VICTORIAN CERTIFICATE OF EDUCATION

2017

STUDENT NAME:

MATHEMATICAL METHODS (CAS)
SAC 1 – EPISODE 2

Harry Potter and the SAC of Fire

2017

Reading Time: 10 minutes
Writing time: 50 minutes

QUESTION AND ANSWER BOOK

Number of questions	Number of questions to be answered	Number of marks
5	5	40
	Total	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved CAS Calculator and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination: blank sheets of paper and/or correction fluid/tape.

Materials supplied

- Question and answer book of 13 pages.
- Formula sheet

Instructions

- Write your name in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

> **Instructions**
>
> Answer **all** questions in the spaces provided
>
> In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
>
> In questions where more than one mark is available, appropriate working **must** be shown.
>
> Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

The beginning of every year at Hogwarts sees all students gathered for dinner in the great hall to see the new students being sorted into houses and new teachers being introduced by the school headmaster, Professor Corkilldore.

This year, as you carefully select your dinner from the variety of meats, vegetables, pies, puddings and breads, Professor Corkilldore announces that there is a new Defence Against the Dark Arts teacher, Professor "Mad Eye" Bohni. He also announces that, for the first time in over 200 years, the Tri-Wizard-SAC will be hosted at Hogwarts. The Tri-Wizard-SAC is a famously difficult SAC that representatives from each of the 3 wizarding academies; Hogwarts, Beauxbatons and Durmstrang, will compete against one another in. The Tri-Wizard-SAC can be so challenging that competitors have been known to die whilst competing in it.

The SAC of fire, held later in the year, will decide which of the students from each school will compete.

Today you find yourself in a Potions class with Professor Severus Shnier. He has asked everyone to attempt to brew an anti-sleeping potion called the Wideye Potion.

Question 1 (6 marks)

As you read up about the Wideye Potion, you discover that it works by keeping a person's eyes wide open so that they don't fall asleep. The amount of ingredients required is determined by how wide you need the lids of the eyes to open. Glancing at your reflection in your brand new, very shiny copper cauldron you come up with the following model for your eyes.
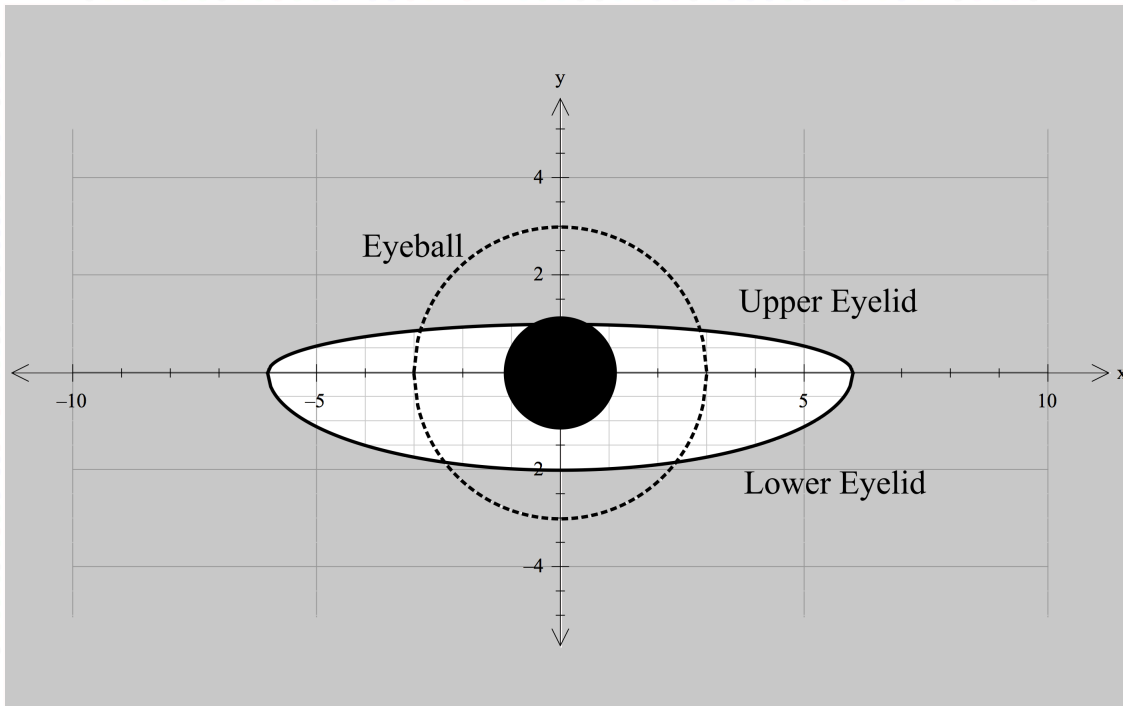
The Eyeball is described by the equation $E(x) = \pm\sqrt{9 - x^2}$.

