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SUZANNE CORY
High School

Name : ________________ Teacher : ______________

MATHEMATICAL METHODS (CAS) UNIT 4

SAC 5 – Analysis Task

Chapters 14, 15, 16 and 17 – Probability and distributions

Reading time: 10 minutes
Writing time: 80 minutes
QUESTION AND ANSWER BOOKLET

Structure of Booklet

Section	*Number of Questions*	*Number of questions to be answered*	*Number of Marks*
1	4	4	60
			Total **60**

- Students are permitted to bring into the test room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference (which will be collected for the duration of the SAC), one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer-based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the test room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- 1 question and answer booklet as well as a formula sheet.

Instructions
- Write your **name** and **teacher** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the test room.

Question 1 (15 marks)
A highway toll booth charges cars that pass through it. The highway toll company has found that the number of passengers in each vehicle varies with accordance with the following probabilities.

Number of passenger, x	0	1	2	3	4
Pr(X=**x**)	0.22	0.16	0.24	K	0.18
	$\frac{22}{100}=\frac{11}{50}$	$\frac{16}{100}=\frac{4}{25}$	$\frac{24}{100}=\frac{6}{25}$	$\frac{2}{100}=\frac{1}{50}$	$\frac{18}{100}=\frac{9}{50}$

a. Find k. (1 mark)

$0.22 + 0.16 + 0.24 + k + 0.18 = 1$

$k = 1 - 0.8 = 0.2$ A1

All probabilities should be correct to four decimal places.

b. Find the probability that a vehicle that passes the toll booth carries (1+1 marks)

i. at least two passengers.

$\Pr(x \geq 2) = 0.24 + 0.2 + 0.18$

$= 0.62$ A1

ii. at most three passengers.

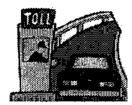
$\Pr(X \leq 3) = 0.22 + 0.16 + 0.24 + 0.2$

or $1 - 0.18 = 0.82$ A1

c. What is the probability that a vehicle carries at least two passengers given that the vehicle carries at most three passengers? (2 marks)

$\Pr(X \geq 2 \mid X \leq 3) = \dfrac{\Pr(x=2) + \Pr(x=3)}{\Pr(x \geq 3)}$ M1

$= \dfrac{0.24 + 0.2}{0.82} = \dfrac{0.44}{0.82} = 0.5366$ A1



d. Find the mean number of passengers per vehicle. (1 mark)

$E(x) = 0 \times 0.22 + 1 \times 0.16 + 2(0.24) + 3(0.2) + 4(0.18)$

$= 1.96$ A1

If ≈ 2 passenger give RE

e. Find $E(X^2)$. (1 mark)

$E(x^2) = 1(0.16) + 4(0.24) + 9(0.2) + 16(0.18)$

$= 5.8$ A1

f. Find the standard deviation, correct to two decimal places, of the number of passengers per vehicle. (2 marks)

$Var(x) = E(x^2) - [E(x)]^2$

$= 5.8 - 1.96^2$ M1

$= 1.9584$

$\sigma = \sqrt{1.9584} = 1.40$ A1

Let A be the event that a vehicle carries between one and three passengers inclusive and B be the event that a vehicle carries more two passengers.

g. Are A and B independent events? Justify. (2 marks)

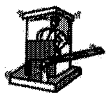
$1 \le A \le 3$ $\Pr(A) = 0.16 + 0.24 + 0.2 = 0.60$ → $\frac{16+24+20}{100} = \frac{60}{100} = \frac{3}{5}$

$B \geq 3$ $\Pr(B) = 0.2 + 0.18 = 0.38$. M1 $\frac{20+18}{100} = \frac{38}{100} = \frac{19}{50}$

$A \cap B = \Pr(3)$ $\Pr(A \cap B) = 0.2$

$0.2 \neq 0.6 \times 0.38$

$0.2 \neq 0.228$



Hence, A and B are not independent (A1)

The toll booth charges \$6 per vehicle (including driver) plus \$2 per passenger. Let *Y* be the toll fee per vehicle.

h. Write down Y in terms of X. (1 mark)

$Y = 2X + 6$ A1

i. Find the expected fee per vehicle. (2 marks)

