VICTORIAN CERTIFICATE OF EDUCATION

2016

MATHEMATICAL METHODS (CAS)

MATH WARS

SOLUTIONS

Episode VI – Return of the Jedpi

Question 1 (6 marks)

Whilst trying to evade the Rangecor's giant claws, Colin manages to find a large bone of length 1.5 meters. Eventually the Rangecor catches Colin in its large claw and raises him up to eat him. The height of the Rangecor's upper jaw, as it opens and closes over a 5 second interval, can be modelled by the equation:

$$U:[0,5]\rightarrow R,\ \mathrm{U}(t)=9+2\sin\left(\frac{\pi t}{5}\right)$$

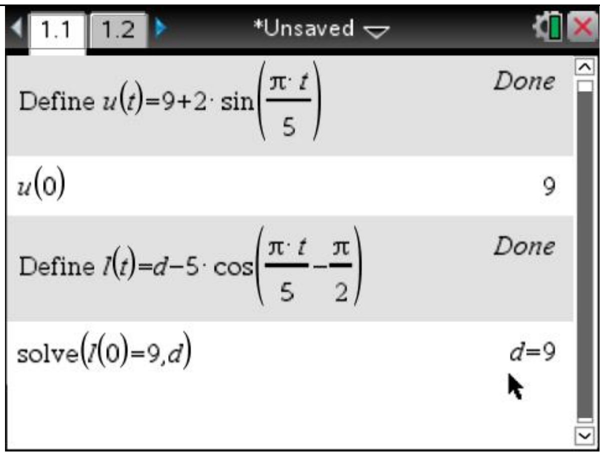
The height of the Rangecor's lower jaw over this same time can be modelled by the equation:

$$L:[0,5]\rightarrow R,\ L(t)=d-5\cos\left(\frac{\pi t}{5}-\frac{\pi}{2}\right)$$

where c and d are positive, real numbers.

a. Given that the Rangecor's mouth is closed (lower jaw and upper jaw are at the same height) at $t=0$ seconds and at $t=5$ seconds, show that $d=9$. 2 marks

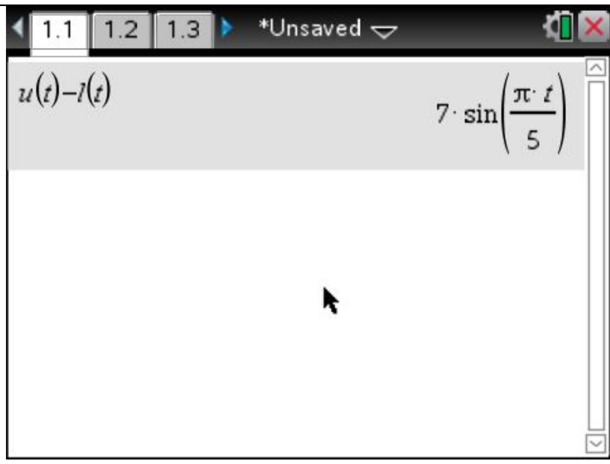
$U(0)=9$	**M1**
$\therefore L(0)=9$ $L(0)=d-5\cos\left(0-\frac{\pi}{2}\right)$ $d-5(0)=9$ $\therefore d=9$ *Substitution of t=0 must be shown ** If students do not state final answer d=9, they should lose final mark	**M2**



b. Colin intends to use the 1.5 meter long bone that he had picked up to wedge the Rangecor's mouth open.

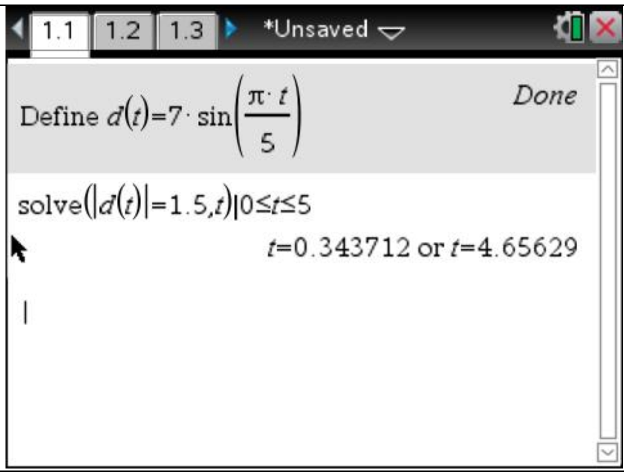
i. Find an equation, $d(t)$, for the vertical distance between the upper and lower jaw at time, t seconds. 1 mark

