MARKING SCHEME 2015 VCE Mathematical Methods Unit 4
SAC 5 Analysis Task Probability Chapters 14 to 18

- Deduct a maximum of one mark overall for rounding error
- Deduct a maximum of one mark overall for units of measurement error
- Deduct a maximum of one mark overall for missing *dx*
- All other marks are deducted per question

Question 1 (19 marks)

a. Show that $k = 0.23$. 1 mark

$0.3 + 0.4 + k + 0.05 + 0.02 = 1$ (M1)

$k = 0.23$

b. For the number of spots on the modified Dalmation dog:

i. find the mean 2 marks

$E(X) = 2 \times 0.3 + 3 \times 0.4 + 4 \times 0.23 + 5 \times 0.05 + 6 \times 0.02$ (M1)

$E(X) = 3.09$ (A1)

ii. find the median 1 mark

3

iii. show that the variance is 0.9019 1 mark

$Var(X) = E(X^2) - [E(X)]^2$

$Var(X) = 2^2 \times 0.3 + 3^2 \times 0.4 + 4^2 \times 0.23 + 5^2 \times 0.05 + 6^2 \times 0.02 - 3.09^2$ (M1)

$Var(X) = 0.9019$

iv. find $Var(3X - 2)$ 2 marks

$= 3^2(0.9019)$ (A1)

$= 8.1171$ (A1)

v. find $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$ **and** interpret your result. 3 marks

$\Pr(\mu - 2\sigma) = 1.191$

$\Pr(\mu + 2\sigma) = 4.989$

$\Pr(1.191 \le X \le 4.989)$

$= \Pr(2 \le X \le 4)$ (M1)

$= 0.93$ (M1)

$\therefore$ 93% of the data lies within 2 standard deviations of the mean OR $\therefore$ 93% of dogs will have between 2 and 4 spots inclusive (A1)

The large population of genetically modified dogs are kept in a secure open range enclosure.

c. If two dogs are randomly chosen from the enclosure, find the probability that

i. each dog has three spots, correct to 2 decimal places — 1 mark

$0.4^2 = 0.16$

ii. the total number of spots on both dogs is equal to seven — 2 marks

$$\begin{aligned}\Pr(\text{total } 7) &= \Pr(2 \text{ and } 5) + \Pr(3 \text{ and } 4) + \Pr(4 \text{ and } 3) + \Pr(5 \text{ and } 2) \\ &= 2(0.3 \times 0.05 + 0.4 \times 0.23) && \text{(M1)} \\ &= 0.214 && \text{(A1)}\end{aligned}$$

iii. one of the dogs has two spots, given that the total number of spots on both dogs is equal to seven, correct to 4 decimal places. — 2 marks

$\Pr(\text{one dog 2 spots} | \text{total spots is 7})$

$$\begin{aligned}&= \frac{\Pr(\text{one dog 2 spots and total spots is 7})}{\Pr(\text{total spots is 7})} \\ &= \frac{\Pr(2 \text{ and } 5) + \Pr(5 \text{ and } 2)}{\Pr(\text{total spots is 7})} \\ &= \frac{2(0.3 \times 0.05)}{0.214} && \text{(M1)} \\ &= 0.1402 && \text{(A1)}\end{aligned}$$

d. Eight modified Dalmation dogs are randomly chosen from the enclosure.

i. Find, correct to 4 decimal places, the probability that a dog chosen at random from the eight dogs has two spots. — 1 mark

0.1977

ii. Find, correct to 4 decimal places, the probability that at least four of the dogs have two spots. — 1 mark

0.1941

e. A random group of genetically modified Dalmation dogs manage to escape from the enclosure. Find the smallest number of dogs in the group if the probability that at least two of the dogs in the group, having less than four spots, is greater than 0.99. — 3 marks

$X \sim Bi(n, 0.7)$

$\Pr(X \geq 2) > 0.99$ } (M1) Either line

$\Pr(X \leq 1) \leq 0.01$

$\binom{n}{1}(0.7)(0.3)^{n-1} + \binom{n}{0}(0.3)^n \leq 0.01$ (M1) Also accept equals sign

$n \geq 6.085$

$\therefore$ 7 dogs (A1)

Question 2 (10 marks)
It is found that when an **unmarried** couple go and see a movie together, there is a 58% chance that they will see a romantic comedy (RomCom) if the previous movie they saw together was a RomCom, and a 46% chance that they will see an action movie if the previous movie they saw together was an action movie.

a. Create a transition matrix for the probabilities of an unmarried couple seeing either a RomCom or an action movie together, given the previous movie seen together. 1 mark

$$T = \begin{bmatrix} 0.58 & 0.54 \\ 0.42 & 0.46 \end{bmatrix} \quad \text{or} \quad T = \begin{bmatrix} 0.46 & 0.42 \\ 0.54 & 0.58 \end{bmatrix}$$

