2018 SAC 3 PREP 2 - SOLUTIONS

Question 1

At a cinema, the time T minutes, that customers have to wait in order to buy their tickets has a continuous probability density function given by

$$f(t)=\begin{cases} bt^2 & \text{for} \quad 0 \le t \le 4 \\ c(8-t) & \text{for} \quad 4 < t \le 8 \\ 0 & \text{otherwise} \end{cases}$$

a. Show that $b=\dfrac{3}{160}$ and $c=\dfrac{3}{40}$.

Continuous so

$16b = 4c$

$c = 4b$

$\int_0^4 bt^2dt + \int_4^8 c(8-t)dt = 1$

$\int_0^4 bt^2dt + \int_4^8 4b(8-t)dt = 1$

$\left[\dfrac{bt^3}{3}\right]_0^4 + \left[32bt - 2bt^2\right]_4^8 = 1$

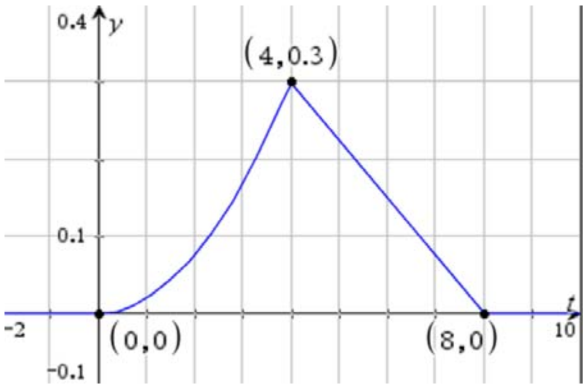
$\dfrac{64b}{3} - 0 + 128b - 96b = 1$

$\dfrac{160b}{3} = 1$

$c = 4\left(\dfrac{3}{160}\right) = \dfrac{3}{40}$

3 marks

b. Sketch the graph of $y = f(t)$ on the axes provided, clearly labelling the scale.



Don't forget the zero sections

3 mark

c. Use calculus to find the probability that a customer waits more than six minutes to buy a ticket.

$\Pr(T>6) = \int_6^8 f(t)dt$

$\Pr(T>6) = \left[\dfrac{3}{40}\left(8t - \dfrac{t^2}{2}\right)\right]_6^8$

$\Pr(T>6) = \dfrac{12}{5} - \dfrac{9}{4}$

$\Pr(T>6) = \dfrac{3}{20}$

3 marks

d. Find the mean time that customers have to wait in order to buy their tickets, give your answer, correct to one decimal place.

$E(T)=\int_{-\infty}^{\infty} t \times f(t)dt$

$E(T)=\frac{22}{5}$

$E(T)=4.4\text{ minutes}$

2 marks

e. Find the median time, correct to two decimal places, that customers have to wait in order to buy their tickets.

$\frac{1}{2}=\int_{-\infty}^{m} f(t)dt$

$m=4.35\text{ minutes}$

2 marks

f. State one reason why the mean and the median have similar values.

This may occur as the distribution is nearly symmetric around $t = 4$.

2 marks

The running times of movies shown at the cinema are normally distributed with a mean of 94 minutes, with a standard deviation of 10 minutes.

g. Find the probability that the running time of a movie is more than 109 minutes, give your answer correct to three decimal places.

Let Y = running time of a movie in minutes

$Y \sim N(94,10^2)$ $\Pr(X > 109) = 0.067$

1 mark

h. A certain cinema complex has four different movies playing. Find the probability that at least two of the movies have a running time of more than 109 minutes, give your answer correct to three decimal places.

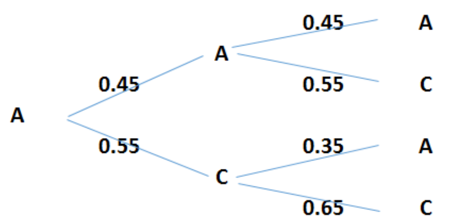
$Y \sim Bi(4,0.067)$ $\Pr(Y \geq 2) = 0.025$

2 marks

Pete goes to the movies once every week. He likes to see only action or comedy movies.
If he sees an action movie one week, the probability that he sees an action movie the following week is 0.45, while if he sees a comedy one week, the probability that he sees an

action movie the following week is 0.35. Suppose he has just seen an action movie.

i. What is the probability that of the next three movies he sees, exactly two are comedies, give your answer correct to three decimal places.



Note: I have drawn the next 2 movies.

$\Pr(2\ comedies) = \Pr(ACC) + \Pr(CAC) + \Pr(CCA)$

$\Pr(2\ comedies) = 0.45 \times 0.55 \times 0.65 + 0.55 \times 0.35 \times 0.55 + 0.55 \times 0.65 \times 0.35$

$\Pr(2\ comedies) = 0.392$

2 marks

Total 20 marks

Question 2

At George's garage, the time, X hours, taken to service a car follows the probability density function f where

$$f(x)=\begin{cases}0 & x<2\\ \dfrac{(x-5)^2+1}{12} & 2\le x\le 5\\ 0 & x>5\end{cases}$$

a. Find the mean time taken to service a car at George's garage. 1 mark

$$\text{mean}=E(X)=\int_2^5\left(x\times\frac{(x-5)^2+1}{12}\right)dx=\frac{47}{16}\ hours$$

Last week there were 40 cars serviced at George's garage. Andrew's car was among the fastest 25% of cars to be serviced there last week.

b. What was the maximum time that Andrew's car could have taken to be serviced? Express your answer in hours, correct to 3 decimal places.