$d(t)=U(t)-L(t)$	*
$d(t)=9+2\sin\left(\frac{\pi t}{5}\right)-9+5\cos\left(\frac{\pi t}{5}-\frac{\pi}{2}\right)$	
$d(t)=2\sin\left(\frac{\pi t}{5}\right)+5\cos\left(\frac{\pi t}{5}-\frac{\pi}{2}\right)$	*
$\cos\left(\theta-\frac{\pi}{2}\right)=\sin(\theta)$	
$d(t)=2\sin\left(\frac{\pi t}{5}\right)+5\left(\sin\left(\frac{\pi t}{5}\right)\right)$	*
$d(t)=7\sin\left(\frac{\pi t}{5}\right)$	*
*No working is required to get this mark ** Accept any of the lines indicated by an asterisk '*'	**A1**



ii. Hence, find the time(s), to 3 decimal places, at which the vertical distance between the Rangecor's jaws is equal to the length of the bone. 2 marks

$d(t)=1.5$	**M1**
$t=0.344$ or $t=4.656$ * students should be awarded consequential marks from part i.	**A1**



iii. At what time should Colin jam the bone into the Rangecor's jaws in order to wedge them open? Justify your answer with a reason. 1 mark

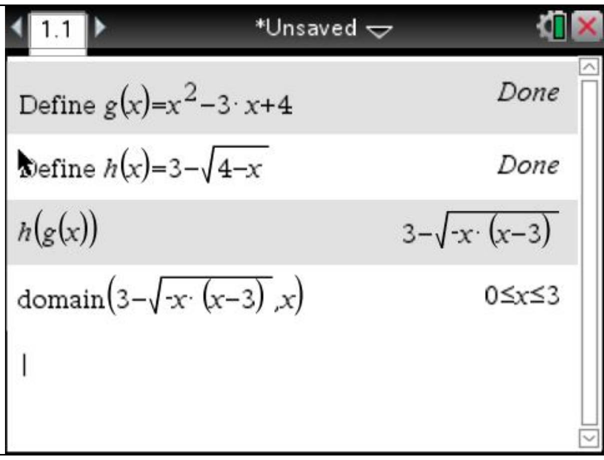
$t=4.656$ seconds It cannot be 0.347 seconds as at this time, the Rangecor is opening it's mouth so Colin will have to wait until the jaw is closing before wedging the bone in. *Or any equivalent or accurate justification. **If no justification is given, student should not be given the mark.	**H1**

Question 2 (10 marks)

Harnath's team arrive in their stolen Exponential Shuttle near the forest moon of Endor. In order to get safe passage to the moon's surface however, they need to pass an Exponential authentication test that involves them solving 3 different problems correctly.

a. Given that $g:[1,a)\rightarrow R,\ g(x)=x^2-3x+4$ and $h:\left[\frac{7}{4},4\right)\rightarrow R,\ h(x)=3-\sqrt{4-x}$, find the maximal value of a for which the function $h(g(x))$ is defined. 2 marks

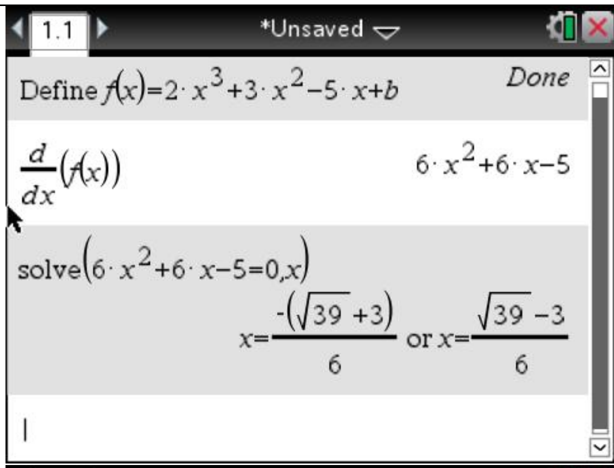
$h(g(x))=3-\sqrt{x(3-x)}$	**M1**
Maximal domain of $h(g(x))$ is $[0,3]$ therefore maximal value of a is 3.	**A1**

