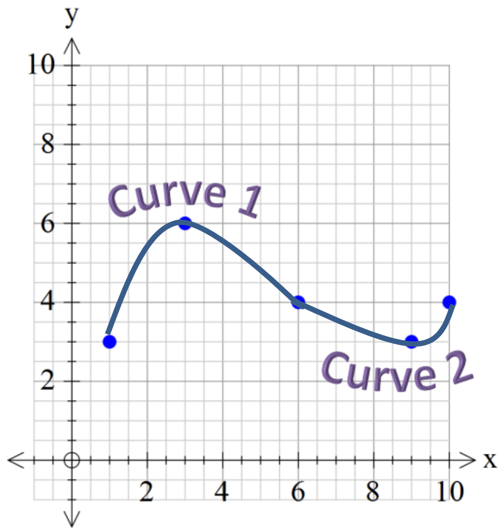
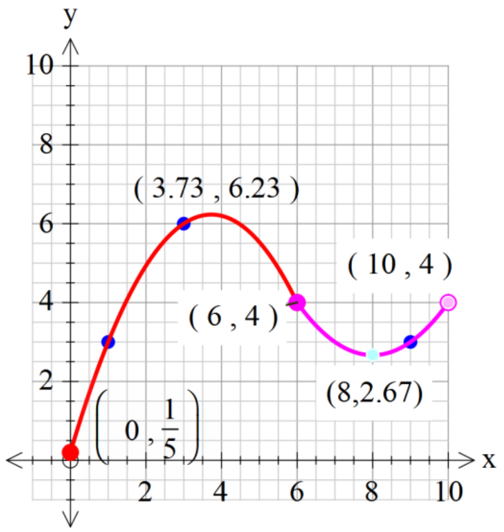
2018 SAC PREP 2 - Solutions

John decides that a more complex situation could be modelled by a piecewise function. He decides to investigate a number of possibilities.



a) Determine an equation of a parabola that passes through the first three points. Also find the equation of the parabola that passes through the last three points. On the axes below accurately sketch the resultant piecewise function. Include a statement of the piecewise function.



$3 = a + b + c \ldots\ldots(1)$

$6 = 9a + 3b + c \ldots(2)$

$4 = 36a + 6b + c \ldots(3)$

$a = \frac{-13}{30}$

$b = \frac{97}{30}$

$c = \frac{1}{5}$

$y = \frac{-13}{30}x^2 + \frac{97}{30} + \frac{1}{5}$

$4 = 100a + 10b + c \ldots\ldots(1)$

$3 = 81a + 9b + c \ldots(2)$

$4 = 36a + 6b + c \ldots(3)$

$a = \frac{1}{3}$

$b = \frac{-16}{3}$

$c = 24$

$y = \frac{1}{3}x^2 - \frac{16}{3}x + 24$

$$f(x) = \begin{cases} \frac{-13}{30}x^2 + \frac{97}{30}x + \frac{1}{5}, & 0 \le x \le 6 \\ \frac{1}{3}x^2 - \frac{16}{3}x + 24, & 6 < x \le 10 \end{cases}$$

b) As the robot is following a computer program it does not cope well when there are breaks in the line it follows or places where the curve is not smooth. Explain why the piecewise function is continuous for $x \in [0,10]$.

$$\lim_{x \to 6^-} f(x) = \lim_{x \to 6^+} f(x) = f(6) = 4$$

As the curve can be drawn without lifting the pencil from the paper it is continuous.

c) Determine if the piecewise function meets smoothly (is differentiable) at $(6,4)$.

$$\lim_{x \to 6^-} f'(x) = \frac{-59}{30}$$

$$\lim_{x \to 6^+} f'(x) = \frac{-4}{3}$$

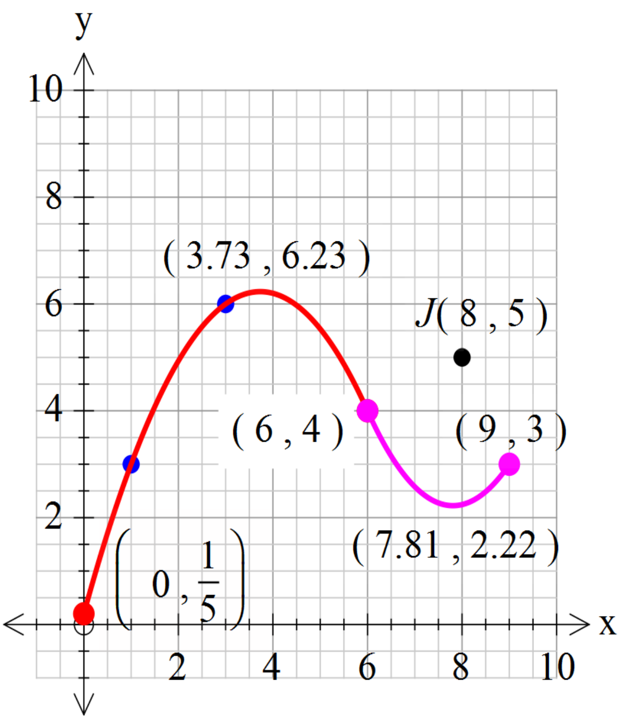
$$\therefore \lim_{x \to 6^-} f'(x) \neq \lim_{x \to 6^+} f'(x) = \frac{-4}{3}$$

