SUZANNE CORY
High School

Name: MARKING SCHEME V3 **Teacher Name:**

2017 MATHEMATICAL METHODS (CAS) UNIT 4

SAC 3 – Probability Analysis Task

TEST CONDITIONS PART 3

Reading time: 10 minutes
Writing time: 70 minutes

QUESTION AND ANSWER BOOKLET

Structure of Book

	Section	*Number of questions*	*Number of marks*
1	Extended Response Questions	4	40
			Total 40

- Students are permitted to bring into the test room: pens, pencils, highlighters, erasers, rulers, sharpeners, one **bound** reference, a CAS and/or graphic and/or scientific calculator.
- Students are NOT permitted to bring into the test room: blank sheets of paper, white out liquid/tape.

Materials supplied

- Question and answer book.
- Working space is provided throughout the book.

Instructions

- Write your **name** and **teacher's name** in the space provided above on this page.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the test room.

Instructions
Answer all questions in the spaces provided. In all questions where a numerical answer is required, express as an exact answer unless stated otherwise. In questions worth more than 1 mark, appropriate working must be shown. Unless otherwise indicated, all diagrams are not drawn to scale.

Matt Skeek is a student and aspiring video game designer! His school project is to design a game which he will then present to a potential video game producer.

Question 1 (11 marks)

Matt lives close by school. He has the option of being taken to school in his father's car, or has the option of walking to school.

The probability density function for the time it takes for him to walk to school is given by

$$w(x) = \begin{cases} \dfrac{-r(x-15)(x-6)^3}{4x^2}, & 6 < x < 14 \\ 0, & otherwise \end{cases}$$

Where x is the time-taken in minutes.

a. Show that the value of r is approximately 0.1961 (to four decimal places). 1 mark

$\int_6^{14} w(x)dx = 1$ M1

$r = 0.1961$

-1 unit mark for no 'dx'

Use this value of r to answer the following questions.

b. Calculate the probability that Matt's walk to school took between 8 and 10 minutes. Express your answer to three decimal places. 1 mark

$\int_8^{10} w(x)dx = 0.197$ A1

c. Calculate the probability that Matt's walk to school took more than 8 minutes, given that he took less than 10minutes. Express your answer to three decimal places. 2 marks

$\Pr(X > 8 | X < 10) = \frac{\Pr(8<X<10)}{\Pr(X<10)}$ M1

Accept $= \frac{0.197}{0.222}$ instead of notation

$= 0.886$ A1

d. Using the answer from (a), calculate the variance of Matt's time-taken to walk to school. Express your answer to three decimal places. 2 marks

$\int_6^{14} x^2 w(x)dx - \left(\int_6^{14} x\, w(x)dx\right)^2$ M1

$= 2.565$ A1

Also accept 2.546, accept notion $E(X^2) - [E(X)]^2$ instead of integrals

e. Using the answer from (a), calculate the median time of Matt's time-taken to walk to school. Express your answer to three decimal places. 1 mark

$\int_6^{m} w(x)dx = 0.5$

$m = 11.458\ minutes$ A1

Not accepting 17 ....

The time it takes for Matt to get to school in his father's car follows a continuous random variable with the density function given as follows:

$$c(x) = \begin{cases} -\dfrac{(x-4)^2}{2} + k, & a < x < b \\ 0, & otherwise \end{cases}$$

Where a, b & k are positive constants such that $a < b$.

It is known that $x = a$ and $x = b$ when $c(x) = 0$.

f. By writing the values of a and b in terms of k or otherwise, show that the value of $k = 0.655$ when corrected to three decimal places. 2 marks

$c(x) = 0$

$a = 4 - \sqrt{2k}, b = 4 + \sqrt{2k}$ M1

$\int_{4-\sqrt{2k}}^{4+\sqrt{2k}} c(x)\, dx = 1$ A1

$k = 0.655$

g. Use the value of k from (g) to find the mean time it takes for Matt to get to school in his father's car. Express your answer in minutes to one decimal place. 2 marks

$\int_{4-\sqrt{2(0.655)}}^{4+\sqrt{2(0.655)}} x\, c(x)dx$ M1

$= 4.0\ minutes$ A1

Question 2 (11 marks)

Correct all answers in this question to four decimal places, unless stated otherwise.