The Upper Eyelid is described by the equation $U(x) = \frac{1}{2}\sqrt{4 - \left(\frac{x}{3}\right)^2}$.

The Lower Eyelid is described by the equation $L(x) = -\sqrt{4 - \left(\frac{x}{3}\right)^2}$.

a.   What is the domain and range of the Eyeball in this model?   2 marks

The first ingredient that you need for the Wideye Potion is some snake fangs. Snake fangs are directly responsible for moving the Upper Eyelid.

b.   What dilation factor is required so that the Upper Eyelid touches the top of the Eyeball? State the new equation of the Upper Eyelid, U_2, after this transformation has been applied   2 marks

You look up the conversion of dilation factors into snake fangs and determine that you will need 6 snake fangs to achieve sufficient widening of the Upper Eyelid.

You work out that the Lower Eyelid will require a dilation factor of $\frac{3}{2}$ from the x axis. You can achieve this by adding some "standard ingredient".

c.   State the new equation of the Lower Eyelid, L_2, after this transformation has been applied   1 mark

You work out that this will equate to 4 measures of "standard ingredient" being added to the cauldron. Unfortunately, you realise that the "standard ingredient" also results in a dilation of the Lower Eyelid from the y axis.

The new equation of the Lower Eyelid, L_3, can be expressed as $L_3 = L_2\left(\frac{x}{5}\right)$.

Fortunately, you know that dried Billywig Stings can be used to apply a dilation from the y axis as well.

d.   What dilation factor would be required in order to undo the dilation from the y axis caused by the standard ingredient added in **part c** to turn L_3 into L_2?   1 mark

This dilation factor means that you will need to include 6 dried Billywig Stings in your potion as well. You put all of your ingredients aside and begin to heat up the water in your cauldron.

You know that in order for your potion to work effectively, you need to ensure that the rate at which the temperature increases over time remains constant. You also know that every time you add an ingredient to the mixture, it will drastically affect the rate at which the temperature changes. You will have to use twigs of wolfsbane that have been infused with mathematical equations to try and counteract the effect of each ingredient as you add it.

Question 2 (11 marks)

The initial temperature of the water in your cauldron is 17° Celsius. You adjust the fire heating your cauldron so that the temperature is increasing at a steady 1° Celsius every minute.

a. **i.** Find an equation for the temperature, T, in $^\circ$ Celsius, t minutes after you begin heating up the cauldron.   1 mark

ii. You need to add the snake fangs when the temperature reaches 25° Celsius. After how many minutes will this occur?   1 mark

The addition of snake fangs alters the temperature of the potion so that it now follows the equation $T_1(t) = T(t) + S(t)$, where $S(t)$ is the temperature change caused by the addition of the snake fangs and is described by the equation:

$$S(t) = 7 - 3e^{5t+2}$$

In order to counteract the addition of snake fangs, you need to infuse a twig of wolfsbane with the equation of the inverse of $S(t)$ and add it to the potion.

b. Find the equation of the inverse, S^{-1}, of $S(t)$.   2 marks

You find the equation of the inverse of S and perform an incantation to infuse the wolfsbane with the mathematical equation. You then proceed to add the wolfsbane and snake fangs at the same time. You are very pleased when the potion turns a nice shade of green and the rate at which the temperature is increasing remains constant.

The next ingredient, the "standard ingredient", already has crushed-up wolfsbane incorporated into it and so it won't alter the rate of change of temperature of the potion. The dried Billywig Stings are a different story however.

