MLC

NAME: ______________________________

Victorian Certificate of Education

2021

MATHEMATICAL METHODS UNITS 3 and 4

Problem Solving/Modelling Task SAC 2

Question and Answer Book

Total marks: 74

SAC Details
This SAC is designed as a Modelling Task of the Calculus (Differential and Integral) Area of Study.
SAC 2 is 15 minutes reading time + 120 minutes. The task assesses the 3 outcomes, with marking allocation as outlined below. It contributes to 25% of the marks for Unit 3 and 4 SAC work combined.

Outcome 1 (7.5%)
Define and explain key concepts as specified in the content from the areas of study and apply a range of related mathematical routines and procedures.

Outcome 2 (10%)
Apply mathematical processes in non-routine contexts and analyse and discuss these applications of mathematics.

Outcome 3 (7.5%)
Select and appropriately use a computer algebra system and other technology to develop mathematical ideas, produce results and carry out analysis in situations requiring problem-solving, modelling or investigative techniques or approaches.

Directions to Students

Instructions:

- Answer **all** questions in the spaces provided.
- Unless otherwise specified, an **exact** answer is required to a question.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Write your **name** in the space provided above on this page.
- All written responses must be in English.

Materials allowed:

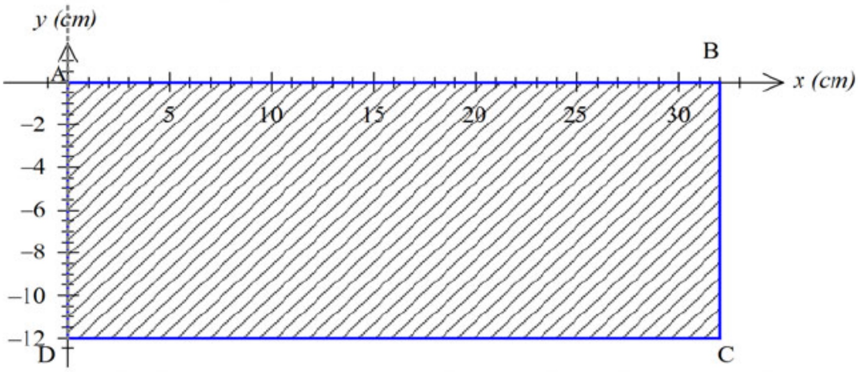
- 1 calculator
- 1 bound reference
- VCAA Mathematical Methods formula sheet.

As part of the VCAL Applied Fashion Design and Technology course, Year 12 students will be designing handbags.

The initial stage of creating a handbag is to develop a 2-dimensional (2D) design of it which then leads to developing prototypes.

Question 1 (11 marks)

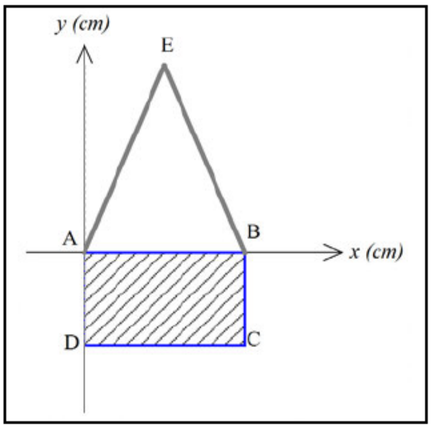
The design of the 2D rectangular cross section of the bag can be modelled using the x and y axis with vertices A(0,0), B(32,0), C(32,-12) and D(0,-12) as represented by the shaded area shown below.



a. Write down the rule and domain that defines the edge labelled as AB.

b. Write down the rule and domain that defines the edge labelled as BC.

Whilst the main part of the handbag must remain a rectangle with the same area, students are encouraged to be creative when it comes to designing the straps or handles. However, all straps end at points A and B as shown.



c. Let $f(x)$ represent a piecewise function that models the straps.

(i) State an appropriate function $f_1(x)$ that meets the requirements.