$E(2x+6) = 2E(x) + 6$ M1

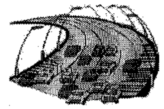
$= 2 \times 1.96 + 6$

$= \$9.92$ A1

↑ um

i. Find the probability that over the next three vehicles, the first two vehicles each carries at least two passengers but the third carries more than two passengers? (1 mark)

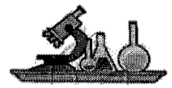
$0.62 \times 0.62 \times 0.38 = 0.1461$ A1



Question 2 (15 marks)

The incubation period, *T* days, for a highly contagious viral disease of domestic fowls, commonly known as Infectious bursal disease (IBD) is 2 to 10 days. Infected birds are depressed, have ruffled feathers, droopy appearance and may be seen pecking at the vent. The probability density function, *S*, of showing the symptom, *T* days, after close contact with a symptomatic fowl, assuming 100% morbidity, is modelled by the following function:

$$S(t) = \begin{cases} -a(t-2)(t-10) & 2 \le t \le 10 \\ 0 & otherwise \end{cases}$$



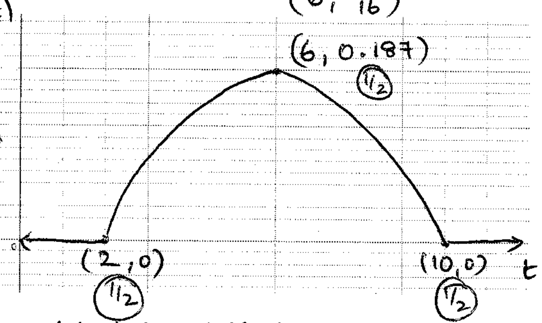
a. Show that $a = \frac{3}{256}$. (1 mark)

$\int_2^{10} -a(t-2)(t-10)\,dt = 1$ A1

b. Sketch the graph of $y = s(t)$ on the grid below. Label axes intercepts and any stationary points with their **exact** coordinates. (3 marks)

$(6, \frac{3}{16})$

accept both .187 and $\frac{3}{16}$



$\frac{1}{2}$ m → axes

$\frac{1}{2}$ m → and ←

$\frac{1}{2}$ m → Shape.

c. Find the **mode** incubation period for this disease. (1 mark)

6 days. A1

Q2(a)

• dx instead of dt (um).

d. Find the **mean** incubation period for this disease, in days. (2 marks)

$$E(t) = \int_2^{10} t \times S(t)\, dt \quad \text{or} \quad \int_2^{10} t \times \frac{3}{256}(t-2)(t-10)\, dt$$ M1

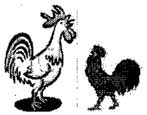
$$= 6 \text{ days}$$ A1

e. Find the **standard deviation** for the incubation period for this disease, in days, correct to two decimal places. (2 marks)

$$Var(t) = E(t^2) - [E(t)]^2$$

$$E(t^2) = \frac{196}{5}$$

$$= \int_2^{10} t^2 \times \frac{3}{256}(t-2)(t-10)\, dt - 6^2$$ M1



$$= \frac{16}{5} \approx 3.2$$

$$\sigma = \sqrt{3.2} = 1.78885$$

$$\approx 1.79$$ A1

f. Find the probability that symptoms will occur within the first 5 days of contracting the disease. (1 mark)

$$\int_2^5 S(t)\,dt = \int_2^5 -\frac{3}{256}(t-2)(t-10)\, dt = \frac{81}{256}$$ A1

$$= 0.3164$$

g. When will 50% of the fowls have shown the initial symptoms? (2 marks)

$$\int_2^m S(t)\,dt = 0.5$$ M1

$t = 6$ days. A1.

um.

Using more recent data, the probability of showing the initial symptom was found to be modelled by a cubic probability density function with a mode of 5.

$$S_2(t) = \begin{cases} -a(t-2)(t-10)(t-b) & 2 \le t \le 10 \\ 0 & otherwise \end{cases}$$

h. Find **a** and **b**. (3 marks)

$$\int_2^{10} -a(t-2)(t-10)(t-b)\, dt = 1$$ M1

$$S_2'(5) = 0.$$ M1

$$a = +\frac{3}{1664} \qquad b = \frac{25}{2}$$ A1

Question 3 (16 marks)
Tasmania is the shooting guard for his basketball team. During his matches, he has many attempts at scoring a goal.