Dried Billywig Stings are quite volatile and will alter the temperature according to the equation:

$$B(x) = -2x^2 + 8x - 2$$

where x is the time, in minutes, after the Billywig Stings have been added to the potion.

c. **i.** Explain why the function $B(x)$ does not have an inverse function, B^{-1}.   1 mark

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You realise that you can still use the wolfsbane to counteract the effect of the Billywig Stings but that you will need to use two separate pieces, each with a different infused equation that you will use at different times. You let the alteration caused by the Wolfsbane be described by the equation $B^{-1}(x)$.

    ii. Find the maximum value of a such that the domain restriction $[0, a]$, when applied to $B(x)$, allows the inverse function, $B^{-1}(x)$ to exist.   1 mark

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    iii. Find the equation of $B^{-1}(x)$ over the domain $[0, a]$.   3 marks

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You infused the first piece of wolfsbane with this equation and add it to the potion with the Billywig Stings so that the Temperature is now described by $B(B^{-1}(x))$.

    iv. At what time, x minutes, after adding the Billywig Stings should the first piece of wolfsbane be removed and swapped with your next piece?   1 mark

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    v. The next piece of wolfsbane is added after the time found in **part iv.** What should $B^{-1}(x)$ be now to counteract the Billywig Stings after this time?   1 mark

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After adding the final twig of wolfsbane to the mixture a puff of orange smoke is released from your cauldron. Professor Shnier races over and tastes your potion. His eyes widen (you're not sure if that's the potion working or if he's surprised by the taste of your potion) and he smirks as he turns away and announces that everyone should pack up their gear as class is almost over. As everyone scrambles to pack everything away Professor Shnier announces that everyone will need to have collected some asphodel, dittany and wiggentree bark to bring to the next potions class.

After lunch, you head off to your next class, Defence Against the Dark Arts, with Professor "Mad Eye" Bohni. As you enter the classroom you find Professor "Mad Eye" Bohni firing spells at students in rapid succession. "You must always be prepared for an attack" he yells as he fires yet another spell. You notice that many of your classmates are slumped at their desks, stupefied by a spell that the Professor had fired at them.

Suddenly, Professor "Mad Eye" Bohni turns towards you and fires a spell. "Solve this!" he cries maniacally.

Question 3 (1 mark)

The function $f(x) = x^{\frac{5}{13}}$ is greater than the function $g(x) = x^{\frac{2}{3}}$ for what values of x?

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You drop your books and solve the spell that was cast at you with a flourish of your wand. You breathe a sigh of relief and go to pick up your books when you notice that the class is suddenly all staring at you, including Professor "Mad Eye" Bohni.

"What's your name?" Professor "Mad Eye" Bohni asks. Before you get a chance to answer he waves his hand and says "Never mind, just leave those books there and come up the front." As you walk towards the front he casts a spell that collects your books and places them neatly on a nearby table. "That was an impressive counter to my spell" he says. "Now today, we will be learning how to manipulate our foes in battle."

Professor "Mad Eye" Bohni points his wand at a nearby unconscious student and an invisible force picks the student up and suspends him in the air at the front of the class. The student's head is now 3 meters above the ground.
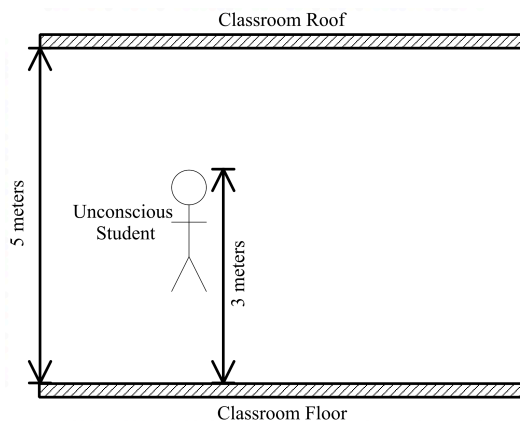
"For your first spell" Professor Bohni tells you, "I want you to move this student upwards until they are touching the roof".