(ii) If the length of the strap pictured, i.e two joined straight lines, AE and EB, has to be 80 cm in total what could the function, $f_2(x)$ be?

d. Give a possible rule with a domain for a different strap design that has endpoints at points A and B and that can be modelled by a:

(i) sine function

(ii) semi circle equation

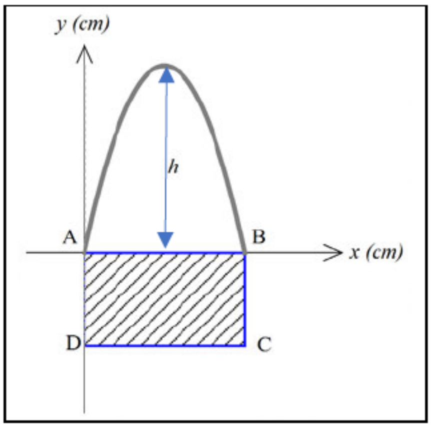
e. **(i)** Use integral calculus to write an expression to find the area bounded by the semi circle strap in **d(ii)** and the top edge of the bag AE. State this area.

(ii) Use the area formula $A = \pi r^2$ to verify the answer in **e(i).**

Question 2 (10 marks)

The conditions for the design of the 2D rectangular cross section of the handbag remain the same such that:

- the bag has vertices A(0,0), B(32,0), C(32,-12) and D(0,-12)
- the strap must terminate at points A and B.



Another design for the handbag strap has a parabolic shape as shown.

The vertical distance between where the strap hangs on the shoulder and the top of the bag (line segment AB) is h cm.

Let $f(x)$ represent a function that models the strap, $f:[0,32] \rightarrow \mathbb{R},\ f(x) = -px(x-32)$.

a. Show that $h = 256\,p$.

b. The arc length of any parabola $y = f(x)$ from $x = p$ to $x = q$ is given by the formula,

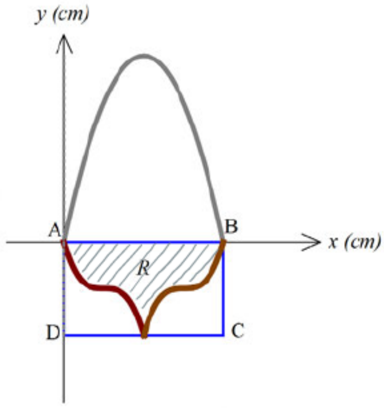
$$ARC\ length = \int_{p}^{q} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$$

Calculate the length of the strap if $h = 40$ correct to 1 decimal place.

The vertices of the bag remain at A(0,0), B(k,0), C(k,-12) and D(0,-12), where $8 \le k \le 20$.

It was decided that part of the bag cover, marked with R on the diagram, will be created using a different material.

c. Find the piecewise function $g(x)$ in terms of k that models the cubic curves which border the shaded region R, as shown.



d. **(i)** State the shaded area for region R in terms of k.

(ii) Give the range of possible area values of region R.

Question 3 (16 marks)

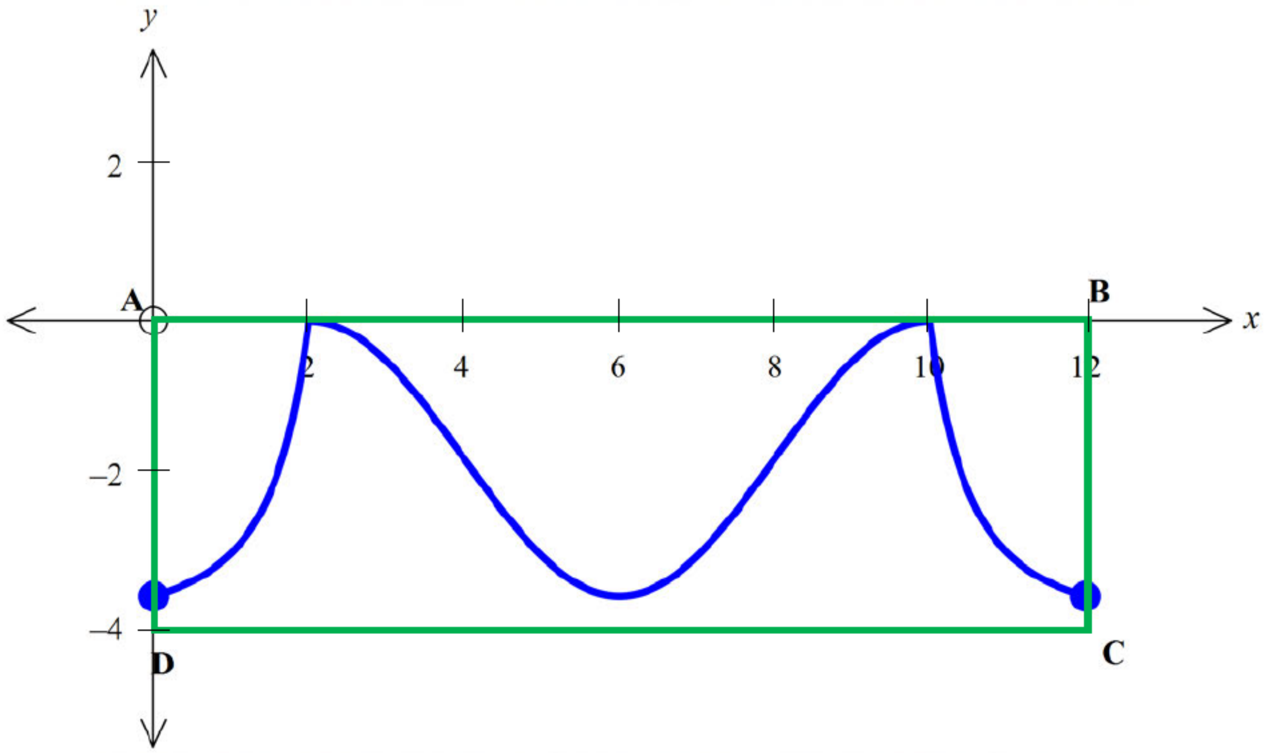
A different design is created for the bag cover as shown below where the measurements are in centimetres. This design will be scaled up when creating the actual bag.

