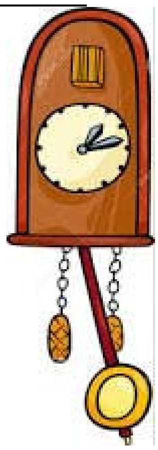
2018 SAC 2 - PREP 1

Question 1



A pendulum's motion relative to an equilibrium point can be modelled by the rule

$$f(x) = 3e^{-x}\cos(x), \text{ for } 0 \le x \le \pi.$$

Where x is the displacement in centimetres from the equilibrium point and t is measured in seconds.

a) Using algebra, find any axial intercepts.

x-int $\quad 0 = 3e^{-x}\cos(x)$

$3e^{-x} \neq 0$ (*statement required*) $\therefore \cos(x) = 0, \; x = \frac{\pi}{2}$

y-int $\quad y = 3e^{0}\cos(0) = 3$

3 marks

b) Use calculus to differentiate $f(x) = 3e^{-x}\cos(x)$, and hence, find any turning points.

$$f'(x) = 3e^{-x} \times -\sin(x) - 3e^{-x} \times \cos(x)$$

$$0 = -3e^{-x}[\sin(x) + \cos(x)]$$

$$-3e^{-x} \neq 0 \qquad \therefore \sin(x) + \cos(x) = 0$$

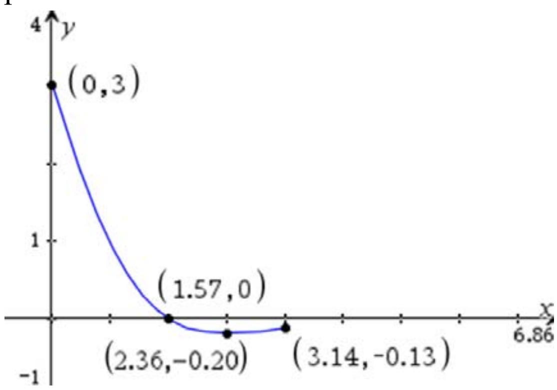
$$\sin(x) = -\cos(x)$$

$$\tan(x) = -1 \quad x = \frac{3\pi}{4}$$

when $x = \frac{3\pi}{4}$ $\; y = 3e^{\frac{-3\pi}{4}}\cos\left(\frac{3\pi}{4}\right) = \frac{-3\sqrt{2}}{2}e^{\frac{-3\pi}{4}}$ or if required as a decimal (2.36,–0.20)

3 marks

c) Sketch the graph of the pendulum's motion. labelling intercepts, turning points and end points.



(For this graph I have chosen to label the points to 2 decimal places – if not specified exact values should be shown). Remember the CAS will display three digits – FLOAT 3 – if you need more detail either use a calculator page or hover over the decimal and press + to increase the number of decimal places

3 marks

d) Determine the greatest distance the end of the pendulum travels before changing direction. After how long does this occur?

$d_{max} = 3 + \frac{3e^{-\frac{3\pi}{4}}\sqrt{2}}{2}\ cm$ $\qquad x = \frac{3\pi}{4} seconds$

2 marks

e) Find the average rate of change of the function between $x = 0$ and $x = \frac{3\pi}{4}$. Express your answer, correct to two decimal places.

$$AROC = \frac{f\left(\frac{3\pi}{4}\right) - f(0)}{\frac{3\pi}{4} - 0} = -1.36 cm/s$$

2 marks

f) Find the instantaneous velocity when $x = \frac{\pi}{2}$ as an exact value.

$$f'\left(\frac{\pi}{2}\right) = -3e^{-\frac{\pi}{2}}\ cm/s$$

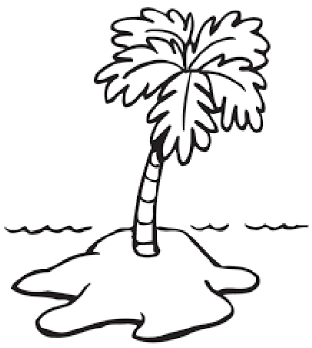
1 mark

g) Determine the period of the pendulum's motion.

$$Period = 2\pi\ \ s$$

1 mark

Question 2



Tom is marooned on an island in a region that has extreme tides. He is able to accurately measure the depth of the water around the island. At midday the depth of the water is 4.8 metres. He has also found that 24 hours separates the high tides. At low tide the depth of the water is 40 cm. The depth of the water can be modelled by the function of the form $h(t) = a\cos(bt) + c, t \geq 0$, where h is the depth of the water at time t hours after 12 noon.