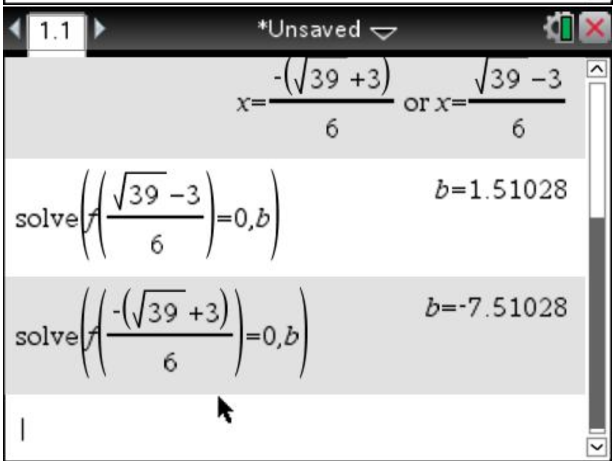


OR

ran $g(x)\subseteq$ dom $h(x)$ $\therefore$ ran $g(x)=\left[\frac{7}{4},4\right)$ $\therefore g(a)=4$	**M1**
therefore maximal value of a is 3.	**A1**

b. State the values of b, **correct to 2 decimal places**, for which the expression $2x^3+3x^2-5x+b=0$ has 3 solutions. 3 marks

<table>
<tr>
<td>Location of the b value that results in one of the turning points touching the x axis.
$b=1.51028$
OR
$b=-7.51028$</td>
<td>M1</td>
<td>


</td>
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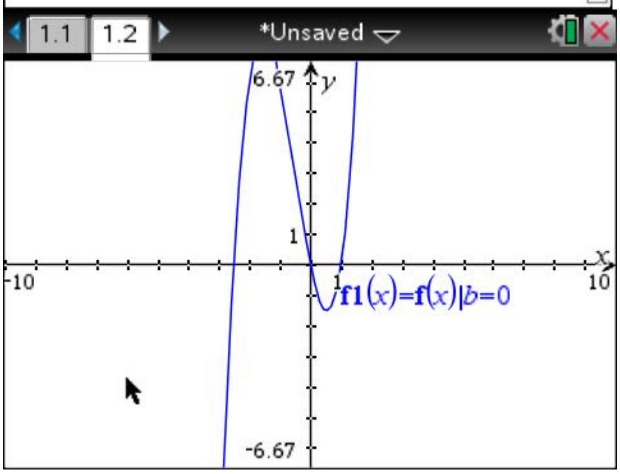
<td>Sketch of $2x^3+3x^2-5x+b$ or evidence of identifying shape of cubic</td>
<td>M2</td>
<td rowspan="2">
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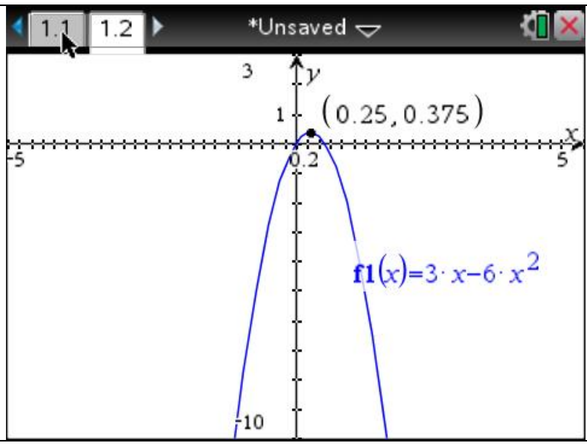
<tr>
<td>$-7.51<b<1.51$
*If students can get the correct answer without the suggested working shown, and demonstrate an approach to generating the solution that is valid, they should be awarded full marks. These suggested solutions are a guide only.</td>
<td>A1</td>
</tr>
</table>

c. Consider the function $f:(c,5]\to R, f(x)=3x-6x^2$

i. Find the minimum value of c for which the inverse function $f^{-1}(x)$ exists. — 2 marks