In the video game that Matt is designing, the character can throw a punch at an enemy creature. Whenever the character tries to throw a punch, the program will generate a random number – called the "roll-score" – normally distributed with a mean of 500 and a standard deviation 45. The character will successfully 'hit' the enemy creature if the roll-score is between 465 and 600.

a. What is the probability on any given punch, the character will score a 'hit'? 1 mark

$X \sim N(500, 45^2)$

$\Pr(465 < X < 600) = 0.7685$ A1

b. What is the probability that on any given punch, the roll-score will be at least 560? 1 mark

$\Pr(X > 560) = 0.0912$ A1

To make the game more sophisticated, the character in the game can score a 'critical', which deals double damage. If the character successfully hits an enemy creature, **and** the roll-score is at least 560, this is known as a 'critical hit'.

c. What is the probability of scoring a critical hit? 1 mark

$\Pr(560 < X < 600) = 0.0781$ A1

d. Given that the character has hit the enemy creature, what is the probability that it is a critical hit? 2 marks

$\Pr(critical\ strike|hit) = \frac{\Pr(560<X<600)}{\Pr(465<X<600)}$ M1

$= 0.1016$ A1

e. What is the probability, that out of 8 punches that the character throws, it will score at least 2 critical hits? 1 mark

$Y \sim Bi(8, 0.0781)$

$\Pr(Y \geq 2) = 0.1246$ A1 (give consequential)

An enemy creature can be killed if the character lands two non-critical hits and a critical hit, or just two critical hits.

f. What is the probability that the character can kill the enemy creature within 3 punches? 2 marks

$\Pr(HHC) + \Pr(CC'C \cup C'CC) + \Pr(CC)$

$= \frac{3!}{2!}(0.6904)^2(0.0781) + 2(0.0781)^2(1 - 0.0781) + (0.0781)^2$ M1

$= 0.1290$ A1

Matt wants to introduce higher difficulty settings for the game.

On a 'medium difficulty' setting, he can change the standard deviation of the roll-score of scoring a successful hit to make it less likely to land a hit, while keeping the mean the same.

g. Would he increase or decrease the standard deviation? Explain your answer.

1 mark

Increase. By increasing the standard deviation, the probability distribution becomes less concentrated about the mean. Thus, the interval over which to obtain a 'hit would decrease with a lower standard deviation. A1

No marks without correct justification.

Matt decides that on a 'hard difficulty' setting, he would change both the mean and the standard deviation of the roll score.

The probability that the roll score is less than 465 is 0.3085, whereas the probability that the roll score is more than 560 is 0.0038.

h. Calculate the mean and standard deviation on 'hard difficulty' setting. Express to the nearest whole number.

2 marks

$\Pr\left(Z < \frac{465-\mu}{\sigma}\right) = 0.3085$

$-0.500 = \frac{465-\mu}{\sigma}$

$\Pr\left(Z > \frac{560-\mu}{\sigma}\right) = 0.0038$

$2.669 = \frac{560-\mu}{\sigma}$ M1

$\mu = 480, \sigma = 30$ A1

Require both equations for method mark

PLEASE TURN OVER

Question 3 (8 marks)

Matt's homegroup is very supportive of his ambition to become a video game designer! In order to help to raise funds for his project, his homegroup have organised a fund-raising raffle.