The design is composed of three curves and can be defined as:

$$f:[0,2]\to\mathbb{R}, f(x)=\frac{4}{(x-3)^2}-4$$

$$g:(2,10]\to\mathbb{R}, g(x)=p-b\sin^2\left(n(x-c)\right)$$

$$h:(10,12]\to\mathbb{R}, h(x)=\frac{r}{(x-s)^2}-4$$



a. If the graph of $h(x)$ is the reflection of the graph of $f(x)$ in the line $x = 6$,

(i) show that $n=\frac{\pi}{8}$

(ii) find the values of b, c, p, r and s.

b. On the graph above, sketch the graph of the gradient function.

c. The curve $y = 4 - \frac{4}{(x-3)^2}$ undergoes the following transformations.

- reflection in the y-axis
- translation of 6 units in the positive direction of x-axis

Show that it becomes $y = 4 - \frac{4}{x^2 - 6x + 9}$.

d. A brand logo (**Figure 1**) is developed and will be embossed on the front panel of the bag as shown on **Figure 2**.

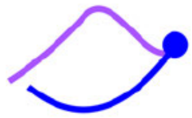


Figure 1

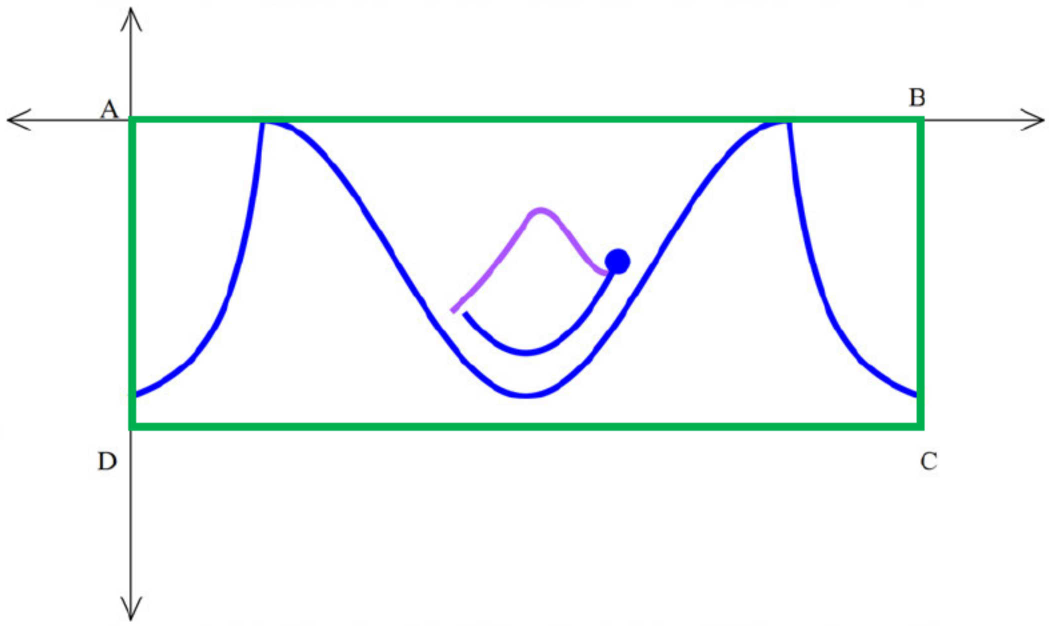


Figure 2

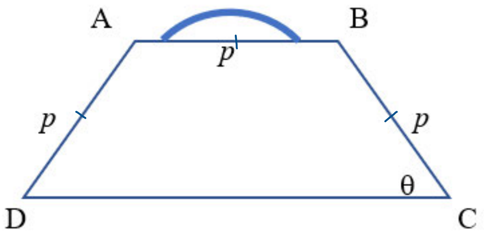
The logo in **Figure 1** consists of three curves with equations shown below. Two of the curves meet smoothly at a point.

$$y_1 = 0.6e^{0.5(x-3)} - 4, \qquad 5 \le x < 6$$
$$y_2 = a(x-6)(x-6.5)(x-7.7) + b, \qquad 6 \le x < 7.4$$
$$y_3 = 0.6(x-6)^2 - 3, \qquad 5.2 \le x \le 7.4$$

Find the values of a and b.

Question 4 (17 marks)

A different bag in the shape of a trapezium ABCD is shown below. The sides AB, BC and DA are of equal length p. The size of the acute angle BCD is θ radians.



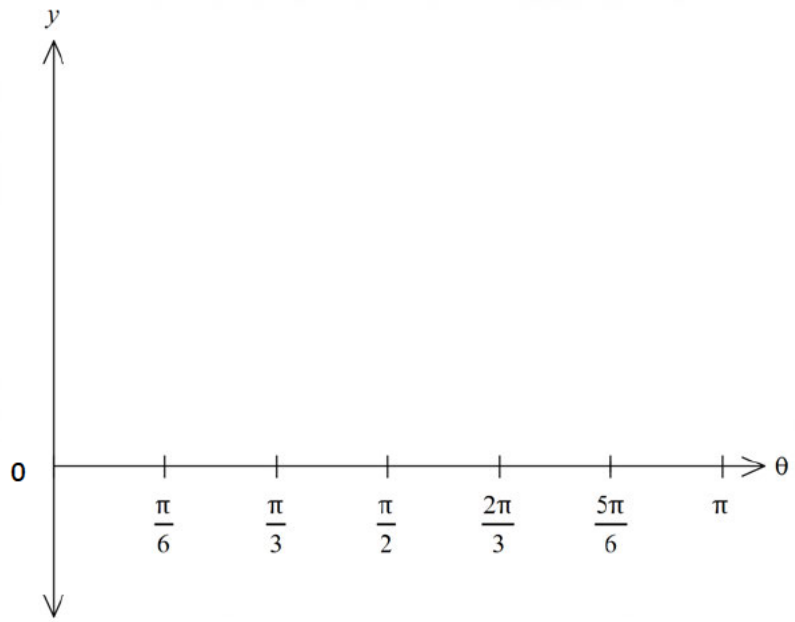
a. Show that the length of the side DC is given by $p(1+2\cos(\theta))$.

b. Show that the area, A, of the trapezium is given by $A(\theta)=p^2\sin(\theta)(1+\cos(\theta))$.