Assume that each attempt at scoring a goal is **independent** of any other attempt. In the long term, his scoring rate has been shown to be 75% (that is, 75 out of 100 attempts to score a goal are successful).

a. What is the probability, correct to four decimal places, that his first 7 attempts at scoring a goal in a match are successful? 1 mark

$0.75^7 = 0.1335$

b. What is the probability, correct to four decimal places, that **exactly** 5 of his first 7 attempts at scoring a goal in a match are successful? 2 marks

$$\Pr(X=5) = \binom{7}{5}(0.75)^5(0.25)^2 \quad \text{(M1) Use of probability mass function}$$

$$= 0.3115 \quad \text{(A1)}$$

c. What is the probability, correct to three decimal places, that his first 4 attempts at scoring a goal are successful, given that exactly 5 of his first 7 attempts at scoring a goal in a match are successful? 2 marks

$$\Pr(\text{score first 4 attempts} \mid X=5) = \frac{\Pr(\text{score first 4 attempts}) \cap \Pr(\text{score 1 of last 3 shots})}{\Pr(X=5)} \quad \text{(M1)}$$

$$= \frac{0.75^4 \times \binom{3}{1}(0.75)(0.25)^2}{0.3115}$$

$$= 0.143 \quad \text{(A1)}$$

d. If Tasmania attempts 4 shots at goal, find the expected number of successful attempts. 1 mark

$\mu = 0.75 \times 4 = 3$

e. Find $E(X^2)$. 1 mark

$$E(X^2) = 1 \times 0.0469 + 2^2 \times 0.2109 + 3^2 \times 0.4219 + 4^2 \times 0.3164$$

$$= 9.75$$

Note: solutions may also use $\left[E(X)^2\right] = E(X^2) - Var(X)$

f. **Hence,** determine the standard deviation of the number of successful attempts, correct to three decimal places. 2 marks

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 9.75 - 3^2$$

$$= 0.75 \quad \text{(M1) Appropriate method shown}$$

$$SD(X) = \sqrt{0.75}$$

$$= 0.866 \quad \text{(A1)}$$

g. Find $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$. Give your answer correct to 2 decimal places. 2 marks

$2\sigma = 1.73$

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) = \Pr(1.27 \le X \le 4.73)$ (M1)

$= \Pr(2 \le X \le 4)$

$= 0.95$ (see comment) (A1)

Note that although $\Pr(2 \le X \le 4) = 0.95$, the calculated answer will not always be 0.95. As an example, to obtain consequential marks for $\Pr(0 \le X \le 6)$, the consequential answer must be 1 and not 0.95.

Assume instead that the success of an attempt to score a goal depends only on the success or otherwise of his previous attempt at scoring a goal.

If an attempt at scoring a goal in a match is successful, then the probability that his next attempt at scoring a goal in the match is successful is 0.87. However, if an attempt at scoring a goal in a match is unsuccessful, then the probability that his next attempt at scoring a goal in the match is successful is 0.69.

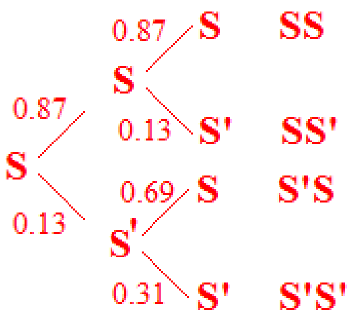
His first attempt at scoring a goal in a match is successful.

h. What is the probability, correct to four decimal places, that:

i. his next 7 attempts at scoring a goal in the match will be successful? 1 mark

$0.87^7 = 0.3773$

ii. **exactly** 1 of his next 2 attempts at scoring a goal in the match will be successful? 2 marks



$\Pr(X = 1) = 0.87 \times 0.13 + 0.13 \times 0.69$ (M1)

$= 0.2028$ (A1)

Before leaving a training session, Tasmania decides that he will not leave until he makes a shot from the half-court line. Assume that each attempt at making a shot from the half-court line is independent of any other attempt, and that the probability that he will make a half-court shot is 0.04.

i. What is the minimum number of shots that Tasmania needs to attempt so that the probability of making a shot is greater than 0.95? 2 marks

$\Pr(X > 0) > 0.95$

$1 - \Pr(X = 0) > 0.95$

$-\Pr(X = 0) > -0.05$

$\Pr(X = 0) < 0.05$

$\binom{n}{0}(0.96)^n < 0.05$ (M1)

$n > 73.39$

$\therefore$ 74 shots (A1)

OR

There is a mistake in the question as it should ask ".. so that the probability of making at least one shot is greater than 0.95?". Due to this the following working will also be accepted:

$\Pr(X = 1) > 0.95$ (M1)

$\binom{n}{1}(0.04)(0.96)^{n-1} > 0.95$ (M1)

Note, however, that there is no solution to the above equation. Hence, any further working that explicitly states a solution will have a mark deducted for further engagement.