Matt's homegroup consist of 9 girls and 16 boys. A group of three students were to be selected at random in order to run the raffle on a particular night. p represents the proportion of girls in the class.

a. What are the possible values of the sample proportion $\hat{p}$? 1 mark

$\hat{p} = 0, \frac{1}{3}, \frac{2}{3}, 1$ A1

b. Construct the probability distribution table for $\hat{P}$. 1 mark

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{28}{115}$	$\frac{54}{115}$	$\frac{144}{575}$	$\frac{21}{575}$

-1 exact, if not in exact form.
Give consequential based on table

c. Find $\Pr(\hat{P} > 0.3)$ 1 mark

$\Pr(\hat{P} > 0.3) = 1 - \Pr(\hat{P} = 0)$

$= \frac{87}{115}$ A1

d. Find $\Pr(\hat{P} > 0.3 \mid \hat{P} < 0.5)$ 2 marks

$\Pr(\hat{P} > 0.3 | \hat{P} < 0.5) = \frac{\Pr(0.3<\hat{P}<0.5)}{\Pr(\hat{P}<0.5)}$ M1

$= \frac{\Pr\left(\hat{P}=\frac{1}{3}\right)}{\Pr\left(\hat{P}=\frac{1}{3}\right)+\Pr(\hat{P}=0)}$

$= \frac{27}{41}$ A1

Matt's homegroup is 95% sure that 35% to 45% of students will participate the fund-raising raffle.

e. What sample size is needed for this level of confidence? 2 marks

$z = 1.96, \hat{p} = 0.4$

$0.35 = 0.4 - 1.96\sqrt{\frac{0.4\times 0.6}{n}}$ M1

$n = 369$ A1

f. Explain in words what would happen to the margin of error if the sample size was doubled? 1 mark

The margin of error will decrease, by a factor of $\sqrt{2}$. A1
Student need to provide a numerical answer.

Question 4 (10 marks)

In the fund-raising raffle that the homegroup organised, there is a chance to win a big prize and a small prize. Each ticket the homegroup sells earns them $10. There is a 15% chance the ticket holder will win a small prize of $40, and a 1% chance to win a big prize of $200.

Let X be the random variable for the amount of **profit** the **homegroup** stands to make from each ticket.
The probability distribution table is given by:

x	10	-30	-190
$\Pr(X = x)$	0.84	0.15	0.01

a. i) Find $E(X)$. 1 mark

$E(X) = 10 \times 0.84 + (-30) \times 0.15 + (-190) \times 0.01$
$E(X) = 2$ A1

ii) Interpret your result in the context of this question. 1 mark

The homegroup makes an average of $2 per ticket sold. A1

b. Calculate the maximum value of the big prize such that there will be no expected funds raised overall. 1 mark

$0 = 10 \times 0.84 + (-30) \times 0.15 + (x) \times 0.01$
$x = -390$

Prize = $400 A1

c. Calculate the minimum number of tickets that one needs to buy in order to ensure the probability of winning two small prizes is greater than 0.25. 2 marks

$X \sim Bi(n, 0.15)$
$\Pr(X = 2) = \frac{n!}{2!(n-2)!}(0.15)^2(0.85)^{n-2} > 0.25$ M1

$n = 9$ tickets A1

Later that month, the same homegroup ran their fund-raising raffle again. However, they found that in the previous raffle, they raised too little funds. Owing to this, they decided to change the probability of winning the small and big prizes. The probability of winning a small prize was a and the probability of winning a big prize was b. The probability of not winning any of prize was 0.9.

The payout for each outcome remained unchanged.

d. Calculate the probabilities a and b if the homegroup expected to raise \$2500 from 1000 raffle tickets. 3 marks

$E(Y) = \frac{2500}{1000} = 2.5$ M1 for correctly calculating expectation

$0.9 + a + b = 1 \ldots (1)$

$0.9 \times 10 + a \times (-30) + b \times (-190) = 2.5 \ldots (2)$ M1 for forming two sim equations

$a = \frac{5}{64}, b = \frac{7}{320}$ A1 (exact form)

Or $a = 0.078125, b = 0.021875$

Let Y be the profit made from each ticket in this scenario.

e. Using linear properties of expected value, calculate $E(2Y + 3)$ for this scenario. 2 marks

$E(2Y + 3) = 2E(Y) + 3$ M1 must show linear properties

$2(2.5) + 3 = 8$ A1

END OF QUESTION AND ANSWER BOOKLET