It is found that when a **married** couple go and see a movie together there is a 92% chance that they will see a RomCom if the previous movie they saw together was an action movie, and a 5% chance that they will see an action movie if the previous movie they saw together was a RomCom. A transition matrix for the probabilities of a married couple seeing either a RomCom or an action movie together given the previous movie seen together is

$$T = \begin{bmatrix} 0.95 & 0.92 \\ 0.05 & 0.08 \end{bmatrix}$$

b. John and Susan are a **married** couple who decide to go and see a movie together. It is equally likely that they will see either a RomCom or an action movie the first time that they go to the movies together. Find the probability that the second time they go to the movies together, John and Susan will see a RomCom 2 marks

$$\text{Pr(RomCom second time)} = T \times \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$\text{Pr(RomCom second time)} = \begin{bmatrix} 0.935 \\ 0.065 \end{bmatrix}$$ (M1) Matrix

$\therefore\ 0.935$ (A1) Answer stated outside of matrix

c. As it happens, the first movie that John and Susan saw in 2015 was an action movie. Find:

i. the probability that the next three movies John and Susan see together are RomComs. 1 mark

$0.92 \times 0.95^2 = 0.8303$

ii. the probability that the fifth movie they see together in 2015 will be a RomCom, correct to 4 decimal places. 2 marks

$$\text{Pr(fifth movie is RomCom)} = T^4 \times \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{Pr(fifth movie is RomCom)} = \begin{bmatrix} 0.9485 \\ 0.0515 \end{bmatrix}$$ (M1) Matrix

$\therefore\ 0.9485$ (A1) Answer stated outside of matrix

iii. the steady state probability that they will see an action movie together, giving your answer as an exact value — 2 marks

$$\text{Steady state} = \frac{0.05}{0.05 + 0.92}$$ (M1) Also accept use of S^m and S^n, where m and n are large numbers with use of ApproxFraction

$$\text{Steady state} = \frac{5}{97}$$ (A1)

On a particular day it was found that 84 married couples saw an action movie together and 212 married couples saw a RomCom together.

d. Find, correct to the nearest whole number, the number of married couples expected to see each type of movie in the long term. — 2 marks

$$\text{RomCom} = \frac{92}{97} \times (84 + 212)$$

$\text{RomCom} = 281$ (A1)

$\text{Action} = 84 + 212 - 281$

$\text{Action} = 15$ (A1)

Question 3 (15 marks)
A social researcher carried out a lifestyle survey on a group of people. A mathematical model was then developed from the result. The time, in hours, spent each week on sport and recreation by people surveyed is a continuous random variable, *T*, with probability density function

$$f(t) = \begin{cases} kt^2 e^{-\frac{t}{2}} & \text{for } 0 \le t \le 8 \\ -\dfrac{rt}{13e^4} + \dfrac{116}{13e^4} & \text{for } 8 < t \le 14.5 \\ 0 & \text{elsewhere} \end{cases}$$

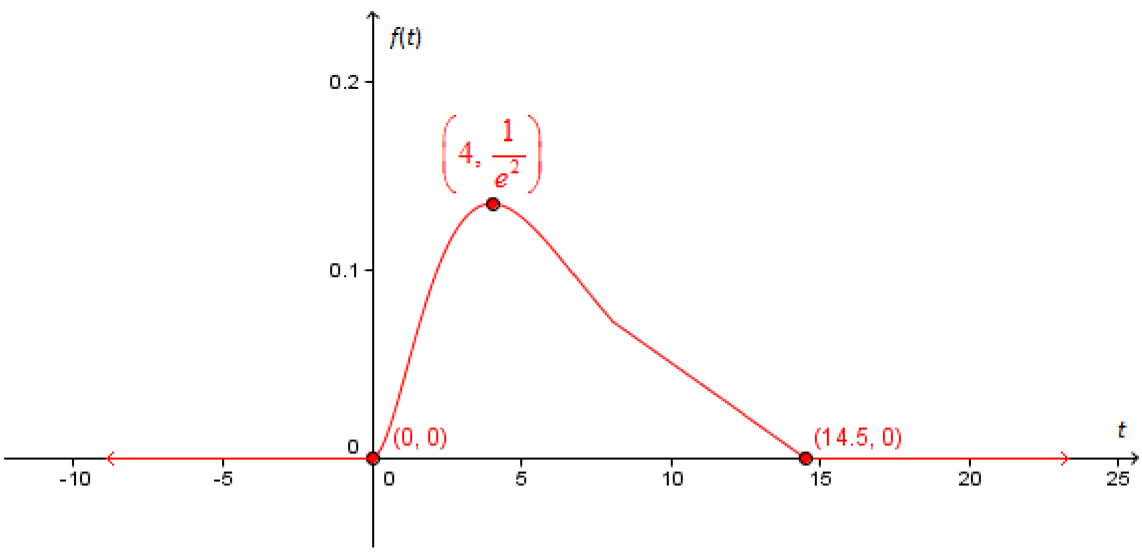
a. Given that $f(t)$ is continuous at $t = 8$, show that $k = \dfrac{1}{16}$ and $r = 8$. 2 marks