Question 4 (14 marks)

For the tomatoes from **Acme** planation

a. use the information provided to complete the transition matrix 1 mark

$$\begin{array}{c} \\ A \\ R \end{array}\begin{array}{c} \begin{array}{cc} A & R \end{array} \\ \begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix} \end{array}$$

b. if the first tomato inspected is accepted, find the probability that the third tomato is rejected 2 marks

$$S_2 = \begin{bmatrix} 0.8 & 0.7 \\ 0.2 & 0.3 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ (M1) Find S_2 *OR* other appropriate method shown

$$S_2 = \begin{bmatrix} 0.78 \\ 0.22 \end{bmatrix}$$

$\therefore$ 0.22 (A1) Answer must be stated independent of matrix

c. if the first tomato inspected is rejected, find the probability that the next two tomatoes are rejected 1 mark

$0.3^2 = 0.09$

d. find the steady state probability that any one of the tomatoes from Acme plantation are accepted 2 marks

$$\Pr(X_n = 0) = \frac{0.7}{0.2 + 0.7}$$ (M1) Substitution of large numbers until matrix repeats also accepted

$$= \frac{7}{9}$$ (A1) Answer must be exact

Tomatoes from a different plantation, **Beep** planation, are inspected. It is found that if the first tomato inspected is accepted, then the probability that the third tomato inspected is accepted is 0.67.

e. Show that the value of p from **Beep** planation is 0.7. 3 marks

$$\begin{bmatrix} p & p-0.1 \\ 1-p & 1.1-p \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$$

(M1) Construct transition matrix
(M1) Construct matrix equation

$$\begin{bmatrix} 1.1p-0.1 \\ -1.1p+1.1 \end{bmatrix} = \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$$

$p = 0.7$ QED

(A1)"Show that" answer mark awarded only if method leads to correct value for p

Let C be the probability that a tomato has unacceptable scarring and let D be the probability that the tomato is of an unacceptable size.

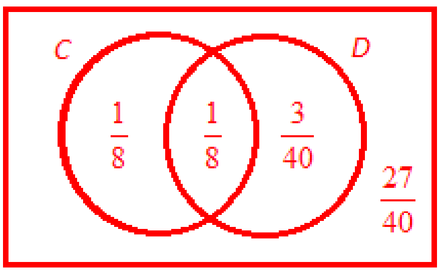
A new shipment of tomatoes from **Coyote** plantation have arrived at Victoria's company, such that $\Pr(C) = \frac{1}{4}$ and $\Pr(D) = \frac{1}{5}$

f. Create a probability table (Karnaugh map) **labelling** all rows and columns, *OR* create a Venn diagram for when $\Pr(C \cap D) = \frac{1}{8}$ 2 marks

	C	C'	
D	$\frac{1}{8}$	$\frac{3}{40}$	$\frac{1}{5}$
D'	$\frac{1}{8}$	$\frac{27}{40}$	$\frac{4}{5}$
	$\frac{1}{4}$	$\frac{3}{4}$	1

Rows and columns labelled	(M½)
"Outer" probabilities shown	(M½)
All 4 "inner" probabilities correct OR 2 or 3 "inner" probabilities correct	(M1) OR (M½)
Total is sum of marks rounded down	(M2)

OR



Rectangle shown and circles labelled	(M½)
"Outer" probability correct	(M½)
All 3 "inner" probabilities correct OR 2 "inner" probabilities correct	(M1) OR (M½)
Total is sum of marks rounded down	(M2)

g. Find $\Pr(C')$ 1 mark

$\frac{3}{4}$

h. Find $\Pr(C \cap D')$ 1 mark

$\frac{1}{8}$

i. Calculate $\Pr(C \cap D')$ when C and D are mutually exclusive events. 1 mark

$\frac{1}{4}$