$\int_2^m \frac{(x-5)^2+1}{12}dx=0.25$ for m. **(1 mark)**

$m=2.3318...$ **(1 mark)**

The maximum time was 2.332 hours (correct to 3 decimal places).

2 marks

The diameters, in mm, of a consignment of 700 cylindrical filters used at the garage are normally distributed with a mean of 24mm and a standard deviation of 0.3mm.

George has found that the widest 20% of these filters cannot be used in Mini cars.

c. What is the maximum diameter of a filter that can be used in a Mini car? Express your answer in mm correct to 2 decimal places. Use a suitable diagram in your solution.

$X\sim N(24,0.3^2)$

$\Pr(X<x)=0.8$

$x=24.252486...$ (using the inverse normal function)

Filters with a maximum diameter of 24.25mm (correct to 2 decimal places) can be used in a Mini car.

2 marks

Filters with a diameter of less than 23.5 mm will fit in any car.

d. How many filters in this consignment will fit in any car? Express the answer correct to the nearest whole number.

$\Pr(X<23.5)=0.047790...$ (using the normal cdf function) **(1 mark)**

$0.047790...\times 700=33.4532...$

So 33 of the filters will fit in any car (to the nearest whole number)(**(1 mark)**

2 marks

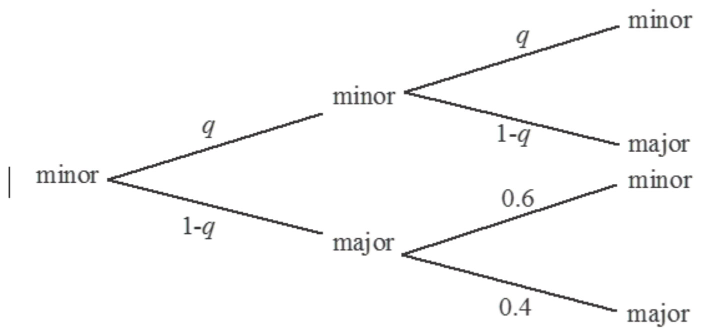
The services George performs at his garage are classified as major or minor services depending on the number of kilometres a car has done.

George starts each service and often hands over to another mechanic to finish a service. He has found over time that if a service he starts is major then the probability that the next service he starts is also major is 0.4.

If a service he starts is minor then the probability that the next service he starts is minor is q where $0 < q < 1$ and q can vary from day to day.

The first service George starts with each day is a minor service.

e. **i.** Use a tree diagram to represent the three services completed by George.



2 marks

ii. Find the probability, expressed in terms of q, that the third service George starts in a day is a major service.

$$\Pr(3^{\text{rd}}\text{ service is major})$$
$$= q(1-q) + (1-q)\times 0.4$$
$$= q - q^2 + 0.4 - 0.4q$$
$$= -q^2 + 0.6q + 0.4$$

2 marks

iii. On a particular day the probability that the third service of the day was a major one was 0.2. Find the value of q on this day expressing your answer correct to 3 decimal places.

$$-q^2 + 0.6q + 0.4 = 0.2$$
$$q = -0.23851\ldots \text{ or } q = 0.83851\ldots$$
$$\text{Since } 0 < q < 1,\ q = 0.839 \text{ (correct to 3 decimal places)}$$

2 marks

The garage receives a daily delivery of parts. The probability that this delivery occurs before noon is 0.8 and the time that the delivery takes place one day is independent of the time it takes place the next day.

f. **i.** The probability that over an n day period the delivery occurred each day before noon is 0.32768. Find the value of n.

1 mark

$$(0.8)^n = 0.32768 \text{ for } n$$
$$n = 5$$

ii. Find the probability that over a 20 day period, the delivery arrives before noon on more than 16 days given that on at least half of the 20 days it arrives before noon. Express your answer correct to 4 decimal places.

$$\Pr(X > 16 \mid X \geq 10)$$
$$= \frac{\Pr(X > 16 \cap X \geq 10)}{\Pr(X \geq 10)}$$
$$= \frac{\Pr(X > 16)}{\Pr(X \geq 10)} \qquad \begin{array}{l}(\text{binom cdf}(20,0.8,17,20))\\(\text{binom cdf}(20,0.8,10,20))\end{array}$$
$$= \frac{0.4114488...}{0.99943...}$$
$$= 0.4117 \text{ (correct to 4 decimal places)}$$

2 marks

George receives a mail delivery each work day. The probability of the mail being delivered before a certain time can be expressed as $\Pr(T < t) = m$, where $m \in (0,1)$.

g. Find an expression that represents the mail being delivered before this certain time on 2 of the five days.

$$\Pr(T < t) = \binom{5}{2} m^2 (1-m)^3$$
$$\Pr(T < t) = 10m^2(1-m)^3$$

2 marks

h. Show that the maximum value this probability can take is 0.3456. Also find the value of *m* for this probability.

$$\frac{dP}{dm} = -10m(m-1)^2(5m-2)$$
$$0 = -10m(m-1)^2(5m-2)$$
$$m = 0, 1, \frac{2}{5}$$
$$m \in (0,1) \quad m = \frac{2}{5}$$
$$\Pr(T < t) = 10\left(\frac{2}{5}\right)^2\left(\frac{3}{5}\right)^3 = 0.3456$$

3 marks

i. If it is known that the delivery times are actually normally distributed with mean of midday (12 o'clock) and standard deviation of 1 hour, find the certain time *t*, correct to the nearest minute.

$X \sim N(12, 1^2)$

$\Pr(\text{T} < t) = 0.4$

$t = 11.746...$ (using the inverse normal function)

$\therefore 11:45$ am

2 marks

Total 24 marks