$$\lim_{x \to 6^-} \left(\frac{d}{dx}(f1(x))\right) \quad \frac{-59}{30}$$

$$\lim_{x \to 6^+} \left(\frac{d}{dx}(f1(x))\right) \quad \frac{-4}{3}$$

To ensure that the robot does not encounter any difficulties John wishes to make the join at $(6,4)$ smooth. To do this he removes the point at (10,4). Curve 1 is to remain unchanged, but Curve 2 will now join smoothly at (6,4) and pass through (9,3).

d) Determine the rule for the piecewise function that passes over the given four points and that joins smoothly at (6,4). Include a statement of the piecewise function. Sketch accurately on the axes below.



$$12a + b = \frac{-59}{30} \ (1)$$

$$3 = 81a + 9b + c \ldots\ldots (2)$$

$$4 = 36a + 6b + c \ldots\ldots (3)$$

$$a = \frac{49}{90}$$

$$b = \frac{-17}{2}$$

$$c = \frac{177}{5}$$

$$f(x) = \begin{cases} \frac{-13}{30}x^2 + \frac{97}{30}x + \frac{1}{5}, & 0 \le x \le 6 \\ \frac{49}{90}x^2 - \frac{17}{2}x + \frac{177}{5}, & 6 < x \le 9 \end{cases}$$

$$\lim_{x \to 6^-} \left(\frac{d}{dx}(f(x))\right) \quad \frac{-59}{30}$$

$$\lim_{x \to 6^+} \left(\frac{d}{dx}(f(x))\right) \quad \frac{-59}{30}$$

John's robot is to demonstrated at a competition. A judge stands at the coordinates $J(8,5)$ to observe the movement of the robot.

e) Mark the position of the judge on the axes above.

The judge is very keen to know when the robot will be the minimum distance away from his location. Note that due to his position he will be closest to the quadratic that has domain $x \in [6,9]$.

f) Let the closest position be represented by the coordinates $A(g,h)$. Write the coordinates of this point in terms of g only.

$$\left(g, \frac{49}{90}g^2 - \frac{17}{2}g + \frac{177}{5}\right)$$

g) Using a suitable method:

- Use of distance formula and calculus
  OR
- The length of the line segment *AJ* will be a minimum when the line *AJ* is **perpendicular** to the curve.
  OR
- Make sure you can do both!

In your answer include the coordinates of the closest point to the judge and the minimum distance the robot is from the judge.

Define $d(g)=\sqrt{(g-8)^2+\left(\frac{49}{90}\cdot g^2-\frac{17}{2}\cdot g+\frac{177}{5}\right.}$ ▸

Done

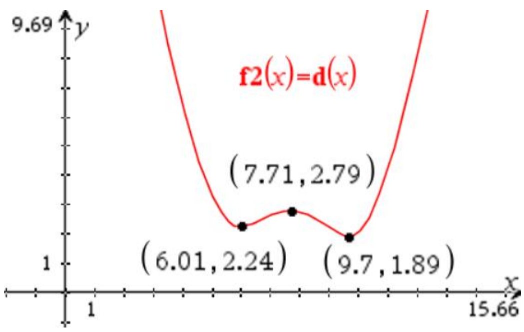
©Derivative=0

$\text{solve}\left(0=\frac{d}{dg}(d(g)),g\right)$

$g=6.00889$ or $g=7.71002$ or $g=9.69946$

Remember the CAS will find all points of zero gradient. Not just the minimum.

©domain is 6≤g≤9	
$d(7.71002)$	2.78612
$d(7.71)$	2.78612
©Check graph	
$d(6.00889)$	2.236
$d(7.71002)$	2.78612



Therefore, the closest point is when $g = 6.01$ where the distance is 2.236 metres.

The coordinates of this point are (6.01,3.98)

Alternatively, you can find when *AJ* is perpendicular to the curve.

$\left(g, \frac{49}{90}g^2 - \frac{17}{2}g + \frac{177}{5}\right)$

We need to equate the gradient of the line segment *AJ* with the negative reciprocal of the gradient at A.

$$\text{Define } h(x)=\frac{49}{90}\cdot x^2-\frac{17}{2}\cdot x+\frac{177}{5} \qquad \textit{Done}$$

$$\text{DelVar } g \qquad \textit{Done}$$

$$\text{solve}\left(\frac{h(g)-5}{g-8}=\frac{-1}{\frac{49\cdot g}{45}-\frac{17}{2}},g\right)$$

$$g=6.00889 \text{ or } g=7.71002 \text{ or } g=9.69946$$

This gives the same results as above so we can evaluate the y-value of the point. For this case we need to use Pythagoras with the two points found to find the distance.

$$\sqrt{(8-6.00889)^2+(5-3.982559361921)^2} \qquad 2.236$$

Remember here not to round off too soon. Keep the values accurate and round off at the end.