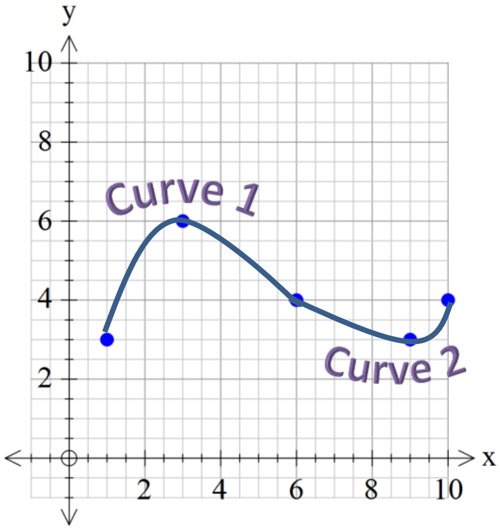
2018 SAC PREP 4 - solutions

Previously John, a robot engineer had a robot follow a curve based on a piecewise function.



He decides that his main criteria is that the join needs to be smooth. He wants to transform the first part of the piecewise function so that the two parts of the piecewise function meets at (x_T, y_T) smoothly. The piecewise functions is:

$$f(x)=\begin{cases} \frac{-13}{30}x^2+\frac{97}{30}x+\frac{1}{5}, & 0\le x\le 6 \\ \frac{1}{3}x^2-\frac{16}{3}x+24, & 6<x\le 10 \end{cases}$$

a) Given that $f_1(x)=\frac{-13}{30}x^2+\frac{97}{30}x+\frac{1}{5}$ and we wish to perform a translation of k units parallel to the y-axis. Create a new path $f_2(x)$ that would join smoothly. Sketch the result on the axes below.

$$Let\ f_3(x)=\frac{-13}{30}x^2+\frac{97}{30}x+\frac{1}{5}+k$$

$$f_2(x)=\frac{1}{3}x^2-\frac{16}{3}x+24$$

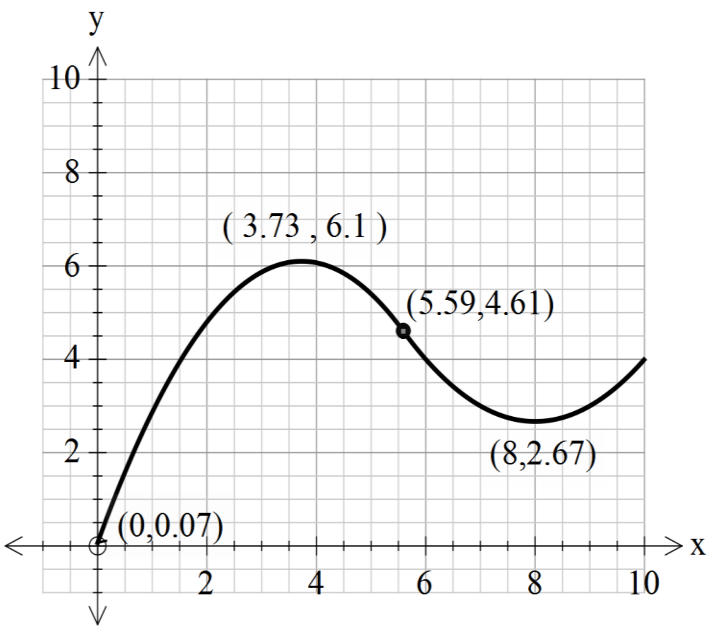
$$f_2(x)=f_3(x)$$

$$-\frac{13}{30}x^2+\frac{97}{30}x+\frac{1}{5}+k=\frac{1}{3}x^2-\frac{16}{3}x+24$$

$$f_3'(x)=f_2'(x)$$

$$\frac{-13}{15}x+\frac{97}{30}=\frac{2}{3}x-\frac{16}{3}$$

$(5.59, 4.18)$ *and* $k=-0.13$



See below for CAS instructions:

Define $f1(x)=\frac{-13}{30}\cdot x^2+\frac{97}{30}\cdot x+\frac{1}{5}$ *Done*

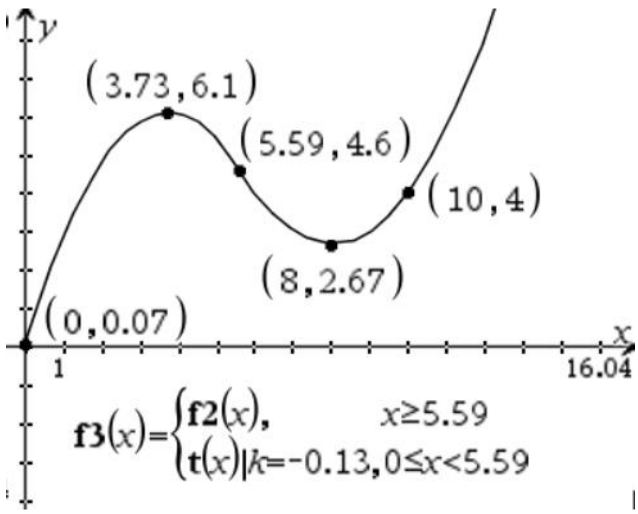
Define $f2(x)=\frac{1}{3}\cdot x^2-\frac{16}{3}\cdot x+24$ *Done*

Define $t(x)=f1(x)+k$ *Done*

$\text{solve}\left(\begin{cases} t(x)=f2(x) \\ \frac{d}{dx}(t(x))=\frac{d}{dx}(f2(x)) \end{cases}, x\right)$

$x=5.58695652174$ and $k=-0.130797101449$

$f2(5.58695652174)$ 4.60759294266



b) Investigate other options of transforming the first part of the piecewise function:

These could include:

Dilation from the x-axis

$$Let\ f_3(x)=k\left(\frac{-13}{30}x^2+\frac{97}{30}x+\frac{1}{5}\right)$$

$$f_2(x)=\frac{1}{3}x^2-\frac{16}{3}x+24$$

$$f_2(x)=f_3(x)$$

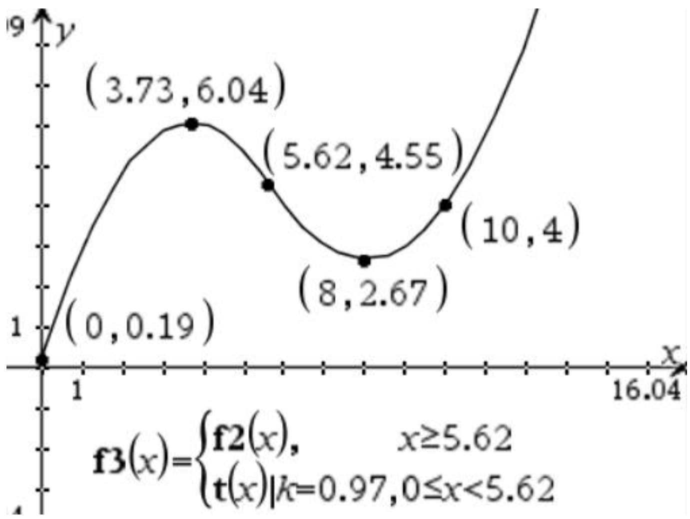
$$k\left(\frac{-13}{30}x^2+\frac{97}{30}x+\frac{1}{5}\right)=\frac{1}{3}x^2-\frac{16}{3}x+24$$

$$f_2'(x)=f_3'(x)$$

$$k\left(\frac{-13}{15}x+\frac{97}{30}\right)=\frac{2}{3}x-\frac{16}{3}$$

$$x=5.62,\ k=0.97$$

Intersection at (5.62,4.56)



John enters a competition where he needs to follow a line in the demonstration area. The top of the line is defined as:

$$y = -0.3(x-4)^2 + 7$$

The distance between the top of the line and the bottom of the line is 0.5 metres.

