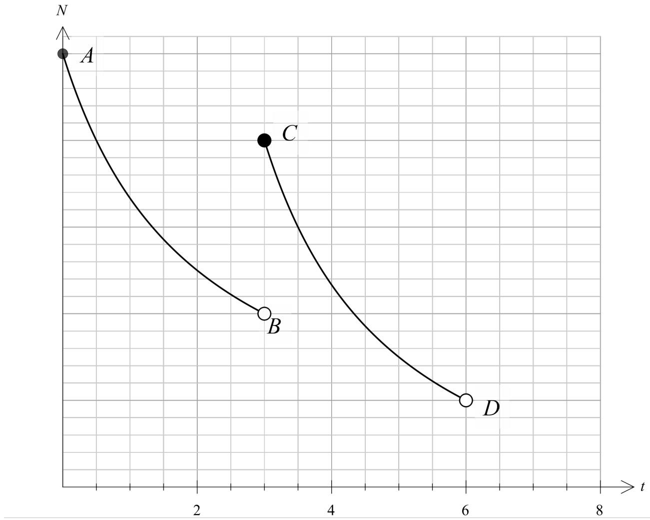
iii. If the pattern continues, and after each tablet wears off, the number of viruses double before the next tablet starts to work, at what time, to the nearest minute, will all of the viruses be removed from Colin's system? 2 marks

$N_{CD}(6) = 50000$ $\therefore N_{EF}(t) = \dfrac{500000}{(t-6)+2} - 150000$	**M1**
$N_{EF}(t) = 0$ $t = 7$ hours and 20 minutes	**A1**



While Colin is recovering from $H \circ th$ cold virus he begins to think about the pattern made by the starting point of each of the curves of N against t when a patient is being treated with Kaldunphloo tablet (see the diagram below).

He begins to think that there might be a way of determining the coordinates of each of the starting points of the curves $N(t)$ for any starting number of viruses, k.



Assume that k is a very large, positive integer value and that the graph of $N(t)$ terminates when $N = 0$.

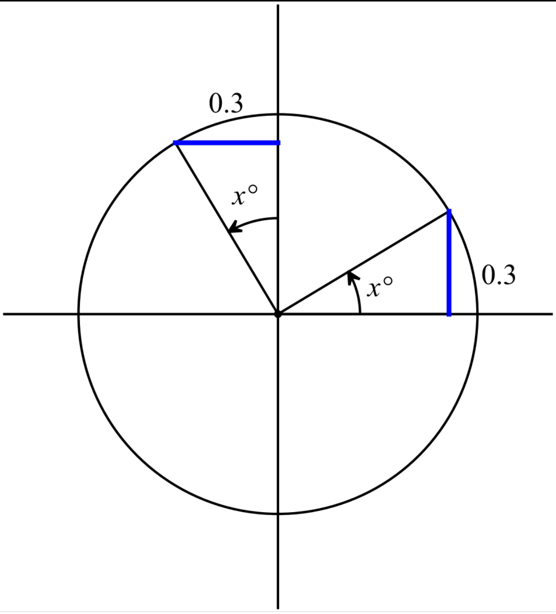
c. The coordinates of the starting points can be expressed as $(3n, y)$ where n is an integer that represents which number curve is being referenced.
Find the general rule that will give the value of y, the starting points of the n^{th} curve given that the initial number of viruses in the system is some number, k. 2 marks

*Any valid approach towards generating a solution similar to the one shown below should be awarded a working mark $A = k$ $B = A - 150{,}000 = k - 150{,}000$ $C = 2B = 2(k - 150{,}000)$ $D = c - 150{,}000 = 2(k - 150{,}000) - 150{,}000$ $E = 2D = 2(2(k - 150{,}000) - 150{,}000)$ ... OR $0 \rightarrow k$ $1 \rightarrow 2k - 2 \times 150{,}000$ $2 \rightarrow 4k - 6 \times 150{,}000$ $3 \rightarrow 8k - 14 \times 150{,}000$ ...	**M1**
for the nth term, starting at $n = 0$ for point A the coordinates are: $\left(3n\ ,\ k2^n - 300{,}000(2^n - 1)\right)$ *OR* $\left(3n\ , k2^n - (300{,}000 \times 2^n + (2^n - 2)k)\right)$ *or any other equivalent solution for y. **Students do not need to give the answer in coordinate form.	**A1**