To assist with this spell, you picture the student on a cartesian plane with the top of the head located at the coordinate (0,0).
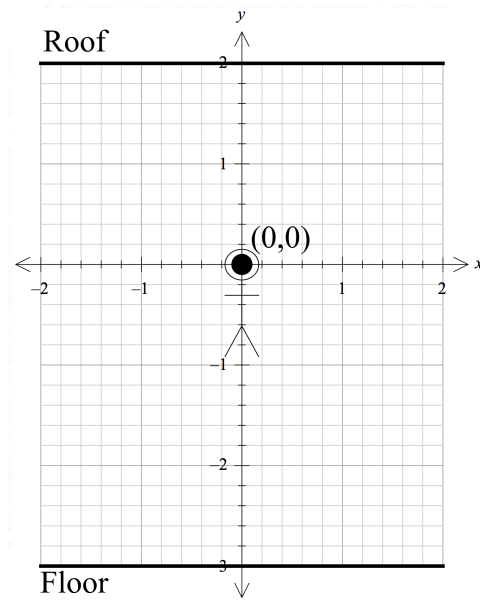
You decide to use a technique that you had been studying over the holidays involving matrices.

Question 4 (5 marks)

a. Let T_1 be the transformation that translates the unconscious student's head from (0,0) to

(0,2). Write $T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ as a matrix equation.   1 mark

"Very good" remarks the Professor. "Now watch carefully what I do…"

Professor "Mad Eye" Bohni, starting with the unconscious student's head located at (0,2), flips the body upside down, moves the body 1 unit to the right and then proceeds to move the body upwards by 3 units.

Let T_2 be the transformations applied by "Mad Eye" Bohni. You jot down the transformations as a matrix equation.

$$T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

b.   i.   What are the coordinates of the head of the unconscious student after these transformations have taken place?   1 mark

"Now, try and undo what I just did" shouts Professor Bohni enthusiastically.

You decide that the easiest way to undo what Professor "Mad Eye" Bohni did was to apply a negative sign to everything that the professor did. Doing so, you get the following transformation matrix equation:

$$T_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = -T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

ii. Starting from coordinates (1,1), what are the coordinates of the location of the head of the unconscious student after these transformations have taken place?   1 mark

"Hmmm, that isn't right" muses Professor "Mad Eye" Bohni. He quickly undoes your transformations and places the head of the unconscious student back at coordinates (1,1). "Try again!" he commands, "But don't use any dilations, I got in trouble at my last school for breaking one student's bones when I squished him too much".

iii. What is a correct sequence of transformations, expressed as a matrix equation, that will move the upside down unconscious student's head from (1,1) to the original position of (0,2) whilst also flipping the unconscious students body the right way up again?   2 marks

You successfully manage to put the unconscious student back at (0,2) and the right way up securing loud cheers from the other students in the class. The cheering, however, wakes up the unconscious student who begins to panic when they realise that they are hovering 5 meters above the ground.

Professor "Mad Eye" Bohni quickly casts a stunning spell and then returns the now stunned student back to their desk. "Just as well." he mutters, "The next set of transformations got me in trouble with the parents at my last school when I used students as the examples… don't want to go through all that paperwork again." He opens the lid of a nearby chest and brings out a set of strange glowing balls.

Just then, Professor Sproutcovska busrts into the room and informs Professor "Mad Eye" Bohni that class actually ended 20 minutes ago and that she has been waiting for her students to arrive in the Herbology classroom to no avail. Whilst Professor "Mad Eye" Bohni

apologises to Professor Sproutcovska, you and your classmates quickly pack up your books and then proceed to follow Professor Sproutcovska to Herbology.

Once at the Herbology Greenhouse, Professor Sproutcovska exaplains that you will be investigating the properties of the "Devil's Snare" plant, a plant that will attempt to strangle anything in it's immediate vicinity.

"The Devil's Snare Plant grows at differing rates" explains Professor Sproutcovska, "You must try and determine when it is most docile and attempt to study it during these times. If you find yourself snared by the plant, don't struggle, that'll just make it worse, instead call for me and I'll help you escape".

Question 5 (17 marks)

You do some research in your textbook "One Thousand Magical Herbs and Fungi" by Phyllida Spore and discover that the activity level, A, of Devil's Snare plants can be modelled by an equation of the form:

$$A(t) = a\cos(bt + c) + d$$

where a, b, c and d are real numbers and t represents the time of day after midnight in hours.