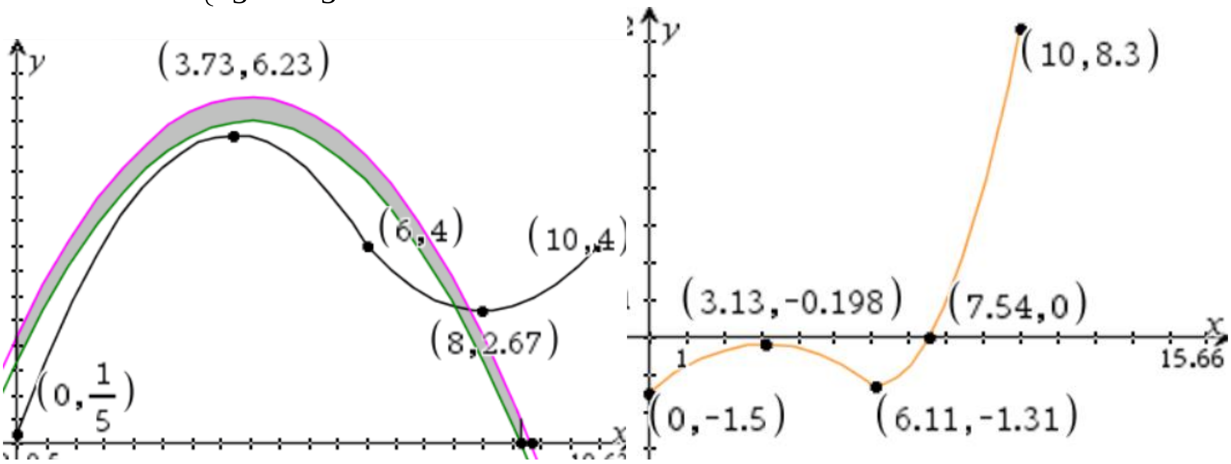
c) State the equation of the bottom of the line.

$$y' = -0.3(x-4)^2 + 6.5$$

d) Determine if John's robot will follow the line. Explain your decision with algebraic or graphical evidence.

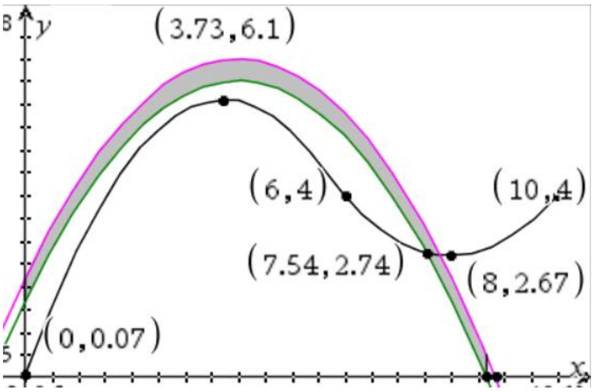
Using the original functions defined at the start of the session

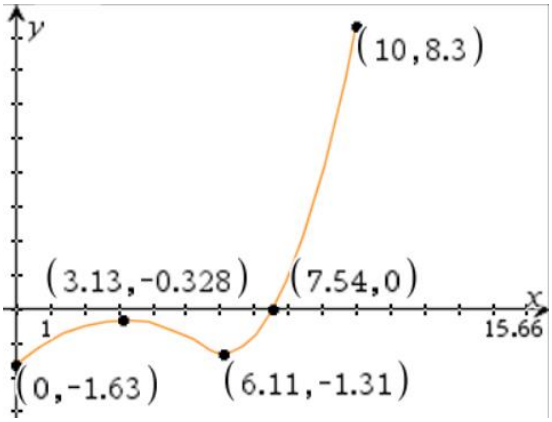
$$f(x) = \begin{cases} \frac{-13}{30}x^2 + \frac{97}{30}x + \frac{1}{5}, & 0 \le x \le 6 \\ \frac{1}{3}x^2 - \frac{16}{3}x + 24, & 6 < x \le 10 \end{cases}$$



- The robot has a continuous path for $x \in (0,10)$
- We know from SAC PREP 2 that the path of the robot is not differentiable at $x = 6$.
- If we consider the graph above we can see that the robot stays below the lower edge of the line until approximately $x = 7.5$. At this point it crosses over and diverges from the line. The second graph is a graph of the vertical distance between the robot and the bottom of the line. This shows a maximum deviation of 1.5 m below and 8.3 above the bottom of the line. Note you could compare the distance between the robot and the curve where the line meets the competition area boundary.

To achieve a curve that is smooth we would need to consider one of the transformed curves calculated above. We can see below that this yields a very similar result. For the translated track:





- The translated track crosses the line at (7.54,2.74) so it is unsuitable.

- The graph drawn to the right is the graph of the vertical distance between the curves. This shows that until x=7.54 the robot is below the bottom line with a maximum distance of 1.63 m from the robot then crosses the line and when x=8.65 (where the line meets the boundary of the competition area the deviation is 2.79 metres.

- In this situation neither graph is perfect.