$\int_0^{14.5} f(t)dt = 1$ or $\int_{-\infty}^{\infty} f(t)dt = 1$ (M1)

$k(8)^2 e^{-\frac{8}{2}} = -\dfrac{r}{13e^4}(8) + \dfrac{116}{13e^4}$ (M1)

$\therefore\ k = \dfrac{1}{16}$ and $r = 8$

b. Sketch the probability density function $f(t)$ on the axes below. Label any intercept(s) and turning point(s) using coordinates. 3 marks



Intercept (0, 0)	(M½)
Intercept (14.5, 0)	(M½)
Turning point $\left(4, \dfrac{1}{e^2}\right)$	(M½)
0 elsewhere shown	(M½)
Correct shape	(M1)
Total is sum of marks rounded down	(M3)

c. Find the median time spent on sport and recreation by this group, in hours, correct to 2 decimal places. 2 marks

$\int_0^m f(t)dt = 0.5$ (M1)

$m = 5.35$ hours (A1)

d. Find the modal time spent on sport and recreation by this group. 1 mark

4 hours

e. A person is chosen at random from the group. Find the probability that the person:

i. spends at least 5 hours on sport and recreation, correct to four decimal places 1 mark

$\int_5^{14.5} f(t)dt = 0.5438$

ii. spends less than 8 hours on sport and recreation, correct to four decimal places 1 mark

$\int_0^8 f(t)dt = 0.7619$

iii. spent less than 8 hours on sport and recreation, given that they spent at least 5 hours on sport and recreation, correct to 2 decimal places. 2 marks

$$\Pr(X < 8 \mid X > 5) = \frac{\Pr(5 < X < 8)}{\Pr(X > 5)}$$

$$\Pr(X < 8 \mid X > 5) = \frac{\int_5^8 f(t)dt}{0.5438} \quad \text{(M1)}$$

$$\Pr(X < 8 \mid X > 5) = 0.56 \quad \text{(A1)}$$

The social researcher decides to interview 6 people from the group to see whether or not the people spend at least 5 hours on sport and recreation each week.

f. Find the probability, correct to two decimal places, that three people spend at least 5 hours on sport and recreation each week. 2 marks

$$\Pr(X = 3) = \binom{6}{3}(0.5438)^3(0.4562)^3 \quad \text{or} \quad X \sim Bi(6, 0.5438) \quad \text{(M1)}$$

$$\Pr(X = 3) = 0.31 \quad \text{(A1)}$$

Question 4 (16 marks)
Records kept by Massivechusettes Institute of Technology suggest that the distribution of the total fees paid by graduating students is normal, with mean \$60000 and standard deviation \$5000.

a. What proportion of the university's graduates pay:

i. less than \$55000, correct to 4 decimal places? 1 mark
0.1587

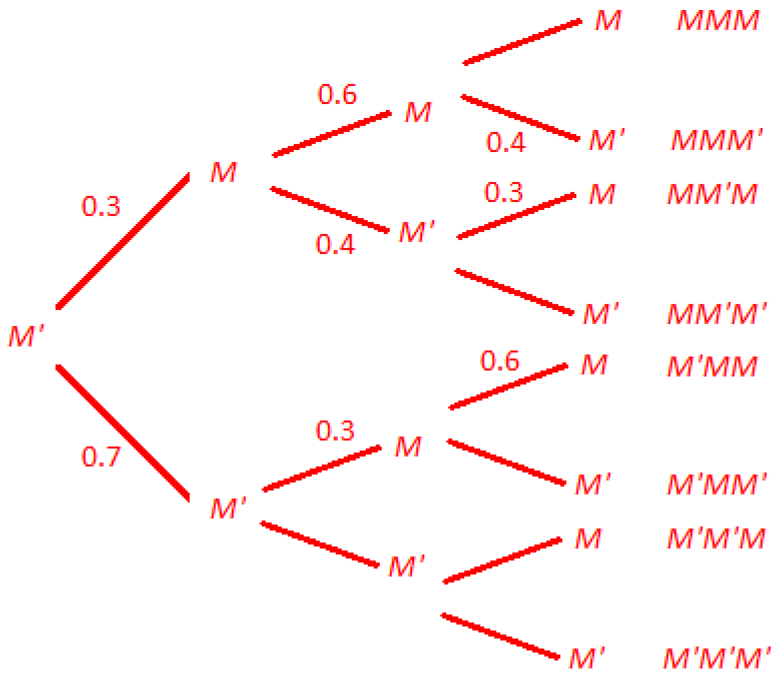
ii. more than \$50000 but less than \$74000, correct to 4 decimal places? 1 mark
0.9747

b. The university's advertising manager wishes to claim that '90% of our graduating students pay less than *c* in total fees. What should *c* be, correct to the nearest dollar? 1 mark
\$66408