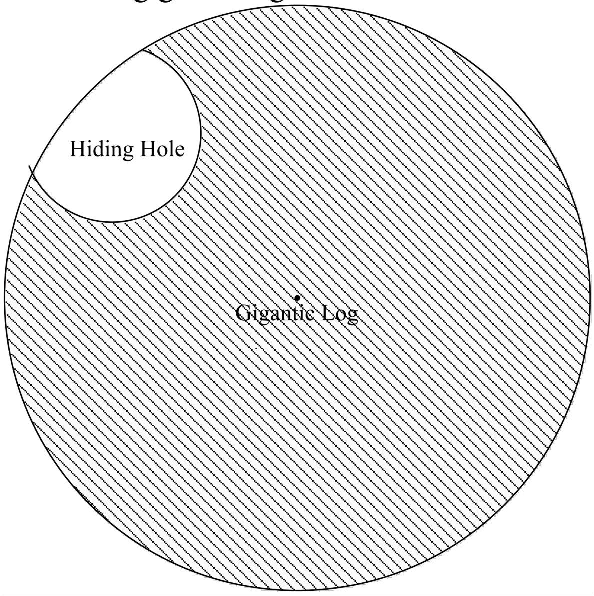
In order for inverse function to exist, $f(x)$ must be one-to-one, therefore, need to find the closest turning point to x=5 $f'(x)=3-12x=0$ $x=\frac{1}{4}$	**M1**	
$\therefore c=\frac{1}{4}$	**A1**	

i. Hence find the rule for $f^{-1}(x)$ and state its domain. — 3 marks

let $y=f^{-1}(x)$ $\therefore f(y)=x$ $x=3y-6y^2$	**M1**
$y=\frac{3\pm\sqrt{9-24x}}{12}$ reject $y=\frac{3-\sqrt{9-24x}}{12}$ as we only want the positive answer $\therefore f^{-1}(x)=\frac{3+\sqrt{9-24x}}{12}$ *Explicit rejection is not required	**A1**
Domain of $f^{-1}(x)$ is $[-135, 0.375)$ OR $\left[-135,\frac{3}{8}\right)$	**A2**

Question 3 (8 marks)

The e^{wok}'s are excellent trap builders and they make use of lots of traps in the forest to fight the Exponential troops. One such trap that the e^{wok}'s have built involves rolling a gigantic log down a slope to squash any Exponential troops. Carved into the log are small holes that allow an e^{wok} that is caught in the path of the log, to hide in and avoid being squashed. The diagram below shows a cross-section of the gigantic log.



The height, H, of the hiding hole above the ground as the gigantic log rolls along the slope can be modelled by the equation:

$$H(t) = 15 + 15\sin(nt)$$

where t is the time in seconds since the log starts rolling.

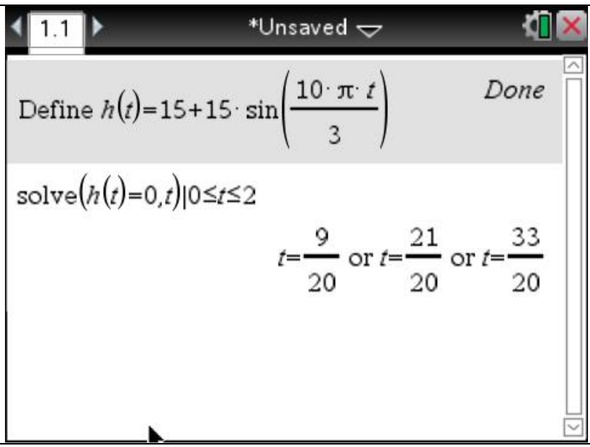
a. It is known that the log completes 5 full revolutions in 3 seconds.

i. Show that the value of n is $\dfrac{10\pi}{3}$ 2 marks

Log completes 5 full revolutions in 3 seconds so: Period $= \dfrac{3}{5}$	**M1**
Period $= \dfrac{2\pi}{n}$ $\therefore n = \dfrac{2\pi}{3/5} = \dfrac{10\pi}{3}$	**M2**

ii. According to the model, at what times in the first 2 seconds is the hiding hole on the ground? 2 marks