Question 3 (6 marks)

$d(y(o))/da$ is a tough master. He begins training Colin by making him run through the swamps of Dagobah whilst carrying $d(y(o))/da$ on his back. After a few days of this intense training, $d(y(o))/da$ begins to integrate asking Colin mathematical questions of varying difficulty into the training.

a. If $\sin(x) = 0.3$ where $0 \le x \le \frac{\pi}{2}$, find $\sin\left(\frac{\pi}{2} + x\right)$ to 3 decimal places. 1 mark



$\sin\left(\frac{\pi}{2} + x\right) = \cos(x)$

$\sin^2(x) + \cos^2(x) = 1$

$0.3^2 + \cos^2(x) = 1$

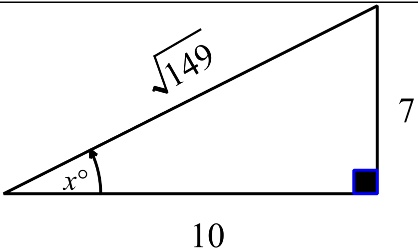
$\cos(x) = \sqrt{1 - 0.09}$

$\cos(x) = \sqrt{0.91} = 0.954$

A1

b. If $\tan(x) = \frac{7}{10}$ and $0 \le x \le \frac{\pi}{2}$, find $\cos(x)$. 1 mark

A1



$\cos(x) = \frac{10}{\sqrt{149}}$

c. Starting with the equation $y = \sqrt{x}$, Colin is asked to apply the following sequence of transformations to find the final equation, y_f.

1. Dilation by factor 4 from the x axis.
2. Reflection in the x axis.
3. Vertical translation of +3 units.
4. Horizontal translation of -6 units.
5. Dilation by factor $\frac{1}{2}$ from the y axis

i. What is the final equation, y_f, according to the described sequence of transformations 2 marks

$y = \sqrt{x}$ $y_1 = 4\sqrt{x}$ $y_2 = -4\sqrt{x}$ $y_3 = 3 - 4\sqrt{x}$ $y_4 = 3 - 4\sqrt{x+6}$ $y_5 = 3 - 4\sqrt{2x+6}$ *Deduct ½ mark for every mistake or error to a maximum of two marks and round down. ** If students write out the transformed equations as shown above, they should be using different names for each line, deduct a mark if this is not done.	**A1** + **A2**

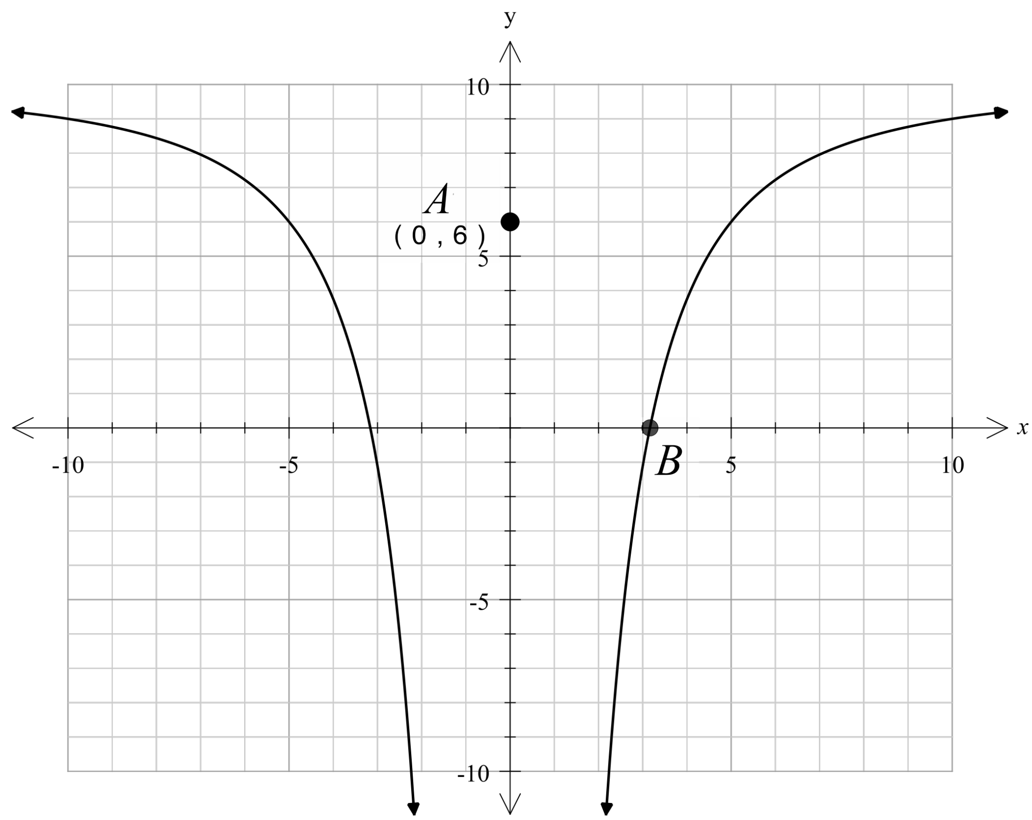
"Noooo!" says $d(y(o))/da$, "The answer should have been $y_f = 12 - 4\sqrt{2x+12}$."