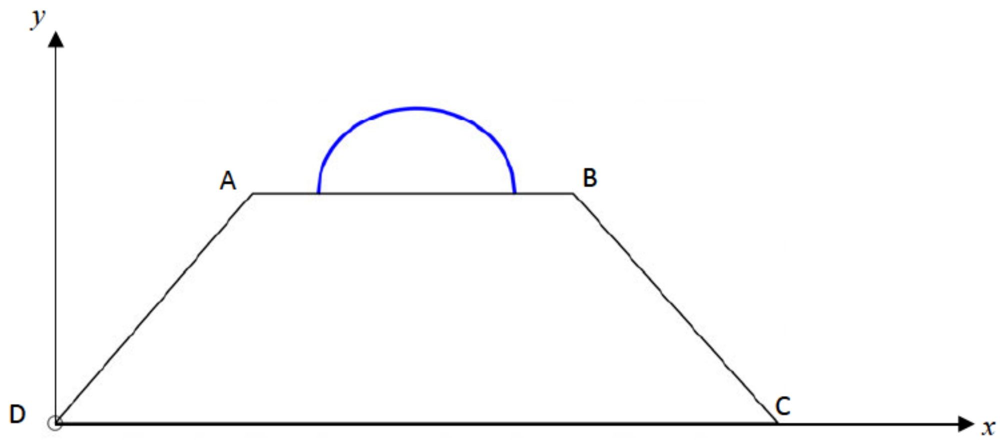
c. **(i)** Give an expression for the derivative, $\frac{dA}{d\theta}$. Show working.

(ii) Hence, find the maximum area and the angle it occurs at.

d. Sketch the graph of $y = A(\theta)$ versus θ over the domain $[0, \pi]$.
Give the coordinates of any turning points in terms of p and the endpoints.



The arched shape of the bag strap can be approximately modelled by the function,
$g : \mathbb{R} \to \mathbb{R}, g(x) = \sqrt{2.25 - (x - 5.5)^2} + 4$



e. There needs to be enough space under the strap to put an arm through.
Use 6 right endpoint rectangles to estimate the area between the strap and the top edge AB of the bag.

f. Find the actual area, to the nearest tenth squared units, between the strap and the top edge AB of the bag.

Question 5 (9 marks)

Students in the school have been asked to vote for their favourite bag design. The teacher has been monitoring the number of votes coming in per hour over a seven day cycle.

With the help of Mrs Pham, a Mathematical Methods teacher, a mathematical model was found for C (the number of votes per hour) as a function of t (time in days).

$C(t) = a(t - t\sin(\pi t)) + 10, \quad \text{where } t \in [0, 7]$

Let $t = 0$ correspond to 8am Monday and $a \in \mathbb{R}$.

a. By choosing different integer values for a, describe the effects of changing a, in the function $C(t)$.

b. Find the smallest feasible value for a correct to 3 decimal places. Justify your choice.

c. In terms of a, find the number of votes that had come in between Thursday 8am and Friday 8am.

Question 6 (11 marks)

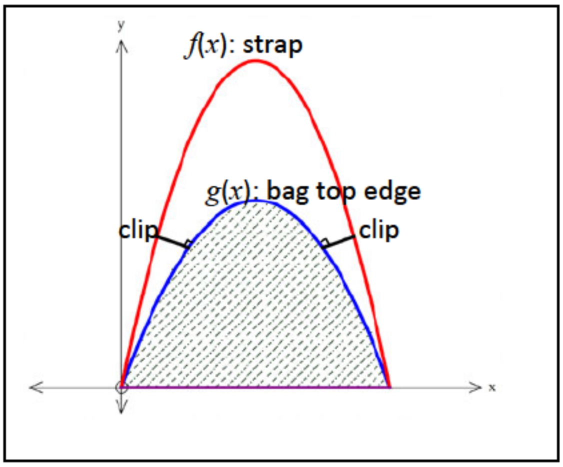
The students decide to make miniature prototypes using a 3D printer. They developed a different bag design as shown below.

Assume $f(x)$ and $g(x)$ to be quadratic functions where $f(x)$ is the function that models the strap and $g(x)$ the function that models the top edge of the bag.

At $(a, g(a))$ and $(b, g(b))$, there are metal clips that attach the straps to the bag and the clips sit at right angles with the edge of the bag, $g(x)$.

Also, $a < b$ and $a > 0$.

Assume that $f(x) = \frac{7x}{4}(4 - x)$ and $g(x) = 4x - x^2$.



a. Choose an appropriate domain for a, justifying your choice.

b. When $a = 1$,

(i) show that the equation of the straight line that models the clip is $x + 2y + 7 = 0$.

(ii) find the point where the clip intersects with $f(x)$.

(iii) At what angle, to the nearest degree, in the positive direction of the x-axis, is the clip attached to $f(x)$?

c. Find the location where the clips intersects with $f(x)$ in terms of a.

End of SAC 2

Spare page for working