$h(t)=15+15\sin\left(\frac{10\pi t}{3}\right)=0$	**M1**
$t=\frac{9}{20},\frac{21}{20}$ or $\frac{33}{20}$ seconds *Units must be given unless stated as t= since t is already defined with units.	**A1**



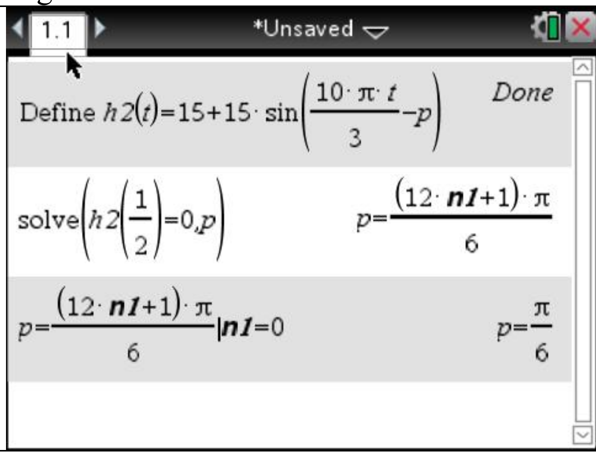
An e^{wok} trap designer notices that the actual times at which the hiding hole first reaches the ground is at $t=0.5$ seconds. The e^{wok} changes the model to the form:

$$H_2(t)=15+15\sin\left(\frac{10\pi t}{3}-p\right)$$

where p is a positive, real number.

iii. Find the smallest value of p that will allow the model to accurately reflect the time at which the hiding hole first reaches the ground. 2 marks

$H_2\left(\frac{1}{2}\right)=15+15\sin\left(\frac{10\pi\left(\frac{1}{2}\right)}{3}-p\right)=0$	**M1**
$p=\frac{\pi}{6}$	**A1**

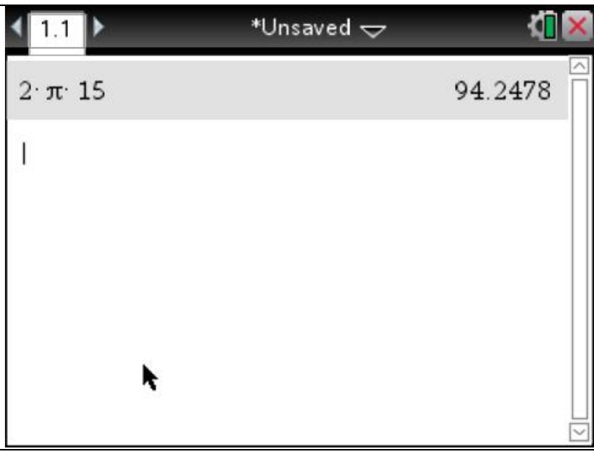


iv. What is the radius of the gigantic log? 1 mark

15 meters	**M1**

v. Using your answer to **part iv.** find the distance, in meters and to 2 decimal places, between successive points where an e^{wok} can make use of the hiding hole being on the ground. 1 mark

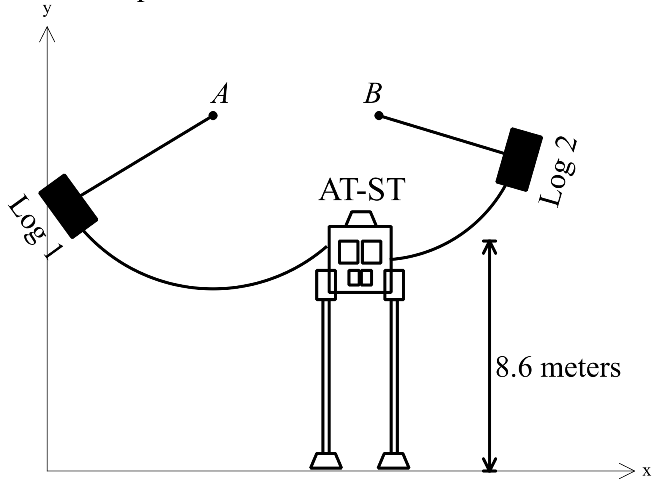
The circumference of the gigantic log is also the distance between successive points on the ground where it will be safe to stand and make use of the hiding hole. $\therefore d=2\pi(15)=30\pi$ $d=94.25$ meters	**A1**



Question 4 (11 marks)

Another one of these traps consists of two large logs that are connected to very tall trees by long ropes. When these are released, the logs swing through the air and collide with the Exponential attack vehicles known as AT-ST's.