Jeremy is practising shots from the free-throw line on the university basketball court. If Jeremy makes a shot, the probability he will make the next shot is 0.6. If he misses a shot, the probability he will make the next shot is 0.3. Jeremy misses his first shot.

c. What is the probability Jeremy will make his next three shots? 1 mark
$0.3 \times 0.6^2 = 0.108$

d. What is the probability Jeremy will make exactly two of his next three shots? 3 marks



$\Pr(\text{make two shots}) = \Pr(MMM') + \Pr(MM'M) + \Pr(M'MM)$

$\Pr(\text{make two shots}) = 0.3 \times 0.6 \times 0.4 + 0.3 \times 0.4 \times 0.3 + 0.7 \times 0.3 \times 0.6$ (M2)

$\Pr(\text{make two shots}) = 0.234$ (A1)

Two marks for appropriate method, one mark for answer.
Note that tree diagram is not essential, but can contribute to appropriate method

e. Given that Jeremy made at least one of the next three shots, what is the probability he made all three shots, correct to 2 decimal places? 2 marks

Let X = number of shots made

$$\Pr(X = 3 \mid X \geq 1) = \frac{\Pr(X = 3)}{\Pr(X \geq 1)}$$

$$\Pr(X = 3 \mid X \geq 1) = \frac{0.108}{1 - 0.7^3} \qquad \text{(M1)}$$

$$\Pr(X = 3 \mid X \geq 1) = 0.16 \qquad \text{(A1)}$$

The ATAR scores of applicants for a particular university course are normally distributed. The university decided to accept the top 15% of applicants. Jeremy was accepted into the university with an ATAR score of 86. Only 10% of applicants had a better score than him. His friend Jody was offered a scholarship. Her ATAR score was 95 and only 1% of applicants had a better score than hers.

f. What was the mean and standard deviation of the ATAR scores of applicants? Give your answers correct to two decimal places. 4 marks

$$\Pr(X < 0.9) = 86$$

$$\Pr(Z < 0.9) = 1.28155$$

$$\Pr(X < 0.99) = 95$$

$$\Pr(Z < 0.99) = 2.32625$$

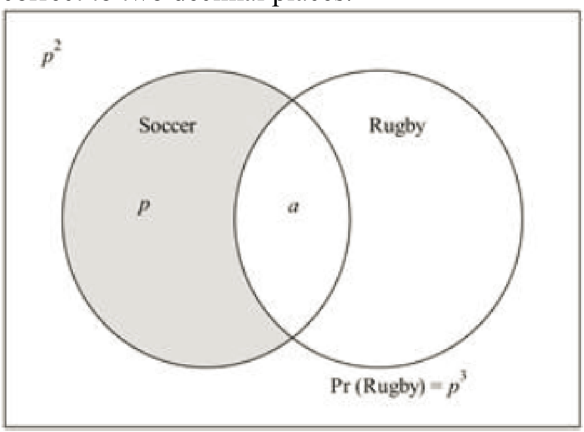
$$1.28155 = \frac{86 - \mu}{\sigma} \qquad \text{(M1)}$$

$$2.32625 = \frac{95 - \mu}{\sigma} \qquad \text{(M1)}$$

$$\mu = 74.96, \ \sigma = 8.61 \qquad \text{(A1) + (A1)}$$

Jody wants to join one of the university's sports clubs. The probability she will be selected for soccer and not rugby is p. The probability she will selected for rugby is p^3 and the probability she will be selected for neither sport is p^2. Being selected for soccer is independent of being selected for rugby.

g. What is the probability Jody will be selected for both sports? Give your answer correct to two decimal places. 3 marks



$$p + p^3 + p^2 = 1 \qquad \text{(M1)}$$

$$p = 0.54 \qquad \text{(M1)}$$

$$\Pr(S \cap R) = a = \Pr(S) \times \Pr(R)$$

$$a = (a + p)p^3$$

$$a = 0.10 \qquad \text{(A1)}$$

END OF QUESTION AND ANSWER BOOKLET