ii. Assuming that the transformations remain the same, what should the order of the transformations above have been in order to result in the answer that $d(y(o))/da$ expected? 2 marks

$y = \sqrt{x}$ 1. Dilation by factor $\frac{1}{2}$ from the y axis ...(5) $y_1 = \sqrt{2x}$ 2. Reflection in the x axis ...(2) $y_2 = -\sqrt{2x}$ 3. Vertical translation of +3 units ...(3) $y_3 = 3 - \sqrt{2x}$ 4. Dilation by factor 4 from the x axis ...(1) $y_4 = 12 - 4\sqrt{2x}$ 5. Horizontal translation of -6 units ...(4) $y_5 = 12 - 4\sqrt{2(x+6)} = 12 - 4\sqrt{2x+12}$ *Accept any answer were 5 comes before 4 AND 2 comes before 3 comes before 1 **Deduct ½ mark for every mistake or error to a maximum of two marks and round down.	**A1** + **A2**

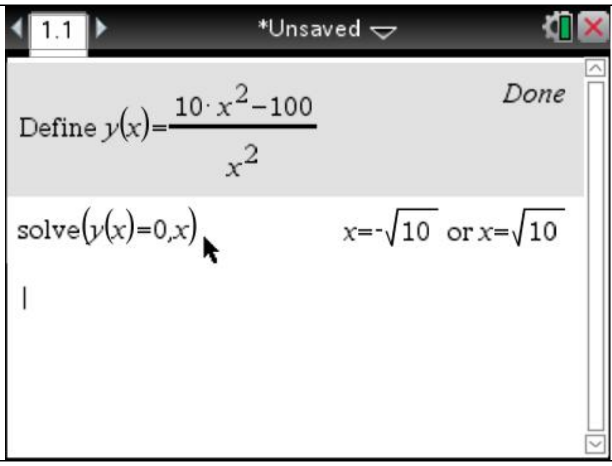
Question 4 (7 marks)

Colin hangs suspended above the lower levels of Cloud City, horrified by Darth Kermond's revelation that he is his father. Instead of giving in to the dark lord, however, he decides to jump from the platform (point A in the diagram below), aiming for the point B in the diagram below, which is located on the horizontal axis.



The equation of the curve upon which point B can be found is, $y = \dfrac{10x^2 - 100}{x^2}$.

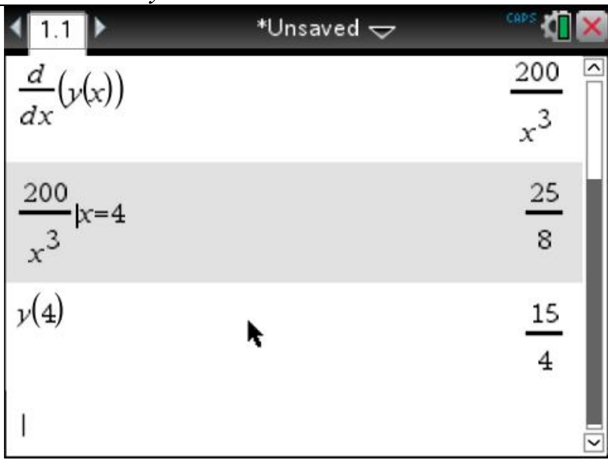
a. Find, correct to 3 decimal places, the coordinates of point B. 1 mark

$\dfrac{10x^2-100}{x^2}=0$ $x=\pm\sqrt{10}$ reject negative solution as it can be seen in the diagram that x is positive $\therefore x=\sqrt{10}\approx 3.162$	**A1**	

b. Due to the pain from losing his hand, Colin can't focus and use the Math as well as he could in the past. As such, he decides to use an approximation technique to find the coordinates of point B.