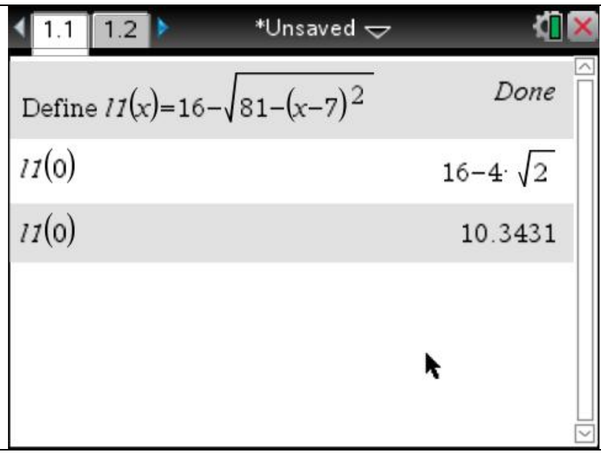
A diagram to show how these traps work is shown below:



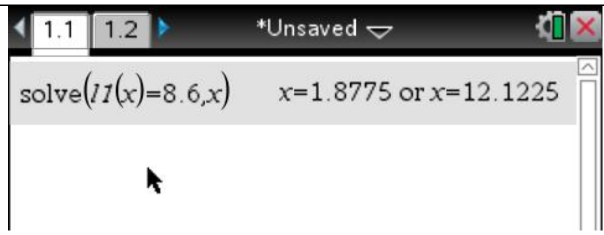
In the particular example that is shown above, the point at which the logs need to hit the AT-ST is 8.6 meters above the ground. Points A and B are at coordinates (7,16) and (11,16) respectively.

a. The path of Log 1 can be described by the equation $L_1(x) = 16 - \sqrt{81 - (x-7)^2}$ where x is the horizontal distance from the point where Log 1 is released and L_1 is the height above ground of Log 1.

i. Find the height, in meters and correct to 2 decimal places, from which Log 1 is initially released. 1 mark

$L_1(0) = 10.34$ meters *Statement of units is not required.	**A1**	

ii. Log 1 will collide with the AT-ST after it has passed its lowest point. Show that the coordinates at which Log 1 collides with the AT-ST are $(12.1,\ 8.6)$. 1 mark

$L_1(x) = 8.6$ $x = 1.88$ OR 12.12 reject x = 1.88 as this is to the right of the minimum point. *both solutions must be shown and rejection of x = 1.88 stated to get this mark	**M1**	

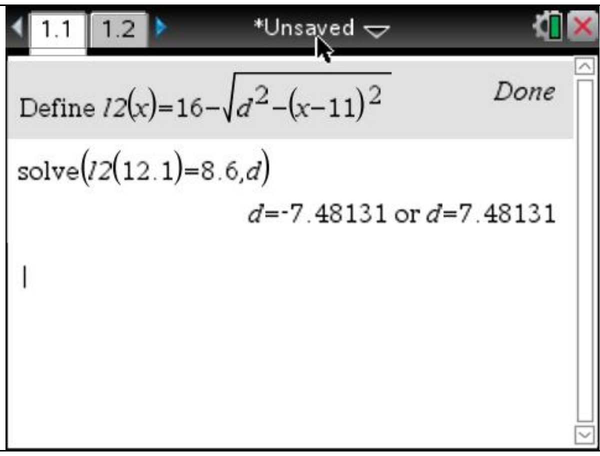
b. The path of Log 2 can be described by a different equation of the form:

$$L_2(x) = f - \sqrt{d^2 - (x-e)^2}$$

i. Explain why, when considering Log 2, $e = 11$ and $f = 16$. 1 mark

e and f reference the location of the centre of the semi-circle that this equation describes. As the centre of the circle is located at (11,16), e must be 11 and f must be 16. *or some other valid justification	**A1**

ii. In the equation for L_2 above, d can be described as the length of the rope connecting Log 2 to point B. If Log 2 collides with the AT-ST at coordinates $(12.1,\ 8.6)$, find the value of d to **2 decimal places**. 2 marks