a) Determine the value of a, b and c. 3 marks

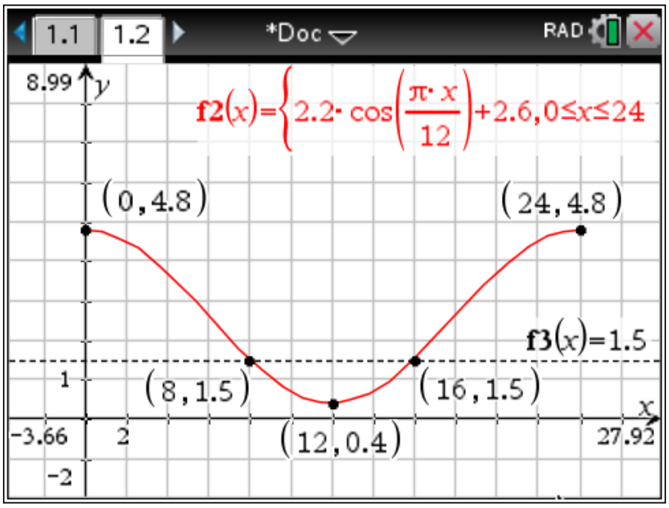
Period = 24 hours $\quad 24 = \frac{2\pi}{n} \quad n = \frac{\pi}{12}$

Amplitude = $\frac{1}{2}(4.8 - 0.40) = 2.2 \quad a = 2.2$

Centre is $\quad \frac{1}{2}(4.8 + 0.40) = 2.6 \quad c = 2.6$

A reef joins the island Tom is on, to a neighbouring island. The depth of the water on the reef is still modelled by the function given above. Tom reasons that he will have a greater chance of being rescued if he can make it to the other island. Being unable to swim, he needs the water to be no deeper than 1.5 metres in order to complete the crossing.

c) Sketch the first period of the graph of *x* against *t*, indicating on your graph the times available to him to complete the crossing.



He can cross between $t = 8 \text{ and } t = 16$. This corresponds to between 8pm and 4am the following morning.

4 marks

d) How much time does have to complete the crossing.

8 hours

1 mark

e) If Tom is afraid of sharks and will only cross if the depth of the water is less than 1 metre, How will this affect the time he has to make the crossing?

$$1 = 2.2\cos\left(\frac{\pi t}{12}\right) + 2.6$$

$$t = 9.11055 \text{ or } \quad t = 14.8895$$

$$\Delta t = 5.7789 \text{ hours}$$

He will have 2 hours and 13 minutes less time to cross to the other island.

3 marks

To his horror, Tom estimates that it will take him more than 10 hours to complete the crossing. He decides to build a fire at the high point of the island to attract a passing ship. Tom calculates that the type of ship that passes the island needs a depth of at least 4.5 metres so that it does not run aground.

f) Between what times during the day can a ship approach the island?

$4.5 = x(t)$

$t = 2.0182 \text{ or } t = 21.9818$

Therefore, remembering that t=0 corresponds to noon each day. The ship can approach from noon to 2:01pm and then again after 9:59 am until the following noon.

(note here that 2.0182 hours is not 2 hours and 2 minutes)

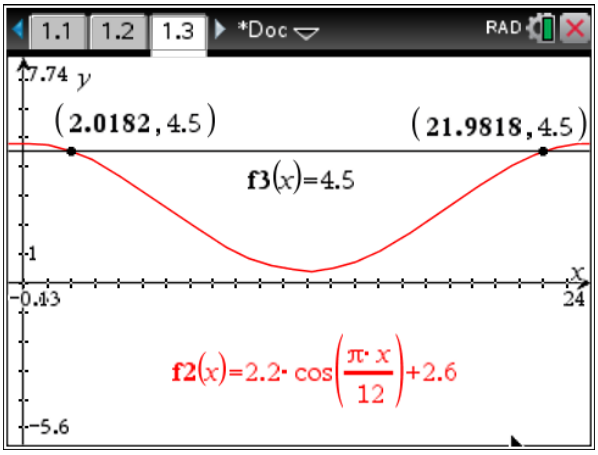
2 marks

g) If the ship must travel 10 km to reach the island, what speed in km/hr must the ship average to save Tom?

$s = \frac{10}{2.0182} = 4.95 km/hr$

Note this would allow the ship to reach Tom before the height of the water is less than 4.5 m. Otherwise it would require a lesser speed.

2 marks



Note: Here I have used Menu, Geometry, Points & Lines, Intersection Points to find the 2 coordinates where the graph intersects with the line $y = 4.5$. I have expanded the decimals for the x-values and then stored the values as t1 and t2 (hence why the numbers are bold. I can then use this definition on a calculator page to find the required speed.

$$\frac{10}{t1} \quad 4.95496902367$$