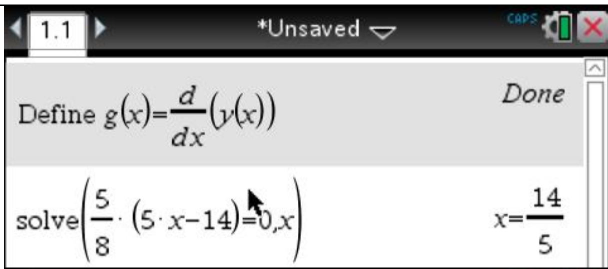
i. Find the equation of the tangent line to the curve y at $x = 4$. 2 marks

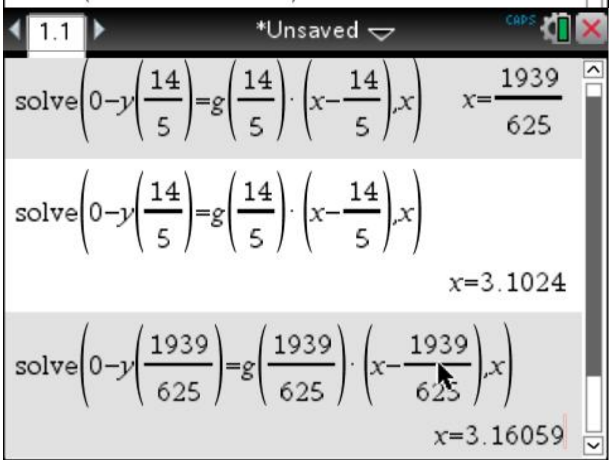
$\frac{dy}{dx} = \frac{200}{x^3}$ at $x = 4$ $\frac{dy}{dx} = \frac{25}{8}$	**M1**
$y - \frac{15}{4} = \frac{25}{8}(x-4)$ $y = \frac{25x}{8} - \frac{50}{4} + \frac{15}{4}$ $y = \frac{25x}{8} - \frac{35}{4}$ OR $y = \frac{5}{8}(5x-14)$	**A1**

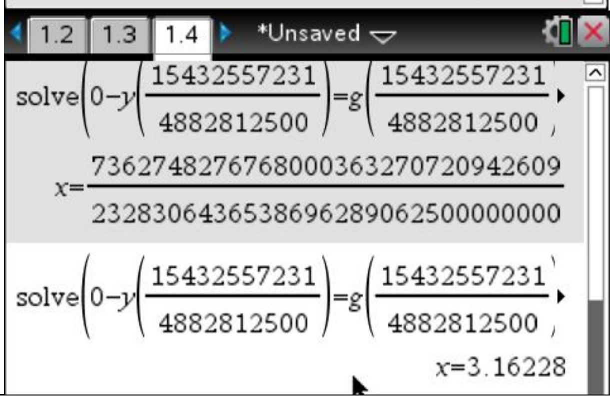


ii. Hence, use Newton's Method to determine the coordinates of point B to 2 decimal places of accuracy. 3 marks

$y = \frac{5}{8}(5x-14) = 0$ $x = \frac{14}{5} = 2.8$	**M1**
$0 - y\left(\frac{14}{5}\right) = \frac{200}{(14/5)^3}\left(x - \frac{14}{5}\right)$ $x = \frac{1939}{625} \approx 3.102$	**M2**
$0 - y\left(\frac{1939}{625}\right) = \frac{200}{(1939/625)^3}\left(x - \frac{1939}{625}\right)$ $x \approx 3.161$ $0 - y(3.161) = \frac{200}{(3.161)^3}(x - 3.161)$ $x \approx 3.162$ Therefore, check indicates that the coordinates of B are: $(3.16, -0.01)$	**A1**







END OF EPISODE V – THE EXPONENTIAL STRIKES BACK