$L_2(12.1) = 16 - \sqrt{d^2 - (12.1-11)^2} = 8.6$	**M1**	
$d = 7.48$	**A1**	

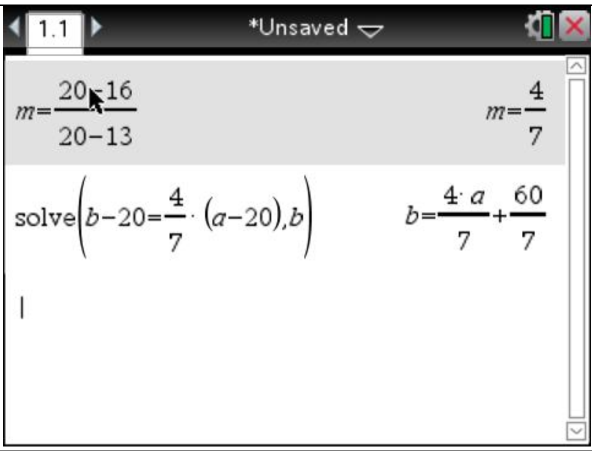
This particular group of e^{wok} ‘s have had to move the location of the original point B to the right, call it point C, due to the branches on which the ropes are tied not being in the correct location.

Cvetkovskbacca and the e^{wok} ‘s spot an AT-ST approaching with it’s top hath open and a Sin Trooper poking his head out of the top of the AT-ST. Cvetkovskbacca sees this as a perfect opportunity to steal the AT-ST and use it against the Exponential Troops.

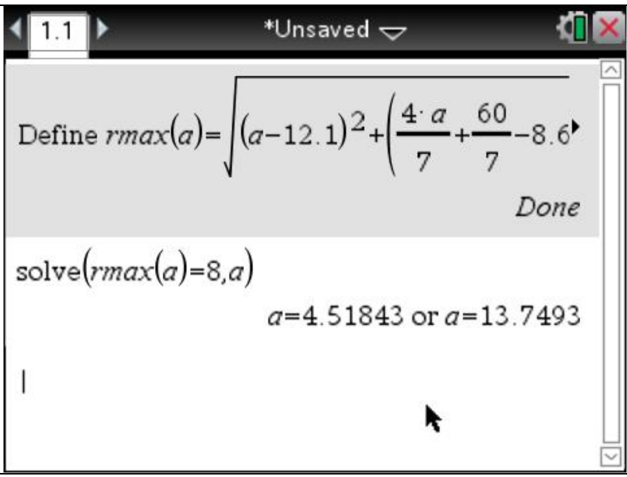
Some quick calculations show that the e^{wok} ‘s and Cvetkovskbacca should aim for Log 2 to hit anywhere in the region of $(12.1,\ 8.6)$ to $(12.1,\ 9.6)$.

c. Consider point C, with general coordinates $(a,\ b)$, to be a point on the straight branch that joins the points $(13,\ 16)$ and $(20,\ 20)$.

i. Find all possible values for the amount of rope, R, needed in order for Log 2 to be able to hit this region, in terms of a only. 4 marks

$b=\dfrac{4a}{7}+\dfrac{60}{7}$	**M1**	
$R_{MAX}=\sqrt{(a-12.1)^2+\left(\dfrac{4a}{7}+\dfrac{60}{7}-8.6\right)^2}$	**A1**	
$R_{MIN}=\sqrt{(a-12.1)^2+\left(\dfrac{4a}{7}+\dfrac{60}{7}-9.6\right)^2}$	**A2**	
$\sqrt{(a-12.1)^2+\left(\dfrac{4a}{7}+\dfrac{60}{7}-9.6\right)^2}\le R\le\sqrt{(a-12.1)^2+\left(\dfrac{4a}{7}+\dfrac{60}{7}-8.6\right)^2}$	**A3**	

ii. The e^{wok} ‘s have 8 meters of rope available to them. What values of a can they use? 2 marks

$R_{MAX}(a)=8$ $a=13.7493$	**M1**	
$13\le a\le 13.7493$	**A1**	

END OF EPISODE VI – RETURN OF THE JEDPI