Phyllida Spore, in her book, mentions that Devil's Snare plants do not like bright lights and instead prefer cool, damp locations. The book provides the following estimates for the values of a, b, c and d.

$$a = 3$$
$$b = \frac{\pi}{12}$$
$$c = -\frac{1}{12}$$
$$d = 5$$

The plant is most docile when the activity level, A, is a minimum.

a.   **i.**   Using the values of a, b, c and d provided by Phyllida Spore's book, state what the minimum activity level, A, of the Devil's Snare plant is.   1 mark

     ii.   Find all times during a day (from midnight to 11:59pm) is the Devil's Snare plant most docile? Give your answer correct to the nearest minute.   2 marks

According to the book, it is considered safe to approach the Devil's Snare plant when its activity level is below 2.6

     iii.   At what times of the day is it safe to approach the Devil's Snare plant? Express your answer correct to the nearest minute.   2 marks

Unfortunately, you quickly realise that the Devil's Snare plant directly in front of you is not an average plant and that these estimated values will not suffice. Upon further reading of Phyllida Spore's book, you discover that each of the terms, *a*, *b*, *c*, and *d* are affected by various environmental factors:

a is related to the humidity level of the environment in which the Devil's Snare plant resides.

b and *c* are both related to the intensity of the light around the plant and *d* is described by the graph shown below.
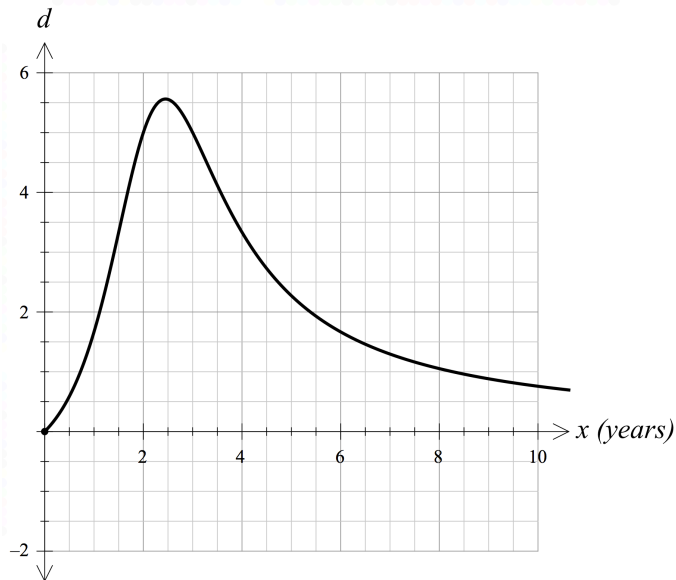
The equation of this graph is:

$$d(x) = \frac{5x}{x^2 - 4x + 6}, x \geq 0$$

where *x* represents the number of years that have passed since the Devil's Snare plant was potted.

You ask Professor Sproutkovska when these Devil's Snare plants were potted and she tells you that they were potted exactly 1 year and 2 months ago by the 3rd year Herbology students.

b.   According to this information, what value of *d* will the Devil's Snare plant in front of you have?   1 mark

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Next you plan to determine the humidity level of the atmosphere around the plant. Every 2 hours, a sprinkler comes on to water the Devil's Snare plants. Whilst the sprinkler is on, the humidity level of the air around the plant can be considered to be 1. It then decreases according to the rule:

$$a: [0,2) \rightarrow R, a(w) = e^{\frac{-w}{3}}$$

where w is the time, in hours, after the sprinkler was turned off.

The sprinkler was last turned off at 1:30pm.

c. **i.** The time is currently 1:43pm. Find the value of a at this time, correct to two decimal places. 1 mark

It is possible to rewrite the equation for a from above, in terms of the time, t, in hours since midnight in the form

$$a: [13.5, 15.5) \rightarrow R, a(t) = e^{-\frac{(t-h)}{3}}$$

where h is a positive, real number.

ii. State the value of h in this case. 1 mark

Finally, you tackle the task of determining the values of b and c that are related to the light intensity around the Devil's Snare plant. You read that the terms b and c can in fact be replaced with a function that represents the light intensity level, L.

The light intensity level, L, in the Herbology Greenhouse varies over the course of a day according to a hybrid function shown below:

$$L(t) = \begin{cases} -\dfrac{1}{18}t^2 + \dfrac{4}{3}t - 6 & , \quad 6 \leq t < 7 \cup 17 \leq t < 18 \\ \dfrac{1}{4}\sin\left(\dfrac{3\pi t}{5}\right) + H & , \qquad 7 \leq t < 17 \\ \qquad 0 & , \qquad \text{elsewhere} \end{cases}$$

where t is the time, in hours, after midnight and H is a positive, real number.

d. **i.** Assuming $L(t)$ is a continuous function, find the value of H. 2 marks

ii.   State the values of t for which the light intensity, L, is a maximum.   2 marks

iii.   What is the maximum light intensity, L, correct to three decimal places?   1 mark

Using your answers to **part c, ii.** and **part d, i.** as well as the approximate value of 2 for d, a new equation for the activity level, A_1, of the Devil's Snare plant at time, t hours after midnight can be written in the form:

$$A_1: [13.5, 15.5) \to R, A_1(t) = a(t)\cos(L(t) \times t) + d.$$

e.   **i.**   According to A_1, what is the activity level of the Devil's Snare plant in front of you at the current time of 1:43pm? Express your answer correct to two decimal places.   2 marks

To be safe, you decide to only approach the Devil's Snare plant when its activity level is below 1.6.

ii.   At what time(s) between 1:30pm and 3:30pm is it safe to approach the Devil's Snare plant? Give your answer correct to the nearest minute.   2 marks

Having determined when it is safe to approach the Devil's Snare plant, you proceed to do so and are quite relieved when you are not strangled to death. You proceed to study the plant and do lots of interesting magical stuff to it until it is no longer safe to continue and then you retreat.

Impressed by your knowledge of when it is safe to approach the plant, Professor Sproutcovska praises you by awarding 20 points to your house.

Having finished classes for the day, you return to your room utterly exhausted from all the mathemagics you have performed over the course of the day. You are just about to lay down on your bed when your best friend bursts into the room and reminds you to hurry downstairs, tonight the SAC of fire is choosing the students who will represent each school in the Tri-Wizard SAC.

Once downstairs, you can feel the excitement bubbling around you. Representatives from both Beauxbatons and Durmstrang are also in the hall and everyone is anxious to find out who will be selected by the SAC of fire.

Professor Corkildore stands up and the entire hall goes silent. "Good evening, ladies and gentlemen, ghosts and – most particularly – guests" he begins. He proceeds to welcome the other schools and a series of ministers from the Ministry of Magic and explains that all eligible students have secretly been assessed in all of their classes so far this year by the impartial SAC of fire.

You suddenly start to get very nervous. You have consistently been achieving very high grades in your classes this year, you thought the SAC of fire would be a separate assessment and not one that was based on what you'd been doing so far. You begin to feel butterflies in your stomach as you try to think of a student who has scored higher than you in your classes.

Before you know it, Professor Corkildore has waved his wand and the first name is revealed to be Viktor Rajewskrum from Durmstrang. The next name that comes out of the SAC of fire is Meagan DelaCross from Beauxbatons.

The next name will be from Hogwarts. You wait for what seems like an eternity, feeling as though you are going to be sick. When Corkildore announces the name of Cedric Diggory, you let out a great sigh, thank goodness it wasn't you. You didn't even think about Cedric but it makes sense, he is regularly getting scores very similar to yours.

Everyone around you is cheering for Cedric and you join in the applause. Suddenly the SAC of fire spits out another name. Terrified that perhaps your name is going to come out again, you sink into your chair.

Professor Corkildore clears his throat and reads out the name "Harry Potter".

This is a very strange outcome, there are only meant to be three champions in the Tri-Wizard SAC but you are quite thankful that it is not you.

You become especially more thankful later in the year when Cedric is killed in the final challenge and Harry Potter is forced to fight the Dark Lord returned.

END OF SAC 1 - EPISODE 2